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Research Article

Dynamics of a Rational Difference Equation

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The main goal of the paper is to investigate boundedness, invariant intervals, semicycles, and global attractivity of all nonnegative solutions of the equation $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-k})/(1 + x_{n-k})$, $n \in \mathbb{N}_0$, where the parameters $\alpha, \beta, \gamma \in [0, \infty)$, $k \geq 2$ is an integer, and the initial conditions $x_{-k}, \ldots, x_0 \in [0, \infty)$. It is shown that the unique positive equilibrium of the equation is globally asymptotically stable under the condition $\beta \leq 1$. The result partially solves the open problem proposed by Kulenović and Ladas in work (2002).

1. Introduction

The aim of this paper is to study the dynamical behavior of the following rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $\alpha, \beta, \gamma \in [0, \infty)$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $k \geq 2$ is an integer, and the initial conditions $x_{-k}, \ldots, x_0 \in [0, \infty)$. The related case where $k = 1$ was investigated in [1].

In 2002, Kulenović and Ladas [1] proposed the following open problem.

Open Problem 1. Assume that $\alpha, \beta, \gamma \in [0, \infty)$ and $k \in \{2, 3, \ldots\}$. Investigate the global behavior of all positive solutions of (1.1).

If we allow the parameters to be satisfied, $\alpha\beta\gamma = 0$, then (1.1) contains, as special cases, six difference equations with positive parameters:

$$x_{n+1} = \frac{\alpha}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

$$x_{n+1} = \frac{\beta x_n}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{1 + x_{n-k}}, \quad n \in \mathbb{N}_0. \tag{1.7}$$

Equations (1.2) and (1.3) can be reduced to (1.5), which was studied in [2]. By the change of variables $x_n = 1/y_n$, equation (1.4) can be reduced to the linear equation

$$y_n = \frac{1}{r} + \left(\frac{1}{r}\right) y_{n-k}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

whose global behavior of solutions is easily derived. Equation (1.6) is investigated in [3].

Equation (1.7) is essentially similar to the Riccati equation. In fact, if $\{x_n\}_{n=-k}^{\infty}$ is a positive solution of (1.7), then $\{x_{-k+(k+1)n}\}_{n=0}^{\infty}, \{x_{-k+1+(k+1)n}\}_{n=0}^{\infty}, \ldots, \{x_{(k+1)n}\}_{n=0}^{\infty}$ are $k+1$ solutions of the following Riccati equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \gamma x_n}{1 + x_n}, \quad n = -k, -k+1, -k+2, \ldots, \\ x_{n+1} &= \frac{\alpha + \gamma x_n}{1 + x_n}, \quad n = -k+1, -k+2, \ldots, \\ &\vdots \\ x_{n+1} &= \frac{\alpha + \gamma x_n}{1 + x_n}, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{1.9}$$

respectively.

Therefore, we only pay attention to investigating (1.1) with positive parameters and omit any further discussion of the above (1.2)–(1.7).

For other related results on difference equations, one can refer to [4–18].

Equation (1.1) has a unique equilibrium $\overline{x}$ which is positive and is given by

$$\overline{x} = \frac{\beta + \gamma - 1 + \sqrt{(\beta + \gamma - 1)^2 + 4\alpha}}{2}. \tag{1.10}$$

In [19], the authors investigated the global asymptotic stability of the positive equilibrium $\overline{x}$ of (1.1). We summarize their results in the following theorem.

Theorem 1.1. (i) *Assume that $\beta < 1$. Then the positive equilibrium $\overline{x}$ of* (1.1) *is locally asymptotically stable.*

(ii) *Assume that $\beta < 1$ and $\gamma \geq \alpha/(1-\beta)$. Then for every solution of* (1.1) *with initial conditions in invariant interval $[0, (\gamma-\alpha)/\beta]$, the positive equilibrium point $\overline{x}$ is globally asymptotically stable.*

(iii) *Assume that $\beta < 1$ and $\alpha < \gamma < \alpha/(1-\beta)$. Then for every solution of* (1.1) *with initial conditions in invariant interval $[(\gamma-\alpha)/\beta, (\alpha\beta+\gamma^2-\alpha\gamma)/(\beta+\gamma-\alpha-\beta^2)]$, the positive equilibrium point $\overline{x}$ is globally asymptotically stable.*

Reviewing the proof of Theorem 1.1, one can easily find that the positive equilibrium $\overline{x}$ of (1.1) is locally asymptotically stable when $\beta = 1$. So Theorem 1.1 (i) can be improved as the following theorem.

