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Existence and uniqueness for a class of iterative fractional differential equations

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Abstract

The presence of a self-mapping increases the difficulty in proving the existence and uniqueness of solutions for general iterative fractional differential equations. In this article, we provide conditions for the existence and uniqueness of solutions for the initial value problem. We also determine the Burton stability of such equations. The arbitrary order case is taken in the sense of Riemann-Liouville fractional operators.

1 Introduction

Fractional calculus is an area of specialization in mathematics that is concerned with differential and integral operators of arbitrary orders. Fractional derivatives are considered excellent tools for the description of nonlinearity. Fractional derivatives are mainly used with classical integer order patterns, in which such effects are neglected. The significance of fractional derivatives is manifested in modeling, electrical properties of real materials, capacity of rheological properties of rocks, and many other fields. Fractional derivatives and fractional integrals have various definitions. The most famous operators are Riesz, Grunwald-Letnikov, Riemann-Liouville, and Caputo [1–10].

Existence and uniqueness theory is studied widely in fractional differential equations. Agarwal *et al.* studied the existence of solutions for the Riemann-Liouville operator for a class of integro-differential equations of high fractional order [11]. Moreover, Agarwal *et al.* presented some existence and uniqueness outcomes for a class of fuzzy fractional integral equations utilizing the Schauder fixed point theorem in semilinear Banach spaces [12]. Li *et al.* established the existence of a class of nonlinear fractional q -difference equations with mixed nonlinear assumptions, including the fractional q -derivative of Riemann-Liouville type, employing the Guo-Krasnoselskii fixed point theorem on cones [13]. Ibrahim applied the Tarski fixed point theorem to study the existence of extremal solutions for fractional differential equations with maxima [14]. The existence and uniqueness in complex domains have been imposed considering the Srivastava-Owa fractional operators [15, 16].

In this study, we introduce some conditions for the existence and uniqueness of solutions for the initial value problems of iterative equations. We also determine the Burton stability of such equations. The arbitrary order case is taken in the sense of Riemann-Liouville fractional operators, as well as Caputo derivative.

2 Preliminaries

This section deals with some preliminaries and notations regarding fractional calculus.

Fractional calculus originated from the Riemann-Liouville definition of fractional integral of order α in the form

$$I_a^\alpha \psi(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau,$$

where ψ is continuous.

The fractional (arbitrary) order differential of the continuous function ψ of order $\alpha > 0$ is defined by

$$D_a^\alpha \psi(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \psi(\tau) d\tau.$$

When $a = 0$, we conclude that

$$D^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}, \quad m > -1; 0 < \alpha < 1$$

and

$$I^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}, \quad m > -1; \alpha > 0.$$

Let ψ be a continuously (n) differentiable function. The Caputo derivative can then be realized by the following form:

$$D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \frac{\psi^{(n)}(\zeta)}{(t-\zeta)^{\gamma-n+1}} d\zeta, \quad (n-1) \leq \gamma < n,$$

where n is an integer and γ is a real number.

3 Related work

Differential equations with state-dependent delays attract interest of specialists because they arise from application models, such as the two-body problem of classical electrodynamics, transmission, mechanical models, population models, infection disease, position control, and dynamics of economical systems. As a special type of state-dependent delay differential equations, iterative differential equations have distinct characteristics. Such equations have been investigated in recent years in terms of their smoothness, analyticity, monotonicity, convexity, and numerical solution. In the theory of differential equations, one of the essential and important problems is the initial value problem. Numerous existence results on special iterative differential equations can be found in the literature. In 1984, Eder proved the existence of the unique monotone solution for the second iterative differential equation [17] using the contraction principle as follows:

$$y'(t) = y(y(t))$$

linked with $y(t_0) = t_0$ ($t_0 \in [-1, 1]$). Fečkan investigated the general second iterative differential equation [18]

$$y'(t) = f(y(y(t)))$$

with the initial value $y(0) = 0$. Modified iterative functional differential equations have been recently suggested by Zhang and Gong [19] to determine approximate solutions. They studied the following iterative functional differential equation:

$$\begin{aligned} y'(t) &= f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[n]}(t)), \\ y(t_0) &= y_0, \end{aligned}$$

which satisfies the following result.

