OSCILLATORY MIXED DIFFERENCE SYSTEMS

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The aim of this paper is to discuss the oscillatory behavior of difference systems of mixed type. Several criteria for oscillations are obtained. Particular results are included in regard to scalar equations.

1. Introduction

The aim of this work is to study the oscillatory behavior of the difference system

$$\Delta x(n) = \sum_{i=1}^{\ell} P_i x(n-i) + \sum_{j=1}^{m} Q_j x(n+j), \quad n = 0,1,2,\ldots, \tag{1.1}$$

where $x(n) \in \mathbb{R}^d$, $\Delta x(n) = x(n+1) - x(n)$ is the usual difference operator, $\ell, m \in \mathbb{N}$, and for $i = 1,\ldots,\ell$ and $j = 1,\ldots,m$ P_i and Q_j are given $d \times d$ real matrices. For a particular form of the scalar case of (1.1), the same question is studied in [1] (see also [2, Section 1.16]).

The system (1.1) is introduced in [9]. In this paper the authors show that the existence of oscillatory or nonoscillatory solutions of that system determines an identical behavior to the differential system with piecewise constant arguments,

$$\dot{x}(t) = \sum_{i=1}^{\ell} P_i x([t-i]) + \sum_{j=1}^{m} Q_j x([t+j]), \tag{1.2}$$

where for $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^d$ and $[\cdot]$ means the greatest integer function (see also [8, Chapter 8]).

By a solution of (1.1) we mean any sequence $x(n)$, of points in $\mathbb{R}^d$, with $n = -\ell,\ldots, 0,1,\ldots$, which satisfy (1.1). In order to guarantee its existence and uniqueness for given

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initial values $x_{-\ell},\dots,x_0,\dots,x_{m-1}$, denoting by I the $d\times d$ identity matrix, we will assume throughout this paper that the matrices $P_1,\dots,P_\ell,Q_1,\dots,Q_m$, are such that

$$\det(I-Q_1)\neq 0,\quad \text{if } m=1,$$
$$\det Q_m\neq 0,\quad \text{if } m\geq 2, \tag{1.3}$$
$$P_i=0,\quad \text{for every } i=1,\dots,\ell,$$

with no restrictions in other cases (see [8, Chapter 7] and [9]).

We will say that a sequence $y(n)$ satisfies *frequently* or *persistently* a given condition, (C), whenever for every $\nu\in\mathbb{N}$ there exists a $n>\nu$ such that $y(n)$ verifies (C). When there is a $\nu\in\mathbb{N}$ such that $y(n)$ verifies (C) for every $n>\nu,(C)$ is said to be satisfied *eventually* or *ultimately*.

Upon the basis of this terminology, a solution of (1.1), $x(n)=[x_1(n),\dots,x_d(n)]^T$, is said to be *oscillatory* if each real sequence $x_k(n)$ $(k=1,\dots,d)$ is frequently nonnegative and frequently nonpositive. If for some $k\in\{1,\dots,d\}$ the real sequence $x_k(n)$ is either eventually positive or eventually negative, $x(n)$ is said to be a *nonoscillatory* solution of (1.1). Whenever all solutions of (1.1) are oscillatory we will say that (1.1) is an *oscillatory system*. Otherwise, (1.1) will be said *nonoscillatory*.

Systems of mixed-type like (1.1) can be looked as a discretization of the continuous difference system

$$x(t+1)-x(t)=\sum_{i=1}^{\ell}P_ix(t-i)+\sum_{j=1}^{m}Q_jx(t+j). \tag{1.4}$$

When $Q_m=I$, one easily can see that, through a suitable change of variable, this system is a particular case of the delay difference system

$$x(t)=\sum_{i=1}^{p}A_jx(t-r_j), \tag{1.5}$$

where the A_j are $d\times d$ real matrices and the r_j are real positive numbers.

As is proposed in [8, Section 7.11], we will investigate, here, conditions on the matrices P_i and Q_j $(i=1,\dots,\ell$, and $j=1,\dots,m)$ which make the system (1.1) oscillatory. For that purpose we will develop the approach made in [3], motivated by analogues methods used in [6, 7] for obtaining oscillation criteria regarding the continuous delay difference system (1.5).

We notice that for mixed-type differential difference equations and the differential analog of (1.4), those methods seem not to work in general. In fact, for such equations the situation is essentially different since one cannot ensure, as for (1.5), that the corresponding Cauchy problem will be well posed, or guarantee an exponential boundeness for all its solutions (see [11]).

According to [9] (or [8, Chapter 7]) the analysis of the oscillatory behavior of the system (1.1) can be based upon the existence or absence of real positive zeros of the characteristic equation

$$\det\left((\lambda-1)I-\sum_{i=1}^{\ell}\lambda^{-i}P_i-\sum_{j=1}^{m}\lambda^{j}Q_j\right)=0. \tag{1.6}$$

That is, letting

$$M(\lambda)=\sum_{i=1}^{\ell}\lambda^{-i}P_i+\sum_{j=1}^{m}\lambda^{j}Q_j, \tag{1.7}$$

one can say that (1.1) is oscillatory if and only if, for every $\lambda\in\mathbb{R}^+=]0,+\infty[$,

$$\lambda-1\notin\sigma(M(\lambda)), \tag{1.8}$$

where for any matrix $C\in\mathbb{M}_d(\mathbb{R})$, the space of all $d\times d$ real matrices, by $\sigma(C)$ we mean its spectral set.