Theorem 1.2. *Assume that $\beta \leq 1$. Then the positive equilibrium $\overline{x}$ of* (1.1) *is locally asymptotically stable.*

By the change of variables $x_n = \gamma y_n$, (1.1) reduces to the difference equation

$$y_{n+1} = \frac{q + r y_n + y_{n-k}}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

where $p = 1/\gamma$, $q = \alpha/\gamma^2$, and $r = \beta/\gamma$. Its unique positive equilibrium is

$$\overline{y} = \frac{(r+1-p) + \sqrt{(r+1-p)^2 + 4q}}{2}. \tag{1.12}$$

Motivated by the above open problem, the purpose of this paper is to investigate the boundedness, local stability, invariant intervals, semicycles, and global attractivity of all nonnegative solutions of (1.11). We show that the unique positive equilibrium of (1.11) is a global attractor when $p \geq r$ and our result solves the open problem when $\beta \leq 1$.

The organization of this paper is as follows. In Section 2, some basic definitions and lemmas regarding difference equations are given. The boundedness and invariant intervals of (1.11) are discussed in Section 3. The semicycle analysis of (1.11) is presented in Section 4. The main results are formulated and proved in Section 5, where the global asymptotic stability of (1.11) is investigated.

2. Some Lemmas

In this section, we recall some definitions and lemmas which will be useful in the sequel.

Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for initial conditions $x_{-k}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

An internal $J \subseteq I$ is called an invariant interval of (2.1) if

$$x_{-k}, \dots, x_0 \in J \Longrightarrow x_n \in J \quad \forall n \in \mathbb{N}. \tag{2.2}$$

That is, every solution of (2.1) with initial conditions in J remains in J.

Definition 2.1. Let $\overline{x}$ be an equilibrium point of (2.1).

(i) The equilibrium point $\overline{x}$ of (2.1) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_{-k}, \dots, x_0 \in I$ with $\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta$, one has $|x_n - \overline{x}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ of (2.1) is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for all $x_{-k}, \dots, x_0 \in I$ with $\sum_{i=-k}^{0} |x_i - \overline{x}| < \gamma$ one has $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of (2.1) is called a global attractor if for every $x_{-k}, \dots, x_0 \in I$ one has $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of (2.1) is called globally asymptotically stable if it is locally asymptotically stable and is a global attractor.

(v) The equilibrium point $\overline{x}$ of (2.1) is called unstable if it is not locally stable.

The following lemmas can be found in [3, 20], respectively; also see [19, 21–24].

Lemma 2.2 (see [3]). *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties.*

(i) *$f(x, y)$ is nondecreasing in each of its arguments.*

(ii) *The equation*

$$f(x, x) = x \tag{2.3}$$

has a unique positive solution in the interval $[a, b]$.

Then (2.1) *has a unique positive equilibrium $\overline{x} \in [a, b]$ and every solution of* (2.1) *converges to $\overline{x}$.*

Lemma 2.3 (see [20]). *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties.*

(i) *$f(x, y)$ is a nondecreasing function in x and a nonincreasing function in y.*

(ii) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the following system:*

$$m = f(m, M), \qquad M = f(M, m), \tag{2.4}$$

then $m = M$.

Then (2.1) *has a unique equilibrium point $\overline{x} \in [a, b]$ and every solution of* (2.1) *converges to $\overline{x}$.*

The following result was proved in [25].

Lemma 2.4 (see [25]). *Assume that $a > 0$ and $A > b > 0$ hold. Then the positive equilibrium $\overline{x}$ of the difference equation*

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

is a global attractor of all positive solutions.