Lemma 3.1 *Let*

$$\varphi_M = \{y \in C^0([t_0 - h, t_0 + h]) : |y(t) - y(s)| \leq M|t - s|, \forall t, s \in [t_0 - h, t_0 + h]\},$$

where $M < 1$. If $f, g \in \varphi_M$, then

$$\|f^{[i]} - g^{[i]}\|_{[t_0-h, t_0+h]} \leq \frac{1-M^i}{1-M} \|f - g\|_{[t_0-h, t_0+h]}, \quad i = 1, 2, \dots$$

Ibrahim utilized nonexpansive operators to establish the existence and uniqueness of iterative fractional differential equations of various forms [20, 21]. Recently, Ibrahim and Darus established sufficient conditions for fractional differential equations as follows:

$$D^\nu y(t) = f(t, y(t), y(\beta t), y(y(t))), \quad \beta \in (0, 1],$$

with the initial value $y(0) = y_0$, see [22].

In this study, we aim to establish the existence and uniqueness of the fractional iterative

$$D^\nu y(t) = f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[n]}(t)) \quad (1)$$

subject to the initial value

$$y(t_0) = y_0,$$

where $y^{[j]}(t) := y(y^{[j-1]}(t))$ indicates the j th iterate of self-mapping y , where $j = 1, 2, \dots, n$.

4 Existence and uniqueness

We start with the following result.

Theorem 4.1 *Suppose that $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is continuous. If positive r exists, such that*

$$(1 - M_1)r \geq l_0, \quad l_0 > 0, \quad (2)$$

where $M_1 = \|f\|_{\bar{B}(z_0, r)} \leq 1$ and $\bar{B}(z_0, r)$ denotes the closed ball centered at $z_0 = (t_0, y_0, \dots, y_0)$ with radius r , then Eq. (1) has a solution defined on $[t_0 - l, t_0 + l]$ for any $l \in [l_0/(1 - M_1), r]$.

Proof The existence of solutions of Eq. (1) is equivalent to finding a continuous solution of the integral equation

$$y(t) = y_0 + \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} ds. \quad (3)$$

Define a set φ_{M_1} as follows:

$$\varphi_{M_1} = \left\{ y \in C^0([t_0 - l, t_0 + l]) : y(t_0) = y_0, |y(t) - y(s)| \leq M_1 \frac{|t-s|^\gamma}{\Gamma(\gamma+1)}, \right. \\ \left. \forall t, s \in [t_0 - l, t_0 + l] \right\}$$

for any $l \in [l_0/(1-M_1), r]$. Then, for $y \in \varphi_{M_1}$, we prove that $y^{[j]}(t)$ ($j = 2, 3, \dots, n$) are well defined on $[t_0 - l, t_0 + l]$. It suffices to establish

$$|y^{[j]}(t) - y_0| \leq l \quad (4)$$

for $j \in \mathbb{N}$ by induction. In fact, for $t_0 \leq s \leq t$, we have

$$|y(t) - y_0| \leq |y(t) - y(s)| + |y(s) - y_0| \leq M_1 l + M_1 \frac{|t-s|^\gamma}{\Gamma(\gamma+1)} \leq l.$$

We let $|y^{[j]}(t) - y_0| \leq l$ for a positive integer $j \geq 1$, then

$$|y^{[j+1]}(t) - y_0| \leq M_1 |y^{[j]}(t) - y(s)| + |y(s) - y_0| \leq l.$$

On the basis of induction, (4) holds and $y^{[j]}([t_0 - l, t_0 + l])$ are well defined for any $y \in \varphi_{M_1}$. In the sequel we apply the Schauder fixed point theorem to prove the existence of the continuous solution of Eq. (3). To this end, we define the integral operator $\mathcal{G} : \varphi_{M_1} \rightarrow C^0([t_0 - l, t_0 + l])$ by

$$\mathcal{G}y(t) := y_0 + \frac{1}{\Gamma(\gamma)} \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) (t-s)^{\gamma-1} ds. \quad (5)$$