Based upon this characterization we will use, as in [3], the so-called logarithmic norms of matrices. For that purpose, we recall that to each induced norm, $\|\cdot\|$, in $\mathbb{M}_d(\mathbb{R})$, we can associate a logarithmic norm $\mu:\mathbb{M}_d(\mathbb{R})\to\mathbb{R}$, which is defined through the following derivative:

$$\mu(C)=\left(\|I+tC\|\right)'|_{t=0}, \tag{1.9}$$

where $C\in\mathbb{M}_d(\mathbb{R})$. As is well known, the logarithmic norm of any matrix $C\in\mathbb{M}_d(\mathbb{R})$ provides real bounds of the set $\operatorname{Re}\sigma(C)=\{\operatorname{Re}z:z\in\sigma(C)\}$, which enables us to handle condition (1.8) in a more suitable way. Those bounds are given in the first of the following elementary properties of any logarithmic norm (see [4, 5]):

(i) $\operatorname{Re}\sigma(C)\subset[-\mu(-C),\mu(C)]$ $(C\in\mathbb{M}_d(\mathbb{R}))$;

(ii) $\mu(C_1)-\mu(-C_2)\le\mu(C_1+C_2)\le\mu(C_1)+\mu(C_2)$ $(C_1,C_2\in\mathbb{M}_d(\mathbb{R}))$;

(iii) $\mu(\gamma C)=\gamma\mu(C)$, for every $\gamma\ge 0$ $(C\in\mathbb{M}_d(\mathbb{R}))$.

In regard to a given finite sequence of matrices, $C_1,\dots,C_\nu$, in $\mathbb{M}_d(\mathbb{R})$, and on the basis of a logarithmic norm, μ, we can define other matrix measures with some relevance in the sequel such as

$$a(C_k)=\mu\left(\sum_{i=1}^{k}C_i\right),\quad b(C_k)=\mu\left(\sum_{i=k}^{\nu}C_i\right),\quad \text{for } k=1,\dots,\nu. \tag{1.10}$$

In the same context, these measures give rise to the matrix measures α and β considered in [10] as follows:

$$\begin{gathered}\alpha(C_1)=a(C_1)=\mu(C_1),\qquad \alpha(C_k)=a(C_k)-a(C_{k-1}),\quad \text{for } k=2,\dots,\nu;\\ \beta(C_\nu)=b(C\nu)=\mu(C_\nu),\qquad \beta(C_k)=b(C_k)-b(C_{k+1}),\quad \text{for } k=1,\dots,\nu-1.\end{gathered} \tag{1.11}$$

In the sequel whenever the values $a(-C_k)$, $b(-C_k)$, $\alpha(-C_k)$, and $\beta(-C_k)$ are considered, we are implicitly referring to the values above with respect to the finite sequence $-C_1,\dots,-C_\nu$.

Notice that by the property (ii) above, these measures are related with the corresponding logarithmic norm μ in the following way:

$$a(C_k) \le \sum_{i=1}^{k} \mu(C_i), \qquad b(C_k) \le \sum_{i=k}^{\nu} \mu(C_i), \tag{1.12}$$

$$\alpha(C_k) \le \mu(C_k), \qquad \beta(C_k) \le \mu(C_k), \tag{1.13}$$

for every $k = 1,\dots,\nu$.

With respect to the measures α and β the following lemma holds.

Lemma 1.1. *Let $C_1,\dots,C_\nu$, be a finite sequence of $d \times d$ real matrices.*

(a) *If $\gamma_1 \ge \cdots \ge \gamma_\nu \ge 0$ is a nonincreasing finite sequence of nonnegative real numbers, then*

$$\mu\left(\sum_{i=1}^{\nu} \gamma_i C_i\right) \le \sum_{i=1}^{\nu} \gamma_i \alpha(C_i). \tag{1.14}$$

(b) *If $0 \le \gamma_1 \le \cdots \le \gamma_\nu$ is a nondecreasing finite sequence of nonnegative real numbers, then*

$$\mu\left(\sum_{i=1}^{\nu} \gamma_i C_i\right) \le \sum_{i=1}^{\nu} \gamma_i \beta(C_i). \tag{1.15}$$

Proof. We will prove only inequality (1.14). Analogously one can obtain (1.15).

Applying the property (ii) of the logarithmic norms, one has

$$\mu\left(\sum_{i=1}^{\nu} \gamma_i C_i\right) = \mu\left(\gamma_\nu \sum_{i=1}^{\nu} C_i + \sum_{i=1}^{\nu-1} (\gamma_i - \gamma_\nu) C_i\right) \le \gamma_\nu \mu\left(\sum_{i=1}^{\nu} C_i\right) + \mu\left(\sum_{i=1}^{\nu-1} (\gamma_i - \gamma_\nu) C_i\right). \tag{1.16}$$

