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On a boundary value problem of a class of generalized linear discrete-time systems

Ioannis K Dassios

Correspondence: jdasios@math.uoa.gr
Department of Mathematics, University of Athens, Panepistimioupolis, Athens, Greece

Abstract

In this article, we study a boundary value problem of a class of generalized linear discrete-time systems whose coefficients are square constant matrices. By using matrix pencil theory, we obtain formulas for the solutions and we give necessary and sufficient conditions for existence and uniqueness of solutions. Moreover, we provide some numerical examples. These kinds of systems are inherent in many physical and engineering phenomena.

Keywords: linear difference equations, boundary value problem, matrix pencil, discrete time system, matrix difference equations

1 Introduction

Linear matrix difference equations (LMDEs) are systems in which the variables take their values at instantaneous time points. Discrete time systems differ from continuous time ones in that their signals are in the form of sampled data. With the development of the digital computer, the discrete time system theory plays an important role in control theory. In real systems, the discrete time system often appears when it is the result of sampling the continuous-time system or when only discrete data are available for use. LMDEs are inherent in many physical, engineering, mechanical, and financial/actuarial models. In this article, our purpose is to study the solutions of generalized linear discrete-time boundary value problems into the mainstream of matrix pencil theory. A boundary value problem consists of finding solutions which satisfies an ordinary matrix difference equation and appropriate boundary conditions at two or more points. Thus, we consider

$$FY_{k+1} = GY_k \tag{1}$$

with known boundary values of type

$$AY_{k_0} + BY_{k_N} = D \tag{2}$$

where $F, G, A, B, \in \mathcal{M}(m \times m; \mathcal{F}), Y_k, D \in \mathcal{M}(m \times 1; \mathcal{F})$ (i.e., the algebra of square matrices with elements in the field $\mathcal{F}$). For the sake of simplicity, we set $\mathcal{M}_m = \mathcal{M}(m \times m; \mathcal{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathcal{F})$.

Systems of type (1) are more general, including the special case when $F = I_n$, where I_n is the identity matrix of $\mathcal{M}_n$.

The matrix pencil theory has extensively been used for the study of linear difference equations with time invariant coefficients, see for instance [1-5]. A matrix pencil is a family of matrices $sF - G$, parametrized by a complex number s. When G is square and $F = I_n$, where I_n is the identity matrix, the zeros of the function det $(sF - G)$ are the eigenvalues of G. Consequently, the problem of finding the nontrivial solutions of the equation

$$sFX = GX \tag{3}$$

is called the generalized eigenvalue problem. Although the generalized eigenvalue problem looks like a simple generalization of the usual eigenvalue problem, it exhibits some important differences. In the first place, it is possible for det $(sF - G)$ to be identically zero, independent of s. Second, it is possible for F to be singular, in which case the problem has infinite eigenvalues. To see this, write the generalized eigenvalue problem in the reciprocal form

$$FX = s^{-1}GX \tag{4}$$

If F is singular with a null vector X, then $GX = \mathbb{O}$, so that X is an eigenvector of the reciprocal problem corresponding to eigenvalue $s^{-1} = 0$; i.e., $s = \infty$. It might be thought that infinite eigenvalues are special, unhappy cases to be ignored in our perturbation problem but that is a misconception (see also [6-9]).

2 Mathematical background and notation

This brief section introduces some preliminary concepts and definitions from matrix pencil theory, which are being used throughout the article. Linear systems of type (1) are closely related to matrix pencil theory, since the algebraic, geometric, and dynamic properties stem from the structure by the associated pencil $sF - G$.

Definition 2.1. Given $F,G \in M_{nm}$ and an indeterminate $s \in F$, the matrix pencil $sF - G$ is called regular when $m = n$ and det $(sF - G) \neq 0$. In any other case, the pencil will be called singular.

Definition 2.2. The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}_m$ and $Q \in \mathcal{M}_m$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G} \tag{5}$$

In this article, we consider the case that pencil is *regular*.

The class of $sF - G$ is characterized by a uniquely defined element, known as a complex Weierstrass canonical form, $sF_w - Q_w$, see [5], specified by the complete set of invariants of the pencil $sF - G$.