3. Boundedness and Invariant Intervals

The following result about boundedness of (1.11) can be found in [19].

Theorem 3.1. *Every solution of* (1.11) *is bounded from above and from below by positive constants.*

Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of (1.11). Then the following identities are easily established:

$$y_{n+1} - 1 = \frac{r(y_n - ((p-q)/r))}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

$$y_{n+1} - \frac{p-q}{r} = \frac{r(y_n - ((p-q)/r)) + (1 - ((p-q)/r))(p + y_{n-k})}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

$$y_{n+1} - \frac{q}{p-r} = \frac{r(y_n - (q/(p-r))) + (1 - (q/(p-r)))y_{n-k}}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

$$y_{n+1} - y_n = \frac{(p-r)((q/(p-r)) - y_n) + (1 - y_n)y_{n-k}}{p + y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

If $p = q + r$, then the unique equilibrium is $\overline{y} = 1$ and (3.1) and (3.4) become

$$y_{n+1} - 1 = \frac{r(y_n - 1)}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

$$y_{n+1} - y_n = \frac{(1 - y_n)(q + y_{n-k})}{p + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

respectively.

If $p = r$, then the unique equilibrium is $\overline{y} = (1 + \sqrt{1 + 4q})/2$ and (3.4) becomes

$$y_{n+1} - y_n = \frac{q + (1 - y_n)y_{n-k}}{p + y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{3.7}$$

Set

$$f(x, y) = \frac{q + rx + y}{p + y}. \tag{3.8}$$

Then we have the following monotone character for the function $f(x, y)$.

Lemma 3.2. *Let $f(x,y)$ be defined by* (3.8). *Then the following statements hold true.*

(i) *Assume that $p > q$. Then $f(x,y)$ is strictly increasing in each of its arguments for $x < (p-q)/r$ and it is strictly increasing in x and decreasing in y for $x \geq (p-q)/r$.*

(ii) *Assume that $p \leq q$. Then $f(x,y)$ is strictly increasing in x and decreasing in y for $x \geq 0$.*

Proof. By calculating, the partial derivatives of the function $f(x,y)$ are

$$f'_x(x,y) = \frac{r}{p+y}, \qquad f'_y(x,y) = \frac{p-q-rx}{(p+y)^2}, \tag{3.9}$$

from which (i) and (ii) easily follow. □

3.1. The Case $p > r$

Lemma 3.3. *Assume that $p > q + r$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11). *Then the following statements are true.*

(i) *If for some $N \geq 0$, $y_N < (p-q)/r$, then $y_n < 1$ for $n > N$.*

(ii) *If for some $N \geq 0$, $y_N = (p-q)/r$, then $y_{N+1} = 1$.*

(iii) *If for some $N \geq 0$, $y_N > (p-q)/r$, then $y_{N+1} > 1$.*

(iv) *If for some $N \geq 0$, $y_N \geq q/(p-r)$, then $y_n > q/(p-r)$ for $n > N$.*

(v) *If for some $N \geq 0$, $y_N \leq q/(p-r)$, then $y_{N+1} > y_N$.*

(vi) *If for some $N \geq 0$, $y_N \geq 1$, then $y_{N+1} < y_N$.*

(vii) *Equation* (1.11) *possesses an invariant interval $[q/(p-r),(p-q)/r]$ and $\overline{y} \in (q/(p-r),(p-q)/r)$; moreover, the interval $[q/(p-r),1]$ is also an invariant interval of* (1.11) *and $\overline{y} \in (q/(p-r),1)$.*

Proof. (i)–(vi) Clearly, in this case $q/(p-r) < 1 < (p-q)/r$ holds. The statements directly follow by using the identities (3.1), (3.3) and (3.4).