On the other hand, since

$$\begin{aligned}\sum_{i=1}^{\nu-1} (\gamma_i - \gamma_\nu) C_i = {} & (\gamma_1 - \gamma_2) C_1 + (\gamma_2 - \gamma_3) C_1 + (\gamma_3 - \gamma_4) C_1 + \cdots + (\gamma_{\nu-1} - \gamma_\nu) C_1 \\ & + (\gamma_2 - \gamma_3) C_2 + (\gamma_3 - \gamma_4) C_2 + \cdots + (\gamma_{\nu-1} - \gamma_\nu) C_2 + \cdots \\ & + (\gamma_{\nu-2} - \gamma_{\nu-1}) C_{\nu-2} + (\gamma_{\nu-1} - \gamma_\nu) C_{\nu-2} + (\gamma_{\nu-1} - \gamma_\nu) C_{\nu-1},\end{aligned} \tag{1.17}$$

and $\gamma_{i+1} \le \gamma_i$, for every $i = 1,\dots,\nu-1$, we have by the properties (ii) and (iii) of the logarithmic norms,

$$\mu\left(\sum_{i=1}^{\nu}\gamma_i C_i\right) \le \gamma_\nu \mu\left(\sum_{i=1}^{\nu} C_i\right) + (\gamma_{\nu-1}-\gamma_\nu)\mu\left(\sum_{i=1}^{\nu-1} C_i\right) + (\gamma_{\nu-2}-\gamma_{\nu-1})\mu\left(\sum_{i=1}^{\nu-2} C_i\right) + \cdots + (\gamma_2-\gamma_3)\mu\left(\sum_{i=1}^{2} C_i\right) + (\gamma_1-\gamma_2)\mu(C_1). \tag{1.18}$$

Thus

$$\mu\left(\sum_{i=1}^{\nu}\gamma_i C_i\right) \le \gamma_\nu\left[\mu\left(\sum_{i=1}^{\nu} C_i\right) - \mu\left(\sum_{i=1}^{\nu-1} C_i\right)\right] + \gamma_{\nu-1}\left[\mu\left(\sum_{i=1}^{\nu-1} C_i\right) - \mu\left(\sum_{i=1}^{\nu-2} C_i\right)\right] + \cdots + \gamma_2\left[\mu\left(\sum_{i=1}^{2} C_i\right) - \mu(C_1)\right] + \gamma_1\mu(C_1), \tag{1.19}$$

which is equivalent to (1.14). □

In view of the examples which will be given in the sections below we recall the following well-known logarithmic norms of a matrix $C = [c_{jk}] \in \mathbb{M}_d(\mathbb{R})$:

$$\mu_1(C) = \max_{1\le k\le d}\left\{c_{kk} + \sum_{j\ne k}|c_{jk}|\right\}, \qquad \mu_\infty(C) = \max_{1\le j\le d}\left\{c_{jj} + \sum_{k\ne j}|c_{jk}|\right\}, \tag{1.20}$$

which correspond, respectively, to the induced norms in $\mathbb{M}_d(\mathbb{R})$ given by

$$\|C\|_1 = \max_{1\le k\le d}\left\{\sum_{j=1}^{d}|c_{jk}|\right\}, \qquad \|C\|_\infty = \max_{1\le j\le d}\left\{\sum_{k=1}^{d}|c_{jk}|\right\}. \tag{1.21}$$

With respect to the norm $\|C\|_2$ induced by the Hilbert norm in $\mathbb{R}^d$, the corresponding logarithmic norm is given by $\mu_2(C) = \max\sigma((B+B^T)/2)$. For this specific logarithmic norm, some oscillation criteria are obtained in [3].

2. Criteria involving the measures α and β

By (1.8) and the property (i) of the logarithmic norms, we have that (1.1) is oscillatory whenever, for every real positive λ,

$$\lambda - 1 \notin [-\mu(-M(\lambda)), \mu(M(\lambda))]. \tag{2.1}$$

This means that (1.1) is oscillatory if either

$$\mu(M(\lambda)) < \lambda - 1, \quad \forall\lambda \in \mathbb{R}^+, \tag{2.2}$$

or

$$\mu(-M(\lambda)) < 1 - \lambda, \quad \forall\lambda \in \mathbb{R}^+. \tag{2.3}$$

Depending upon the choice of the matrix measures proposed, one can obtain several different conditions regarding the oscillatory behavior of (1.1).

Theorem 2.1. *If for every $i = 1,\dots,\ell$, and $j = 1,\dots,m$,*

$$\alpha(P_i) \le 0, \qquad \beta(Q_j) \le 0, \tag{2.4}$$

$$\beta(P_i) \le 0, \qquad \alpha(Q_j) \le 0, \tag{2.5}$$

$$\sum_{i=1}^{\ell} \frac{(i+1)^{i+1}}{i^i}\beta(P_i) < -1, \tag{2.6}$$

then (1.1) is oscillatory.

Proof. By the property (ii) of the logarithmic norms, one has

$$\mu(M(\lambda)) \le \mu\left(\sum_{i=1}^{\ell}\lambda^{-i}P_i\right) + \mu\left(\sum_{j=1}^{m}\lambda^{j}Q_j\right). \tag{2.7}$$

For every real $\lambda \in]1,+\infty[$, inequalities (1.14) and (1.15) and assumption (2.4) imply that

$$\mu(M(\lambda)) \le \sum_{i=1}^{\ell}\lambda^{-i}\alpha(P_i) + \sum_{j=1}^{m}\lambda^{j}\beta(Q_j) \le 0. \tag{2.8}$$

Then, for every real $\lambda > 1$, we conclude that

$$\mu(M(\lambda)) < \lambda - 1, \tag{2.9}$$

since in that case $\lambda - 1 > 0$.