This is the set of *elementary divisors* (e.d.) obtained by factorizing the invariant polynomials $f_i(s,\hat{s})$ into powers of homogeneous polynomials irreducible over field F. In the case where $sF - G$ is a regular, we have e.d. of the following type:

- e.d. of the type s^p *are called zero finite elementary divisors* (z. f.e.d.)
- e.d. of the type $(s - a)^\pi$, $a \neq 0$ *are called nonzero finite elementary divisors* (nz. f.e.d.)
- e.d. of the type $\hat{s}^q$ are called *infinite elementary divisors* (i.e.d.).

Let B_1, B_2, ..., B_n be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is the block diag $\{B_1, B_2, ..., B_n\}$.

Then, the complex Weierstrass form sF_w - Q_w of the regular pencil sF - G is defined by $sF_w - Q_w := sI_p - J_p \oplus sH_q - I_q$, where the first normal Jordan-type element is uniquely defined by the set of f.e.d.

$$(s - a_1)^{p1}, ..., (s - a_v)^{pv}, \sum_{j=1}^{v} p_j = p \tag{6}$$

of sF - G and has the form

$$sI_p - J_p := sI_{p_1} - J_{p_1}(a_1) \oplus \cdots \oplus sI_{p_v} - J_{p_v}(a_v). \tag{7}$$

and also the q blocks of the second uniquely defined block sH_q - I_q correspond to the i.e.d.

$$\widehat{s}^{q_1}, \ldots, \widehat{s}^{q_\sigma}, \quad \sum_{j=1}^{\sigma} q_j = q \tag{8}$$

of sF - G and has the form

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \cdots \oplus sH_{q_\sigma} - I_{q_\sigma}. \tag{9}$$

Thus, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\widetilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, then

$$H_q^{\widetilde{q}} = \mathbb{O}$$

We denote with O the zero matrix. $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \in \mathcal{M}_{p_j}, \tag{10}$$

$$J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & \ldots & 0 & 0 \\ 0 & a_j & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & a_j & 1 \\ 0 & 0 & \ldots & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j} \tag{11}$$

$$H_{q_j} = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}. \tag{12}$$

3 Main results-Solution space form of a consistent boundary value problem

In this section, the main results for a consistent boundary value problem of types (1) and (2) are analytically presented. Moreover, it should be stressed that these results offer the necessary mathematical framework for interesting applications.

Definition 3.1. The boundary value problem (1) and (2) is said to be consistent if it possesses at least one solution.

Consider the problem (1) with known boundary conditions (2). From the regularity of sF - G, there exist nonsingular $\mathcal{M}(m \times m, F)$ matrices P and Q such that (see also Section 2),

$$PFQ = F_w = I_p \oplus H_q \tag{13}$$

and

$$PGQ = G_w = J_p \oplus I_q \tag{14}$$

where $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined by (10), (11), (12) and moreover

$$I_p = I_{p_1} \oplus \cdots \oplus I_{p_\nu} \quad J_p = J_{p_1}(a_1) \oplus \cdots \oplus J_{p_\nu}(a_\nu) \quad H_q = H_{q_1} \oplus \cdots \oplus H_{q_\sigma} \quad I_q = I_{q_1} \oplus \cdots \oplus I_{q_\sigma} \tag{15}$$

Note that $\sum_{j=1}^{\nu} p_j = p$ and $\sum_{j=1}^{\sigma} q_j = q$, where $p + q = n$.

Lemma 3.1. System (1) is divided into two subsystems:

$$Z_{k+1}^p = J_p Z_k^p, \tag{16}$$

and the subsystem

$$H_q Z_{k+1}^q = Z_k^q \tag{17}$$

Proof. Consider the transformation

$$Y_k = QZ_k \tag{18}$$

Substituting the previous expression into (1) we obtain

$$FQZ_{k+1} = GQZ_k$$

whereby, multiplying by P, we arrive at

$$F_w Z_{k+1} = G_w Z_k$$

Moreover, we can write Z_k as $Z_k = \begin{bmatrix} Z_k^p \\ Z_k^q \end{bmatrix}$, where $Z_k^p \in \mathcal{M}_{p1}$ and $Z_k^q \in \mathcal{M}_{q1}$. Taking into account the above expressions, we arrive easily at (16) and (17).