(vii) By Lemma 3.2 (i) the function $f(x,y)$ is strictly increasing in each of its arguments for $x < (p-q)/r$. Given that $y_{-k},\ldots,y_{-1},y_0 \in [q/(p-r),(p-q)/r]$, then we get

$$\begin{aligned} y_1 = f(y_0,y_{-k}) \geq f\left(\frac{q}{p-r},\frac{q}{p-r}\right) &= \frac{q+(qr/(p-r))+(q/(p-r))}{p+(q/(p-r))} = \frac{q(p+1)}{p^2-pr+q} > \frac{q}{p-r}, \\ y_1 = f(y_0,y_{-k}) \leq f\left(\frac{p-q}{r},\frac{p-q}{r}\right) &= \frac{q+(p-q)+(p-q)/r}{p+((p-q)/r)} = 1 < \frac{p-q}{r}, \end{aligned} \tag{3.10}$$

which implies that $y_1 \in [q/(p-r),1] \subset [q/(p-r),(p-q)/r]$. By the induction, $y_n \in [q/(p-r),1] \subset [q/(p-r),(p-q)/r]$ for every $n \in \mathbb{N}$, as claimed.

On the other hand, $p > q + r$ implies that

$$\overline{y} = \frac{(r+1-p)+\sqrt{(r+1-p)^2+4q}}{2} < \frac{(r+1-p)+\sqrt{(r+1-p)^2+4(p-r)}}{2} = 1. \tag{3.11}$$

Furthermore, $\overline{y}$ is the unique positive root of the quadratic equation

$$y^2 + (p - r - 1)y - q = 0. \tag{3.12}$$

Since

$$\left(\frac{q}{p-r}\right)^2 + (p - r - 1)\frac{q}{p-r} - q = \frac{q(q + r - p)}{(p-r)^2} < 0, \tag{3.13}$$

then we have that $\overline{y} > q/(p - r)$. That is, $\overline{y} \in (q/(p - r), 1) \subset [q/(p - r), (p - q)/r]$, finishing the proof. □

When $p = q + r$, identities (3.5) and (3.6) imply that the following results hold.

Lemma 3.4. *Assume that $p = q + r$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11). *Then the following statements are true.*

(i) *If for some $N \geq 0$, $y_N > 1$, then $y_n > 1$ for $n > N$.*

(ii) *If for some $N \geq 0$, $y_N = 1$, then $y_n = 1$ for $n > N$.*

(iii) *If for some $N \geq 0$, $y_N < 1$, then $y_n < 1$ for $n > N$.*

(iv) *If for some $N \geq 0$, $y_N > 1$, then $y_{N+1} < y_N$.*

(v) *If for some $N \geq 0$, $y_N < 1$, then $y_{N+1} > y_N$.*

Lemma 3.5. *Assume that $r < p < q + r$ holds and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of equation* (1.11). *Then the following statements are true*

(i) *If for some $N \geq 0$, $y_N \leq q/(p - r)$, then $y_n < q/(p - r)$ for $n > N$.*

(ii) *If for some $N \geq 0$, $y_N \geq q/(p - r)$, then $y_{N+1} < y_N$.*

(iii) *If for some $N \geq 0$, $y_N \leq 1$, then $y_{N+1} > y_N$.*

(iv) *Equation* (1.11) *possesses an invariant interval $[1, q/(p - r)]$ and $\overline{y} \in (1, q/(p - r))$. Further, if $p \leq q$, then $y_n \geq 1$ for all $n \in \mathbb{N}$, and if $p > q$, then the following statements are also true.*

(v) *If for some $N \geq 0$, $y_N > (p - q)/r$, then $y_n > 1$ for $n > N$.*

(vi) *If for some $N \geq 0$, $y_N = (p - q)/r$, then $y_{N+1} = 1$.*

(vii) *If for some $N \geq 0$, $y_N < (p - q)/r$, then $y_{N+1} < 1$.*

Proof. In this case $(p-q)/r < 1 < q/(p-r)$ holds.

(i)–(iii) Using the identities (3.3) and (3.4), one can see that the results follow.