Let now $0 < \lambda \le 1$. From (2.7) and inequalities (1.14) and (1.15), we obtain

$$\mu(M(\lambda)) \le \sum_{i=1}^{\ell}\lambda^{-i}\beta(P_i) + \sum_{j=1}^{m}\lambda^{j}\alpha(Q_j), \tag{2.10}$$

and by assumption (2.5) we have

$$\mu(M(\lambda)) \le \sum_{i=1}^{\ell}\lambda^{-i}\beta(P_i). \tag{2.11}$$

But as

$$\max_{\lambda>1}\left(\frac{\lambda^{-i}}{\lambda - 1}\right) = -\frac{(i+1)^{i+1}}{i^i}, \tag{2.12}$$

we conclude that, for every real $0 < \lambda \le 1$,

$$\sum_{i=1}^{\ell}\lambda^{-i}\beta(P_i) \le -(\lambda - 1)\sum_{i=1}^{\ell}\frac{(i+1)^{i+1}}{i^i}\beta(P_i). \tag{2.13}$$

Thus by (2.6),

$$\mu(M(\lambda)) \leq -(\lambda-1)\sum_{i=1}^{\ell}\frac{(i+1)^{i+1}}{i^i}\beta(P_i) < \lambda - 1, \tag{2.14}$$

also for every real $0 < \lambda \leq 1$. □

As a corollary of Theorem 2.1, we obtain the following statement.

Corollary 2.2. *Under (2.4) and (2.5), if*

$$\sum_{i=1}^{\ell}\beta(P_i) < -\frac{1}{4}, \tag{2.15}$$

then (1.1) is oscillatory.

Proof. Since $(i+1)^{i+1}/i^i \geq 4$ for every positive integer, the condition (2.15) implies (2.6). □

The condition (2.15) is a result of (2.6) through a substitution involving the lower index of the family of matrices P_i. A condition involving the largest index, m, of the family of matrices Q_j is stated in the following theorem.

Theorem 2.3. *Under (2.4) and (2.5), if $\beta(P_i) \neq 0$, for some $i = 1,\dots,\ell$, and*

$$\left(m\frac{\sum_{j=1}^{m}\alpha(Q_j)}{\sum_{i=1}^{\ell}\beta(P_i)}\right)^{1/(m+1)}\left(\sum_{i=1}^{\ell}\beta(P_i)\right)\left(\frac{1}{m}+1\right) \leq -1, \tag{2.16}$$

then (1.1) is oscillatory.

Proof. As in the proof of Theorem 2.1, we have

$$\mu(M(\lambda)) < \lambda - 1, \tag{2.17}$$

for every real $\lambda > 1$.

Recalling inequality (2.10), we obtain by (2.5), for every real $0 < \lambda \leq 1$,

$$\mu(M(\lambda)) \leq \lambda^{-1}\sum_{i=1}^{\ell}\beta(P_i) + \lambda^m\sum_{j=1}^{m}\alpha(Q_j), \tag{2.18}$$

since $\lambda^{-i} \geq \lambda^{-1}$ and $\lambda^j \geq \lambda^m$. The function

$$f(\lambda) = \lambda^{-1}\sum_{i=1}^{\ell}\beta(P_i) + \lambda^m\sum_{j=1}^{m}\alpha(Q_j) \tag{2.19}$$

is strictly concave and

$$f(\lambda) \leq \left(m\frac{\sum_{j=1}^{m}\alpha(Q_j)}{\sum_{i=1}^{\ell}\beta(P_i)}\right)^{1/(m+1)}\left(\sum_{i=1}^{\ell}\beta(P_i)\right)\left(\frac{1}{m}+1\right). \tag{2.20}$$

By (2.16) we have then, for every real $0 < \lambda \le 1$, $\mu(M(\lambda)) \le -1 < \lambda - 1$, and consequently condition (2.2) is fulfilled and system (1.1) is oscillatory. □

By use of (2.3), the following theorem is stated.

Theorem 2.4. *If for every $i = 1,\dots,\ell$ and $j = 1,\dots,m$,*

$$\alpha(-P_i) \le 0, \qquad \beta(-Q_j) \le 0, \tag{2.21}$$

$$\alpha(-Q_j) \le 0, \qquad \beta(-P_i) \le 0, \tag{2.22}$$

$$\sum_{j=1}^{m} \frac{j^j}{(j-1)^{j-1}} \beta(-Q_j) < -1, \tag{2.23}$$

then (1.1) is oscillatory.

Proof. For every $\lambda \ge 1$, as in (2.8), we have

$$\mu(-M(\lambda)) \le \sum_{i=1}^{\ell} \lambda^{-i} \alpha(-P_i) + \sum_{j=1}^{m} \lambda^j \beta(-Q_j), \tag{2.24}$$

and by (2.21)

$$\mu(-M(\lambda)) \le \sum_{j=1}^{m} \lambda^j \beta(-Q_j). \tag{2.25}$$

Since for $j > 1$,

$$\max_{\lambda > 1} \left(\frac{\lambda^j}{1-\lambda} \right) = -\frac{j^j}{(j-1)^{j-1}}, \tag{2.26}$$

and for $j = 1$,

$$\sup_{\lambda > 1} \left(\frac{\lambda}{1-\lambda} \right) = -1, \tag{2.27}$$

we can conclude (under the convention $0^0 = 1$) that

$$\sum_{j=1}^{m} \lambda^j \beta(-Q_j) < (\lambda - 1) \sum_{j=1}^{m} \frac{j^j}{(j-1)^{j-1}} \beta(-Q_j), \tag{2.28}$$

for every real $\lambda \ge 1$. So by (2.23), we obtain

$$\mu(-M(\lambda)) < (\lambda - 1) \sum_{j=1}^{m} \frac{j^j}{(j-1)^{j-1}} \beta(-Q_j) \le 1 - \lambda, \tag{2.29}$$

for every real $\lambda \ge 1$.