Proposition 3.2. The subsystem (16) has general solution

$$Z_k^p = J_p^{k-k_0} C \tag{19}$$

where $\sum_{j=1}^{\nu} p_j = p$ and $C \in \mathcal{M}_{m1}$ constant.

Proof. See [2,3].

Proposition 3.3. The subsystem (17) has the unique solution

$$Z_k^q = \mathbb{O} \tag{20}$$

Proof. Let q_* be the index of the nilpotent matrix H_q, i.e. ($H_q^{q_*} = \mathbb{O}$), we obtain the following equations

$$\begin{aligned} H_q Z_{k+1}^q &= Z_k^q \\ H_q^2 Z_{k+1}^q &= H_q Z_k^q \\ H_q^3 Z_{k+1}^q &= H_q^2 Z_k^q \\ &\vdots \\ H_q^{q_*} Z_{k+1}^q &= H_q^{q_*-1} Z_k^q \end{aligned}$$

and

$$\begin{aligned} H_q Z_{k+1}^q &= Z_k^q \\ H_q^2 Z_{k+2}^q &= H_q Z_{k+1}^q \\ H_q^3 Z_{k+3}^q &= H_q^2 Z_{k+2}^q \\ &\vdots \\ H_q^{q_*} Z_{k+q_*}^q &= H_q^{q_*-1} Z_{k+q*-1}^q \end{aligned}$$

The conclusion, i.e., $Z_k^q = \mathbb{O}$, is obtained by repetitive substitution of each equation in the next one, and using the fact that $H_q^{q_*} = \mathbb{O}$.

The boundary value problem

A necessary and sufficient condition for the boundary value problem to be consistent is given by the following result

Theorem 3.1. The boundary value problem (1), (2) is consistent, if and only if

$$D \in colspan[AQ_p + BQ_p J^{k_N - k_0}] \tag{21}$$

Where $Q_p \in \mathcal{M}_{mp}$. The matrix Q_p has column vectors the p linear independent eigenvectors of the finite generalized eigenvalues of sF-G (see [1] for an algorithm of the computation of Q_p).

Proof. Let $Q = [Q_p Q_q]$, where $Q_p \in \mathcal{M}_{mp}$ and $Q_q \in \mathcal{M}_{mq}$; Combining propositions (3.2) and (3.3), we obtain

$$Y_k = QZ_k = [Q_p Q_q] \begin{bmatrix} J_p^{k-k_0} C \\ \mathbb{O} \end{bmatrix}$$

or

$$Y_k = Q_p J_p^{k-k_0} C. \tag{22}$$

The solution exists if and only if

$$\begin{aligned} D &= AY_{k_0} + BY_{k_N} \\ D &= [AQ_p + BQ_p J_p^{k_N - k_0}]C \end{aligned}$$

or

$$D \in colspan[AQ_p + BQ_p J_p^{k_N - k_0}]$$

It is obvious that, if there is a solution of the boundary value problem, it needs not to be unique. The necessary and sufficient conditions, for uniqueness, when the problem is consistent, are given by the following theorem.

Theorem 3.2. Assume the boundary value problem (1), (2). Then when it is consistent, it has a unique solutions if and only if

$$rank[AQ_p + BQ_pJ^{k_N-k_0}] = p \tag{23}$$

Then the formula of the unique solution is

$$Y_k = Q_pJ_p^{k-k_0}C$$

where C is the solution of the equation

$$[AQ_p + BQ_pJ_p^{k-k_0}]C = D \tag{24}$$

Proof. Let the boundary value problem (1), (2) be consistent, then from Theorem 3.1 and (22) the solution is

$$Y_k = Q_pJ_p^{k-k_0}C$$

with

$$D = AY_{k_0} + BY_{k_N}$$

and

$$[AQ_p + BQ_pJ_p^{k_N-k_0}]C = D$$

It is clear that for given A, B, D the problem (1), (2) has a unique solution if and only if the system (24) has a unique solution. Since $(AQ_p + BQ_pJ_p^{k_N-k_0}) \in \mathcal{M}_{mp}$, the solution is unique for system (24) if and only if the matrix $AQ_p + BQ_pJ_p^{k_N-k_0}$ is left invertible. This fact is equivalent to:

$$rank[AQ_p + BQ_pJ_p^{k_N-k_0}] = p$$

Then the formula of the unique solution is

$$Y_k = Q_pJ_p^{k-k_0}C$$

where C is the solution of the equation

$$(AQ_p + BQ_pJ_p^{k_N-k_0})C = D$$

Other type of boundary conditions

Assume that the matrix difference equation (1) has a different type of boundary conditions. Let the boundary conditions be

$$\begin{aligned} KY_{k_0} &= S \\ LY_{k_N} &= T \end{aligned} \tag{25}$$

where $K, L, S, T \in \mathcal{M}(m \times m; \mathcal{F})$. Then we can state the following theorem.

Theorem 3.3. The boundary value problem (1), (25) is consistent, if and only if

$$S, T \in colspan[KQ_p] = colspan[LQ_pJ^{k_N-k_0}] \tag{26}$$

Moreover when it is consistent, it has a unique solution if and only

$$rank[KQ_p] = rank[LQ_pJ^{k_N-k_0}] = p \tag{27}$$

and the linear system

$$\begin{aligned} KQ_pC &= S \\ LQ_pJ_p^{k_N-k_0}C &= T \end{aligned} \tag{28}$$

gives a unique solution for the constant column C.

Proof. From (22) and (25) the solution exists if and only if

$$\begin{aligned} S &= KQ_pC \\ T &= LQ_pJ_p^{k_N-k_0}C \end{aligned}$$

or

$$S, T \in colspan[KQ_p] = colspan[LQ_pJ^{k_N-k_0}]$$

It is obvious that a consistent solution of the boundary value problem (1), (25), is unique if and only if the system (28) gives a unique solution for C. Since $KQ_p, LQ_pJ_p^{k_N-k_0} \in \mathcal{M}_{mp}$, the solution is unique if and only if the matrices $KQ_p, LQ_pJ_p^{k_N-k_0}$ are left invertible or $rank[KQ_p] = rank[LQ_pJ_p^{k_N-k_0}] = p$.

4 Numerical example

Consider the boundary value problem (1), (2), where

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

and A, B the identity and zero matrices, respectively. The invariants of $sF - G$ are $s - 1$, $s - 2$, $s - 3$ (finite elementary divisors) and $\hat{s}^3$ (infinite elementary divisor of degree 3). Then

$$J_3^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

and the columns of Q_p are the eigenvectors of the generalized eigenvalues 1, 2, 3, respectively. Then

$$AQ_p + BQ_pJ^{k_N-k_0} = \begin{bmatrix} 3 & -5 & 3 & -5 & 3 & -5 \\ 1 & -1 & 2 & -2 & 4 & -4 \\ 1 & -1 & 3 & -3 & 9 & -9 \end{bmatrix}^T \tag{29}$$

where $()^T$ is the transpose tensor.

4.1 Example 1

Let

$$D = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 0 \\ -10 \\ 8 \end{bmatrix}$$

Then

$$D \in colspan[AQ_p + BQ_pJ_p^{k_N-k_0}]$$

and by calculating C from (24) we get

$$C = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$$

and the unique solution of the system by substituting in (22) is

$$Y_k = \begin{bmatrix} 3 - 2^k - 3^k \\ -5 + 2^k + 3^k \\ 3 - 2^{k+1} - 3^{k+1} \\ -5 + 2^{k+1} + 3^{k+1} \\ 3 - 2^{k+2} - 3^{k+2} \\ -5 + 2^{k+2} + 3^{k+2} \end{bmatrix}$$

4.2 Example 2

Let

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Then

$$D \notin colspan[AQ_p + BQ_pJ^{k_N-k_0}]$$

and the problem is not consistent.

5 Conclusions

The aim of this article was to give necessary and sufficient conditions for existence and uniqueness of solutions for generalized linear discrete-time boundary value problems of a class of linear rectangular matrix difference equations whose coefficients are square constant matrices. By taking into consideration that the relevant pencil is regular, we use the Weierstrass canonical form to decompose the difference system into two sub-systems. Afterwards, we provide analytical formulas when we have a consistent problem. Moreover, as a further extension of this article, we can discuss the case where the pencil is singular. Thus, the Kronecker canonical form is required. For all these, there is some research in progress.

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Competing interests
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