(iv) By Lemma 3.2 the function $f(x,y)$ is strictly increasing in x and decreasing in y in $[1, q/(p-r)]$. Then we can get

$$\begin{gathered} y_1 = f(y_0, y_{-k}) \le f\left(\frac{q}{p-r}, 1\right) \le f\left(\frac{q}{p-r}, \frac{p-q}{r}\right) = \frac{pqr + (p-r)(p-q)}{(p-r)(pr+p-q)} < \frac{q}{p-r}, \\ y_1 = f(y_0, y_{-k}) \ge f\left(1, \frac{q}{p-r}\right) \ge f\left(\frac{p-q}{r}, \frac{q}{p-r}\right) = 1 > \frac{p-q}{r}, \end{gathered} \tag{3.14}$$

which implies that $y_1 \in [1, q/(p-r)]$. By the induction, $y_n \in [1, q/(p-r)]$ for every $n \in \mathbb{N}$.

On the other hand, $p < q + r$ implies that

$$\overline{y} = \frac{(r+1-p) + \sqrt{(r+1-p)^2 + 4q}}{2} > \frac{(r+1-p) + \sqrt{(r+1-p)^2 + 4(p-r)}}{2} = 1. \tag{3.15}$$

Then as same as the argument in Lemma 3.3 (vii) it can be proved that $\overline{y} \in (1, q/(p-r))$.

Further, if $p \le q$, then the identity (3.1) implies that $y_n \ge 1$ for all $n \in \mathbb{N}$, and if $p > q$, the results (v)–(vii) can also follow from (3.1).

The proof is complete. □

3.2. The Case $p \le r$

Lemma 3.6. *Assume that $p \le r$. Then the interval $[1, \infty)$ is an invariant interval of* (1.11) *and $\overline{y} > 1$. Further, if $p > q$, then the following statements are also true.*

(i) *If for some $N \ge 0$, $y_N > (p-q)/r$, then $y_n > 1$ for $n > N$.*

(ii) *If for some $N \ge 0$, $y_N = (p-q)/r$, then $y_{N+1} = 1$.*

(iii) *If for some $N \ge 0$, $y_N < (p-q)/r$, then $y_{N+1} < 1$.*

Proof. Notice that in this case the inequality $(p-q)/r < 1$ holds; then the results can easily be obtained by using the identity (3.1). Further, $p \le r$ implies that

$$\overline{y} = \frac{(r+1-p) + \sqrt{(r+1-p)^2 + 4q}}{2} \ge \frac{1 + \sqrt{1+4q}}{2} > 1, \tag{3.16}$$

finishing the proof. □

4. Semicycle Analysis

Here, we present some results regarding the semicycle analysis of the solutions of (1.11). Now recall two definitions from [25].

Definition 4.1. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (2.1). A positive semicycle of the solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\begin{aligned} &\text{either} \quad l = -k \quad \text{or} \quad l > -k, \; x_{l-1} < \overline{x}, \\ &\text{either} \quad m = \infty \quad \text{or} \quad m < \infty, \; x_{m+1} < \overline{x}. \end{aligned} \tag{4.1}$$

Definition 4.2. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (2.1). A negative semicycle of the solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\begin{aligned} &\text{either} \quad l = -k \quad \text{or} \quad l > -k, \; x_{l-1} < \overline{x}, \\ &\text{either} \quad m = \infty \quad \text{or} \quad m < \infty, \; x_{m+1} < \overline{x}. \end{aligned} \tag{4.2}$$

The next two lemmas can be found in [26] and [19], respectively.

Lemma 4.3 (see [26]). *Assume that $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$ and that $f(x,y)$ is increasing in both arguments. Let $\overline{x}$ be a positive equilibrium of* (2.1)*. Then except possibly for the first semicycle, every oscillatory solution of* (2.1) *has semicycles of length at most k.*

Lemma 4.4 (see [19]). *Assume that $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$ and that $f(x,y)$ is increasing in x for each fixed y and is decreasing in y for each fixed x. Let $\overline{x}$ be a positive equilibrium of* (2.1)*. If $k \geq 2$, then every oscillatory solution of* (2.1) *has semicycles that are either of length at least $k+1$ or of length at most $k-1$.*

Using the monotonic character of the function $f(x,y)$ from Lemma 3.2, in each of the intervals in Lemmas 3.3–3.6, together with Lemmas 4.3 and 4.4, it is easy to obtain the following results concerning semicycle analysis.