On the other hand, for every $0 < \lambda < 1$, as in (2.10), by (2.22), we have

$$\mu(-M(\lambda)) \leq \sum_{i=1}^{\ell} \lambda^{-i}\beta(-P_i) + \sum_{j=1}^{m} \lambda^{j}\alpha(-Q_j) \leq 0 < 1-\lambda, \tag{2.30}$$

and consequently system (1.1) is oscillatory. □

Corollary 2.5. *Under (2.21) and (2.22), if*

$$\sum_{j=1}^{m} \beta(-Q_j) < -1 \tag{2.31}$$

then (1.1) is oscillatory.

Proof. Clearly (2.31) implies (2.23). □

Remark 2.6. In case of having $m > 1$, (2.31) can be replaced by $\sum_{j=1}^{m}\beta(-Q_j) \leq -1$.

We illustrate these results with the following example.

Example 2.7. Consider system (1.1) with $d = \ell = m = 2$, and

$$P_1 = \begin{bmatrix} -1 & 1 \\ -1 & -4 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -\dfrac{1}{10} & -1 \\ 0 & -1 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} -9 & -2 \\ 3 & -10 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} -8 & 1 \\ -2 & -10 \end{bmatrix}. \tag{2.32}$$

Through the logarithmic norm μ_1, we have

$$\begin{aligned} a(P_1) &= \mu_1(P_1) = 0 = \mu_1(P_2) = b(P_2), \\ a(P_2) &= \mu_1(P_1+P_2) = b(P_1) = -\frac{1}{10}, \\ a(Q_1) &= \mu_1(Q_1) = -6 = \mu_1(Q_2) = b(Q_2), \\ a(Q_2) &= \mu_1(Q_1+Q_2) = b(Q_1) = -16, \end{aligned} \tag{2.33}$$

and consequently

$$\begin{aligned} &\alpha(P_1) = 0, \qquad \alpha(P_2) = -\frac{1}{10}, \qquad \beta(Q_1) = -10, \qquad \beta(Q_2) = -6, \\ &\beta(P_1) = -\frac{1}{10}, \qquad \beta(P_2) = 0, \qquad \alpha(Q_1) = -6, \qquad \alpha(Q_2) = -10. \end{aligned} \tag{2.34}$$

Since

$$\sqrt[3]{2\times 160}\left(-\frac{1}{10}\right)\left(\frac{1}{2}+1\right) \approx -1.0260 < -1, \tag{2.35}$$

we can conclude, by Theorem 2.3, that the correspondent system (1.1) is oscillatory.

Notice that, as

$$\sum_{i=1}^{2} \frac{(i+1)^{i+1}}{i^i} \beta(P_i) = 2^2 \times \left(-\frac{1}{10}\right) - \frac{3^3}{2^2} \times 0 = -\frac{2}{5}, \qquad \sum_{i=1}^{2} \beta(P_i) = -\frac{1}{10}, \tag{2.36}$$

Theorem 2.1 and Corollary 2.2 cannot be applied to this system. The same holds to Theorem 2.4 and Corollary 2.5 since the respective conditions (2.21) and (2.22) are not fulfilled.

Through the application of inequalities (1.13), from Theorem 2.1, Corollary 2.2, Theorem 2.4, and Corollary 2.5, the corollaries below extend results contained in [3, Theorem 2].

Corollary 2.8. *Let $\mu(P_i) \le 0$, $\mu(Q_j) \le 0$, for every $i = 1,\dots,\ell$, and $j = 1,\dots,m$. If one of the inequalities*

$$\sum_{i=1}^{\ell} \frac{(i+1)^{i+1}}{i^i} \mu(P_i) < -1, \qquad \sum_{i=1}^{\ell} \mu(P_i) < -\frac{1}{4}, \tag{2.37}$$

is satisfied, then system (1.1) is oscillatory.

Corollary 2.9. *Let for every $i = 1,\dots,\ell$, and $j = 1,\dots,m$, $\mu(-P_i) \le 0$, $\mu(-Q_j) \le 0$. If one of the inequalities*

$$\sum_{j=1}^{m} \frac{j^j}{(j-1)^{j-1}} \mu(-Q_j) < -1, \qquad \sum_{j=1}^{m} \mu(-Q_j) < -1, \tag{2.38}$$

is verified, then system (1.1) is oscillatory.

Example 2.10. Consider system (1.1) with $d = 2$, $\ell = 3$, $m = 2$,

$$P_1 = \begin{bmatrix} -2 & -1 \\ 1 & -7 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} -1 & 1 \\ 0 & -5 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}. \tag{2.39}$$

With respect to the logarithmic norm μ_1, we have

$$\mu_1(P_1) = -1, \qquad \mu_1(P_2) = 0, \qquad \mu_1(P_3) = \mu_1(Q_1) = -1, \qquad \mu_1(Q_2) = -1, \qquad \mu_1(P_1) + \mu_1(P_2) + \mu_1(P_3) = -2. \tag{2.40}$$

Then the corresponding system (1.1) is oscillatory by Corollary 2.8. Remark that Corollary 2.9 cannot be used in this case.

When $d = 1$, one has $\mu(c) = c$, for every logarithmic norm, μ, and any real number, c. As a consequence also $\alpha(c) = \beta(c) = c$. So, all the results involving logarithmic norms and the matrix measures α and β can easily be adapted to the scalar case of (1.1), that is, to the equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j), \tag{2.41}$$

where p_i and q_j are real numbers, for $i = 1,\ldots,\ell$, and $j = 1,\ldots,m$.

Remark 2.11. The scalar case correspondent to Corollary 2.9 is in certain a sense an extension of [1, Theorem 6] (or [2, Theorem 1.16.7]).