Clearly,

$$\mathcal{G}y(t_0) = y_0 + \frac{1}{\Gamma(\gamma)} \int_0^{t_0} f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) (t-s)^{\gamma-1} ds \quad (6)$$

for any $y \in \varphi_{M_1}$. In view of

$$\begin{aligned} & \| (t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[n]}(t)) - (t_0, y_0, y_0, \dots, y_0) \| \\ &= \max \{ |t - t_0|, |y^{[1]}(t) - y_0|, |y^{[2]}(t) - y_0|, \dots, |y^{[n]}(t) - y_0| \} \\ &\leq \max \{ |t - t_0|, M_1 |t - t_0|, M_1 |y^{[1]}(t) - y_0|, \dots, M_1 |y^{[n-1]}(t) - y_0| \} \\ &\leq \max \{ l, M_1 l, M_1 l, \dots, M_1 l \} \leq l \leq r, \end{aligned}$$

we get

$$\begin{aligned} |\mathcal{G}y(t_1) - \mathcal{G}y(t_2)| &\leq \left| \int_0^{t_1} f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) \left| \frac{(t_1 - s)^{\gamma-1}}{\Gamma(\gamma)} \right| ds \right. \\ &\quad \left. - \int_0^{t_2} f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) \left| \frac{(t_2 - s)^{\gamma-1}}{\Gamma(\gamma)} \right| ds \right| \\ &\leq M_1 \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)}, \quad t_1, t_2 > s. \end{aligned}$$

Then φ_{M_2} is a nonempty convex and compact subset of the Banach space $C^0([a, b])$. We consider the mapping $\mathcal{T} : \varphi_{M_2} \rightarrow C^0([a, b])$ defined by

$$\mathcal{T}y(t) := y_0 + \frac{1}{\Gamma(\gamma)} \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s)) (t - s)^{\gamma-1} ds. \quad (7)$$

To show \mathcal{T} is a self-mapping, we note that for any $t_1, t_2 \in [t_0 - l, t_0 + l]$. Therefore Eqs. (5)-(7) yield $\mathcal{G}y \in \varphi_{M_1}$; (i.e., \mathcal{G} is a self-mapping operator). It remains to show that \mathcal{G} is continuous. For this purpose, taking any $y_1, y_2 \in \varphi_{M_1}$, we obtain

$$\begin{aligned} |\mathcal{G}y_1(t) - \mathcal{G}y_2(t)| &\leq \int_0^t |f(s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s)) \\ &\quad - f(s, y_2^{[1]}(s), y_2^{[2]}(s), \dots, y_2^{[n]}(s))| \left| \frac{(t - s)^{\gamma-1}}{\Gamma(\gamma)} \right| ds, \quad t > s. \end{aligned}$$

By Lemma 3.1, we conclude that

$$\begin{aligned} &\| (s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s)) - (s, y_2^{[1]}(s), y_2^{[2]}(s), \dots, y_2^{[n]}(s)) \| \\ &= \max \{ |y_1^{[1]}(s) - y_2^{[1]}(s)|, |y_1^{[2]}(s) - y_2^{[2]}(s)|, \dots, |y_1^{[n]}(s) - y_2^{[n]}(s)| \} \\ &\leq \max \left\{ \|y_1 - y_2\|_{[t_0-l, t_0+l]}, \frac{1 - M_1^2}{1 - M_1} \|y_1 - y_2\|_{[t_0-l, t_0+l]}, \dots, \frac{1 - M_1^n}{1 - M_1} \|y_1 - y_2\|_{[t_0-l, t_0+l]} \right\} \\ &= \frac{1 - M_1^n}{1 - M_1} \|y_1 - y_2\|_{[t_0-l, t_0+l]} < \frac{1}{1 - M_1} \|y_1 - y_2\|_{[t_0-l, t_0+l]}. \end{aligned}$$