Theorem 4.5. *Assume that $p > r$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11)*. Then the following statements are true.*

(i) *If $p > q + r$, then, except possibly for the first semicycle, every oscillatory solution of* (1.11) *which lies in the invariant interval $[q/(p-r), (p-q)/r]$ has semicycles of length at most k.*

(ii) *If $p = q + r$, then* (1.11) *does not have oscillatory solution with $y_i - 1 \geq 0$ or $y_i - 1 < 0$, $i = -k, \dots, 0$.*

(iii) *If $r < p < q + r$, then, except possibly for the first semicycle, every oscillatory solution of* (1.11) *which lies in the invariant interval $[1, q/(p-r)]$ has semicycles that are either of length at least $k+1$ or of length at most $k-1$.*

Theorem 4.6. *Assume that $p \le r$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11). *Then, except possibly for the first semicycle, every oscillatory solution of* (1.11) *which lies in the invariant interval* $[1, \infty)$ *has semicycles that are either of length at least $k+1$ or of length at most $k-1$.*

5. Global Asymptotic Stability for the Case $p \ge r$

In this section, we discuss the global attractivity of the positive equilibrium of (1.11). We show that $\overline{y}$ is a global attractor of all nonnegative solutions of (1.11) when $p \ge r$. Further, the unique positive equilibrium $\overline{x}$ of (1.1) is globally asymptotically stable when $\beta \le 1$.

Theorem 5.1. *Assume that $p \ge r$. Then the unique positive equilibrium $\overline{y}$ of* (1.11) *is a global attractor.*

The proof is finished by considering the following four cases; see Theorems 5.3, 5.5, 5.7, and 5.9.

Theorem 5.2. *Assume that $p > q + r$ holds, and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11). *If $y_0 \in [q/(p-r), (p-q)/r]$, then $y_n \in [q/(p-r), (p-q)/r]$ for $n \in \mathbb{N}$. Furthermore, every nonnegative solution of* (1.11) *lies eventually in the interval $[q/(p-r), (p-q)/r]$.*

Proof. Firstly, note that in this case $q/(p-r) < 1 < (p-q)/r$ holds.

If $y_0 \in [q/(p-r), (p-q)/r]$, then by Lemma 3.3 (i) and (iv), we have that $q/(p-r) < y_n < 1 < (p-q)/r$ for $n \ge 1$, the first assertion follows.

To complete the proof it remains to show that when $y_0 \notin [q/(p-r), (p-q)/r]$ there exists $N > 0$ such that $y_N \in [q/(p-r), (p-q)/r]$. There are two cases to be considered.

Case 1 ($y_0 \in ((p-q)/r, \infty)$). Lemma 3.3 (ii) and (iii) implies that $y_1 < 1$. If $y_1 \le (p-q)/r$, then the proof follows from the first assertion. Now assume for the sake of contradiction that all terms of $\{y_n\}$ never enter the interval $[q/(p-r), (p-q)/r]$; then $\{y_n\}$ would lie in the interval $((p-q)/r, \infty)$ for $n \in \mathbb{N}$. Using Lemma 3.3 (vi), we obtain $y_n > y_{n+1} > (p-q)/r$ for $n \ge 1$, from which it follows that the sequence $\{y_n\}$ is strictly decreasing in the interval $((p-q)/r, \infty)$. Hence, $\lim_{n\to\infty} y_n$ exists and $\lim_{n\to\infty} y_n \ge (p-q)/r$, which is a contradiction, because, in view of Lemma 3.3 (vii), (1.11) has no equilibrium points in the interval $((p-q)/r, \infty)$.