3. The measures a and b

Through the use of the matrix measures a and b, different criteria are obtained through the following theorems.

Theorem 3.1. *If for every $i = 1,\ldots,\ell$, and $j = 1,\ldots,m$,*

$$a(P_i) \le 0, \qquad b(Q_j) \le 0, \tag{3.1}$$

$$a(Q_j) \le 0, \qquad b(P_i) \le 0, \tag{3.2}$$

$$b(P_1) < 0, \qquad \sum_{i=1}^{\ell} b(P_i) \le -1, \tag{3.3}$$

then (1.1) is oscillatory.

Proof. Recall inequality (2.8) and notice that for every real λ,

$$\begin{aligned}\sum_{i=1}^{\ell} \lambda^{-i}\alpha(P_i) &= \lambda^{-1}a(P_1) + \sum_{i=2}^{\ell} \lambda^{-i}\left[a(P_i) - a(P_{i-1})\right]\\ &= \sum_{i=1}^{\ell} \lambda^{-i}a(P_i) - \sum_{i=1}^{\ell-1} \lambda^{-(i+1)}a(P_i)\\ &= \sum_{i=1}^{\ell-1} \lambda^{-i}(1-\lambda^{-1})a(P_i) + \lambda^{-\ell}a(P_\ell),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sum_{j=1}^{m} \lambda^{j}\beta(Q_j) &= \sum_{j=1}^{m-1} \lambda^{j}\left[b(Q_j) - b(Q_{j+1})\right] + \lambda^m b(Q_m)\\ &= \sum_{j=1}^{m} \lambda^{j}b(Q_j) - \sum_{j=2}^{m} \lambda^{(j-1)}b(Q_j)\\ &= \lambda b(Q_1) + \sum_{j=2}^{m} \lambda^{j}(1-\lambda^{-1})b(Q_j).\end{aligned} \tag{3.5}$$

Therefore, for every $\lambda > 1$, we have by (3.1)

$$\sum_{i=1}^{\ell} \lambda^{-i}\alpha(P_i) \le 0, \qquad \sum_{j=1}^{m} \lambda^{j}\beta(Q_j) \le 0, \tag{3.6}$$

taking into account that $\lambda^{-i}(1-\lambda^{-1}) > 0$, for $i = 1,\dots,\ell-1$, and $\lambda^{j}(1-\lambda^{-1}) > 0$, for $j = 2,\dots,m$. Thus, for every $\lambda > 1$, we obtain $\mu(M(\lambda)) \le 0$ and in consequence

$$\mu(M(\lambda)) < \lambda - 1. \tag{3.7}$$

Recalling now inequality (2.10), first observe that, analogously,

$$\begin{aligned} \sum_{i=1}^{\ell} \lambda^{-i}\beta(P_i) &= \sum_{i=1}^{\ell} \lambda^{-i}b(P_i) - \sum_{i=2}^{\ell} \lambda^{-(i-1)}b(P_i) \\ &= \lambda^{-1}b(P_1) + \sum_{i=2}^{\ell} \lambda^{-i}(1-\lambda)b(P_i), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \sum_{j=1}^{m} \lambda^{j}\alpha(Q_j) &= \sum_{j=1}^{m} \lambda^{j}a(Q_j) - \sum_{j=1}^{m-1} \lambda^{(j+1)}a(Q_j) \\ &= \lambda^{m}a(Q_m) + \sum_{j=1}^{m-1} \lambda^{j}(1-\lambda)a(Q_j). \end{aligned} \tag{3.9}$$

Therefore, letting $0 < \lambda \le 1$, (3.2) implies that

$$\sum_{i=1}^{\ell} \lambda^{-i}\beta(P_i) \le \sum_{i=1}^{\ell} b(P_i) - \lambda \sum_{i=2}^{\ell} b(P_i), \tag{3.10}$$

since $\lambda^{-i} \ge 1$ for every $i = 1,\dots,\ell$. On the other hand, as $\lambda^{j}(1-\lambda) \ge 0$ for every $j = 1,\dots,m-1$, we have again by (3.2)

$$\sum_{j=1}^{m} \lambda^{j}\alpha(Q_j) \le 0. \tag{3.11}$$

Thus

$$\mu(M(\lambda)) \le \sum_{i=1}^{\ell} b(P_i) - \lambda \sum_{i=2}^{\ell} b(P_i), \tag{3.12}$$

for every $0 < \lambda \le 1$. If the sum $\sum_{i=2}^{\ell} b(P_i) = 0$, then we obtain by (3.3)

$$\mu(M(\lambda)) \le \sum_{i=1}^{\ell} b(P_i) \le -1 < \lambda - 1 \tag{3.13}$$

for every $0 < \lambda \leq 1$. Otherwise the right-hand term of (3.12) is the straight line determined by the points $(0, \sum_{i=1}^{\ell} b(P_i))$ and $((\sum_{i=1}^{\ell} b(P_i))/(\sum_{i=2}^{\ell} b(P_i)), 0)$, which stays under the straight line $\lambda - 1$ when λ runs the interval $]0,1]$, taking into account (3.3) and that $(\sum_{i=1}^{\ell} b(P_i))/(\sum_{i=2}^{\ell} b(P_i)) > 1$. Hence, for every $0 < \lambda \leq 1$,

$$\mu(M(\lambda)) < \lambda - 1. \tag{3.14}$$

Thus (1.1) is oscillatory and the proof is complete. □

Theorem 3.2. *Under (3.1) and (3.2), with $b(P_1) < 0$, if*

$$\left(m\frac{a(Q_m)}{b(P_1)}\right)^{1/(m+1)} b(P_1)\left(\frac{1}{m}+1\right) \leq -1, \tag{3.15}$$

then (1.1) is oscillatory.

Proof. For $\lambda > 1$, one can follow the proof of Theorem 3.1.

Let now $0 < \lambda \leq 1$. The equalities

$$\begin{gathered}\sum_{i=1}^{\ell} \lambda^{-i}\beta(P_i) = \lambda^{-1}b(P_1) + \sum_{i=2}^{\ell} \lambda^{-i}(1-\lambda)b(P_i), \\ \sum_{j=1}^{m} \lambda^{j}\alpha(Q_j) = \lambda^{m}a(Q_m) + \sum_{j=1}^{m-1} \lambda^{j}(1-\lambda)a(Q_j)\end{gathered} \tag{3.16}$$

imply

$$\mu(M(\lambda)) \leq \lambda^{-1}b(P_1) + \lambda^{m}a(Q_m), \tag{3.17}$$

for every real $0 < \lambda \leq 1$. The function

$$g(\lambda) = \lambda^{-1}b(P_1) + \lambda^{m}a(Q_m) \tag{3.18}$$

is strictly concave and

$$g(\lambda) \leq \left(m\frac{a(Q_m)}{b(P_1)}\right)^{1/(m+1)} b(P_1)\left(\frac{1}{m}+1\right) \tag{3.19}$$

for every real λ. Then by (3.15),

$$\mu(M(\lambda)) \leq -1 < \lambda - 1, \tag{3.20}$$

for every $0 < \lambda \leq 1$, and (1.1) is oscillatory. □

Theorem 3.3. *If for every $i = 1,\dots,\ell$ and $j = 1,\dots,m$,*

$$a(-P_i) \le 0, \qquad b(-Q_j) \le 0, \tag{3.21}$$

$$a(-Q_j) \le 0, \qquad b(-P_i) \le 0, \tag{3.22}$$

$$b(-Q_1) < 0, \qquad \sum_{j=1}^{m} b(-Q_j) \le -1, \tag{3.23}$$

then (1.1) is oscillatory.

Proof. By (3.4) and (3.5), one has, for every real λ,

$$\begin{aligned} \mu(-M(\lambda)) \le & \sum_{i=1}^{\ell-1} \lambda^{-i}(1-\lambda^{-1})a(-P_i) + \lambda^{-\ell}a(-P_\ell) \\ & + \lambda b(-Q_1) + \sum_{j=2}^{m} \lambda^{j}(1-\lambda^{-1})b(-Q_j). \end{aligned} \tag{3.24}$$

If $\lambda \ge 1$, we have by (3.21)

$$\mu(-M(\lambda)) \le \lambda \sum_{j=1}^{m} b(-Q_i) - \sum_{j=2}^{m} b(-Q_j), \tag{3.25}$$

since $\lambda^j \ge \lambda$ for every $\lambda \ge 1$. If $\sum_{j=2}^{m} b(-Q_j) = 0$, then

$$\mu(-M(\lambda)) \le \lambda \sum_{j=1}^{m} b(-Q_j) \le -\lambda < 1-\lambda. \tag{3.26}$$

Otherwise, for $\lambda \ge 1$, the right-hand term of (3.25) is a half line passing through the point $((\sum_{j=2}^{m} b(-Q_j))/(\sum_{j=1}^{m} b(-Q_i)),0)$, with a slope not larger than the slope of $1-\lambda$. Then taking into account (3.23), one has

$$\frac{\left(\sum_{j=2}^{m} b(-Q_j)\right)}{\left(\sum_{j=1}^{m} b(-Q_i)\right)} < 1, \tag{3.27}$$

and consequently $\mu(-M(\lambda)) < 1-\lambda$, for every $\lambda \ge 1$.

Let now $0 < \lambda < 1$. By (3.8) and (3.9), one obtains

$$\begin{aligned} \mu(-M(\lambda)) \le & \lambda^{-1}b(-P_1) + \sum_{i=2}^{\ell} \lambda^{-i}(1-\lambda)b(-P_i) \\ & + \lambda^{m}a(-Q_m) + \sum_{j=1}^{m-1} \lambda^{j}(1-\lambda)a(-Q_j), \end{aligned} \tag{3.28}$$

and by assumption (3.22), we have

$$\mu(-M(\lambda)) \le 0 < 1-\lambda \tag{3.29}$$

for every $0<\lambda<1$.

Thus (1.1) is oscillatory, which achieves the proof. □

The following example illustrates the use of these results.

Example 3.4. Consider now system (1.1) with $d=2$, $\ell=m=3$,

$$\begin{gathered} P_1=\begin{bmatrix} -\dfrac{2}{15} & -\dfrac{1}{15} \\ 1 & -5 \end{bmatrix}, \qquad P_2=\begin{bmatrix} \dfrac{1}{15} & 0 \\ -1 & 2 \end{bmatrix}, \qquad P_3=\begin{bmatrix} -\dfrac{1}{5} & 0 \\ -2 & -6 \end{bmatrix}, \\ Q_1=\begin{bmatrix} -15 & 0 \\ 1 & -11 \end{bmatrix}, \qquad Q_2=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad Q_3=\begin{bmatrix} -6 & -1 \\ -1 & -10 \end{bmatrix}. \end{gathered} \tag{3.30}$$

By use of the logarithmic norm μ_∞, we obtain

$$\begin{gathered} a(P_2)=a(Q_2)=0, \qquad a(P_3)=b(P_3)=b(P_1)=-\frac{1}{5}, \qquad a(Q_3)=b(Q_1)=-19, \\ b(Q_2)=-4,\ b(Q_3)=-5, \qquad a(Q_1)=-3, \qquad b(P_2)=-\frac{2}{15}, \qquad a(P_1)=-\frac{1}{15}. \end{gathered} \tag{3.31}$$

The condition (3.15) is satisfied, since its left-hand term is equal to

$$\left(3\frac{19}{-1/5}\right)^{1/4}\left(-\frac{1}{5}\right)\left(\frac{1}{3}+1\right) = -\frac{4}{15}\sqrt[4]{285} \approx -1.0957. \tag{3.32}$$

Then the correspondent system (1.1) is oscillatory.