Case 2 ($y_0 \in [0, q/(p-r))$). If there exists $N_0 > 0$ such that $y_{N_0} \in [q/(p-r), (p-q)/r]$, then the proof follows from the first assertion. If there exists $N_1 > 0$ such that $y_{N_1} \in ((p-q)/r, \infty)$, then the proof also follows from Case 1. Now assume for the sake of contradiction that $y_n < q/(p-r)$ for all $n \in \mathbb{N}$, then by Lemma 3.3 (v), we have that $y_n < y_{n+1} < q/(p-r)$ for $n \in \mathbb{N}$, which means that $\lim_{n\to\infty} y_n$ exists and $\lim_{n\to\infty} y_n \le q/(p-r)$; this contradicts Lemma 3.3 (vii).

The proof is complete. □

Theorem 5.3. *Assume that $p > q + r$ holds. Then the unique positive equilibrium $\overline{y}$ of* (1.11) *is a global attractor of all nonnegative solutions of* (1.11).

Proof. Theorem 5.2 and Lemma 3.3 (vii) imply that every nonnegative solution of (1.11) eventually enters the interval $[q/(p-r), (p-q)/r]$. Furthermore, the function $f(x, y)$ is increasing in each of its arguments in $[q/(p-r),(p-q)/r]$ and the equation

$$\frac{q+ry+y}{p+y}=y \tag{5.1}$$

has a unique positive solution on the interval $[q/(p-r),(p-q)/r]$. The proof now immediately follows by applying Lemma 2.2. □

Theorem 5.4. *Assume that $p=q+r$, and $\{y_n\}_{n=-k}^{\infty}$ is a nontrivial nonnegative solution of* (1.11). *Then the sequence $\{y_n\}_{n=0}^{\infty}$ is monotonic and* $\lim_{n\to\infty}y_n=1$.

Proof. In this case, the unique positive equilibrium is $\overline{y}=1$. By Lemma 3.4 it easy to see that if $y_0>1$ then $\{y_n\}_{n=0}^{\infty}$ is decreasing and bounded below by 1, if $y_0<1$ then $\{y_n\}_{n=0}^{\infty}$ is increasing and bounded above by 1, and if $y_0=1$ then $y_n=1$ for $n\in\mathbb{N}$. Hence, in all case the sequence $\{y_n\}_{n=0}^{\infty}$ converges to 1, as desired. □

By Theorem 5.4, it is easy to see that the following result is true.

Theorem 5.5. *Assume that $p=q+r$. Then the unique positive equilibrium $\overline{y}$ is a global attractor of all nonnegative solutions of* (1.11).

Theorem 5.6. *Assume that $r<p<q+r$ holds and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11). *If $y_0\in[1,q/(p-r)]$, then $y_n\in[1,q/(p-r)]$ for $n\in\mathbb{N}$. Furthermore, every nonnegative solution of* (1.11) *lies eventually in the interval* $[1,q/(p-r)]$.

Proof. Firstly, note that in this case $(p-q)/r<1<q/(p-r)$ holds.

If $y_0\in[1,q/(p-r)]$, then by Lemma 3.5 (iv) and (i), we have that $1\le y_n\le q/(p-r)$ for $n\ge 1$; the first assertion follows.

To complete the proof it remains to show that when $y_0\notin[1,q/(p-r)]$ there exists $N>0$ such that $y_N\in[1,q/(p-r)]$. There are two cases to be considered.

Case 1 ($y_0\in[0,1)$). Lemma 3.5 (i) implies that $y_n<q/(p-r)$ for $n\ge 1$. If there exists N such that $y_N>1$, then the proof follows from the first assertion. Now assume for the sake of contradiction that all terms of $\{y_n\}$ never enter the interval $[1,q/(p-r)]$, then $\{y_n\}$ would lie in the interval $[0,1)$ for $n\in\mathbb{N}$. Using Lemma 3.5 (iii), we obtain that $y_n<y_{n+1}<1$ for $n\ge 1$, from which it follows that $\lim_{n\to\infty}y_n$ exists and $\lim_{n\to\infty}y_n\le 1$, which is a contradiction, because in view of Lemma 3.5 (iv), (1.11) has no equilibrium points in the interval $[0,1)$.

Case 2 ($y_0\in(q/(p-r),\infty)$). If there exists $N_0>0$ such that $y_{N_0}\in[1,q/(p-r)]$, then the proof follows from the first assertion. If there exists $N_1>0$ such that $y_{N_1}\in[0,1)$, then the proof also follows from Case 1. Now assume for the sake of contradiction that $y_n>q/(p-r)$ for all $n\in\mathbb{N}$, then by Lemma 3.5 (ii), we have that $y_n>y_{n+1}>q/(p-r)$ for $n\in\mathbb{N}$, from which it follows that $\lim_{n\to\infty}y_n$ exists and $\lim_{n\to\infty}y_n\ge q/(p-r)$. This contradicts Lemma 3.5 (iv).

The proof is complete. □

Theorem 5.7. *Assume that $r<p<q+r$ holds. Then the unique positive equilibrium $\overline{y}$ of* (1.11) *is a global attractor of all nonnegative solutions of* (1.11).

Proof. Theorem 5.6 implies that every nonnegative solution of (1.11) lies eventually in the invariant interval $[1, q/(p-r)]$.

Further, the function $f(x, y)$ is nondecreasing in x and nonincreasing in y in $[1, q/(p-r)]$. Let $m, M \in [1, q/(p-r)]$ be a solution of the system

$$m = \frac{q + rm + M}{p + M}, \qquad M = \frac{q + rM + m}{p + m}, \tag{5.2}$$

then $(m - M)(p - r + 1) = 0$, from which we get that $m = M$, and the proof now follows by applying Lemma 2.3. □

Theorem 5.8. *Assume that $p = r$ holds and that $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of* (1.11)*. Then every nonnegative solution of* (1.11) *lies eventually in the invariant interval* $[1, \infty)$.

Proof. When $p \leq q$,

$$y_{n+1} = \frac{q + ry_n + y_{n-k}}{p + y_{n-k}} \geq \frac{q + y_{n-k}}{p + y_{n-k}} \geq 1, \quad n \in \mathbb{N}_0. \tag{5.3}$$

Hence the assertion is true for the case $p \leq q$. There remains to consider the case $p > q$. In this case $(p - q)/r < 1$ holds.

If $y_0 \geq (p-q)/r$, then by Lemma 3.6 (i) and (ii), we have that $y_n \geq 1$ for $n \in \mathbb{N}$, and the assertion follows.

Given that $y_0 < (p-q)/r$, then by Lemma 3.6 (iii), we have that $y_1 < 1$. If $y_1 \geq (p-q)/r$, then the assertion is true from the above proof. Now assume for the sake of contradiction that $y_n < (p - q)/r$ for all $n \in \mathbb{N}$. Using identity (3.7), we get that $y_n < y_{n+1} < (p - q)/r$ for $n \geq 1$, from which it follows that the sequence $\{y_n\}$ is strictly increasing and there is a finite $\lim_{n\to\infty} y_n \leq (p - q)/r$; this contradicts the fact that $\overline{y} = (1 + \sqrt{1 + 4q})/2 > 1$.

The proof is complete. □

Theorem 5.9. *Assume that $p = r$ holds. Then the unique positive equilibrium $\overline{y}$ of* (1.11) *is a global attractor of all nonnegative solutions of* (1.11).

Proof. Theorem 5.8 implies that there exist a positive integer N such that $y_n \geq 1$ for $n \geq N$. Therefore the change of variables

$$y_n = z_n + 1 \tag{5.4}$$

transforms (1.11) to the difference equation

$$z_{n+1} = \frac{q + pz_n}{p + 1 + z_{n-k}}, \quad n \geq N. \tag{5.5}$$

Clearly, $q > 0$ and $p + 1 > p > 0$ hold, and the remainder proof now is a straightforward consequence of Lemma 2.4. □

In view of Theorem 5.1, we know that the unique positive equilibrium $\overline{x}$ of (1.1) is a global attractor when $\beta \leq 1$. From this and Theorem 1.2, we have the following main result, which partially solves Open Problem 1.

Theorem 5.10. *Assume that $\beta \leq 1$. Then the unique positive equilibrium $\overline{x}$ of* (1.1) *is globally asymptotically stable.*

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