Notice that for this system, Theorem 2.1, Corollary 2.2, and Theorems 2.3 and 3.1 cannot be used since

$$\begin{gathered} \alpha(P_3)=a(P_3)-a(P_2)=\frac{1}{5}, \\ b(P_1)+b(P_2)+b(P_3)=-\frac{1}{5}-\frac{2}{15}-\frac{1}{5}=-\frac{8}{15}. \end{gathered} \tag{3.33}$$

By use of inequalities (1.12), from Theorems 3.1 and 3.3, one can state results involving only the logarithmic norm μ. However, such results are less general than those already described in Section 2. Nevertheless, for the scalar equation (2.41), the correspondent results involving the measures a and b are more general than those obtained with the measures α and β. In fact, notice that for any given finite sequence of real numbers, $c_1,\dots,c_\nu$, we have

$$a(c_k)=\sum_{i=1}^{k}c_i,\qquad b(c_k)=\sum_{i=k}^{\nu}c_i,$$

$$\sum_{k=1}^{\nu}a(c_k)=\nu c_1+(\nu-1)c_2+\cdots+2c_{\nu-1}+c_\nu=\sum_{k=1}^{\nu}(\nu-k+1)c_k, \tag{3.34}$$

$$\sum_{k=1}^{\nu}b(c_k)=\nu c_\nu+(\nu-1)c_{\nu-1}+\cdots+2c_2+c_1=\sum_{k=1}^{\nu}kc_k.$$

Moreover, for the finite sequence, $-c_1,\dots,-c_\nu$, one has

$$a(-c_k)=-a(c_k),\qquad b(-c_k)=-b(c_k), \tag{3.35}$$

and consequently

$$\sum_{k=1}^{\nu}a(-c_k)=-\sum_{k=1}^{\nu}(\nu-k+1)c_k,\qquad \sum_{k=1}^{\nu}b(-c_k)=-\sum_{k=1}^{\nu}kc_k. \tag{3.36}$$

Therefore Theorems 3.1, 3.2, and 3.3 can be, respectively, rewritten, as the following corollaries.

Corollary 3.5. *If*

$$a(p_i)=\sum_{k=1}^{i}p_k\le 0,\quad b(p_i)=\sum_{k=i}^{\ell}p_k\le 0,\quad \text{for every } i=1,\dots,\ell,$$

$$a(q_j)=\sum_{k=1}^{j}q_k\le 0,\quad b(q_j)=\sum_{k=j}^{m}q_j\le 0,\quad \text{for every } j=1,\dots,m, \tag{3.37}$$

$$\sum_{i=1}^{\ell}p_i<0,$$

and either

$$\sum_{i=1}^{\ell}ip_i\le -1, \tag{3.38}$$

or

$$\left(m\frac{\sum_{j=1}^{m}q_j}{\sum_{i=1}^{\ell}p_i}\right)^{1/(m+1)}\left(\sum_{i=1}^{\ell}p_i\right)\left(\frac{1}{m}+1\right)\le -1, \tag{3.39}$$

then (2.41) is oscillatory.

Corollary 3.6. *If for every $i = 1,\dots,\ell$ and $j = 1,\dots,m$,*

$$a(p_i) = \sum_{k=1}^{i} p_k \geq 0, \quad b(p_i) = \sum_{k=i}^{\ell} p_k \geq 0, \quad \text{for every } i = 1,\dots,\ell, \tag{3.40}$$

$$a(q_j) = \sum_{k=1}^{j} q_k \geq 0, \quad b(q_j) = \sum_{k=j}^{m} q_j \geq 0, \quad \text{for every } j = 1,\dots,m, \tag{3.41}$$

$$\sum_{j=1}^{m} q_j > 0, \qquad \sum_{j=1}^{m} j q_j \geq 1, \tag{3.42}$$

then (2.41) is oscillatory.

Example 3.7. The equation

$$\Delta x(n) = -x(n-3) + x(n-2) - x(n-1) - x(n+2) \tag{3.43}$$

is oscillatory, by Corollary 3.5 through condition (3.38).

Example 3.8. Still by Corollary 3.5, the equation

$$\Delta x(n) = -\frac{1}{10} x(n-3) - \frac{1}{5} x(n-1) - 3x(n+1) - 5x(n+2) \tag{3.44}$$

is oscillatory through condition (3.39) since

$$\left(2\frac{-8}{-3/10}\right)\left(-\frac{3}{10}\right)\left(\frac{1}{3}+1\right) \approx -1.5057. \tag{3.45}$$

(Notice that condition (3.38) is not fulfilled in this case.)

Example 3.9. The equation

$$\Delta x(n) = 3x(n-3) - x(n-2) + 2x(n-1) + x(n+1) - x(n+2) + x(n+3) \tag{3.46}$$

is oscillatory, by Corollary 3.6.

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