Given the uniform continuity of f on $\bar{B}(z_0, r)$, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, the inequality

$$\|\mathcal{G}y_1 - \mathcal{G}y_2\| < \varepsilon l$$

holds for $\|y_1 - y_2\|_{[t_0-l, t_0+l]} < \delta$, which yields \mathcal{G} is continuous. Φ_{M_1} is a convex, compact subset of the Banach space $C^0([t_0 - l, t_0 + l])$ and \mathcal{G} is a continuous operator that satisfies all conditions of the Schauder fixed point theorem, \mathcal{G} has a fixed point $g \in \varphi_{M_1}$ and g is a solution for Eq. (1) on the interval $[t_0 - l, t_0 + l]$. This completes the proof. \square

Theorem 4.2 Suppose that $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is continuous and any compact interval $[a, b]$ includes t_0 and y_0 . If

$$M_2 A_{t_0} \leq B_{y_0}, \quad (8)$$

where $A_{t_0} = \max\{t_0 - a, b - t_0\}$, $B_{y_0} = \min\{y_0 - a, b - y_0\}$, $M_2 = \|f\|_{[a,b]^{n+1}}$ and $[a, b]^{n+1}$ denotes the product of $n + 1$ intervals $[a, b]$, then Eq. (1) has a solution defined on $[a, b]$.

Proof Let

$$\varphi_{M_2} = \left\{ y \in C^0([a, b], [a, b]) : y(t_0) = y_0, |y(t) - y(s)| \leq M_2 \frac{|t - s|^\gamma}{\Gamma(\gamma + 1)}, \forall t, s \in [a, b] \right\}, \quad (9)$$

then φ_{M_2} is a nonempty convex and compact subset of the Banach space $C^0([a, b])$. We consider the mapping $\mathcal{T} : \varphi_{M_2} \rightarrow C^0([a, b])$ to be defined by

$$\mathcal{T}y(t) := y_0 + \frac{1}{\Gamma(\gamma)} \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s))(t - s)^{\gamma-1} ds. \quad (10)$$

To show \mathcal{T} is a self-mapping, we note that

$$\begin{aligned} |\mathcal{T}y(t)| &\leq y_0 + \frac{1}{\Gamma(\gamma)} \left| \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s))(t - s)^{\gamma-1} ds \right|, \quad t > s \\ &\leq y_0 + M_2 \frac{(t - t_0)^\gamma}{\Gamma(\gamma + 1)}, \quad t > t_0 \\ &\leq y_0 + M_2 A_{t_0} \leq y_0 + B_{t_0} \leq b, \end{aligned} \quad (11)$$

$$\begin{aligned} |\mathcal{T}y(t)| &\geq y_0 - \frac{1}{\Gamma(\gamma)} \left| \int_0^t f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[n]}(s))(t - s)^{\gamma-1} ds \right|, \quad t > s \\ &\geq y_0 - M_2 \frac{|t - t_0|^\gamma}{\Gamma(\gamma + 1)} \geq y_0 - M_2 A_{t_0} \\ &\geq y_0 - B_{y_0} \geq a. \end{aligned} \quad (12)$$

Clearly, $\mathcal{T}y(t_0) = y_0$. Moreover, for any $t_1, t_2 \in [a, b]$, we obtain

$$\begin{aligned} |\mathcal{T}y(t_1) - \mathcal{T}y(t_2)| &\leq \frac{1}{\Gamma(\gamma)} \int_{t_2}^{t_1} |f(s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s))| (t - s)^{\gamma-1} ds, \quad t > s \\ &\leq M_2 \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)}. \end{aligned} \quad (13)$$

Therefore, (12) and (13) yield that \mathcal{T} maps φ_{M_2} into itself. The definitions of A_{t_0} and B_{y_0} prove that $M_2 \leq 1$, thus for any $y_1, y_2 \in \varphi_{M_2}$, according to Lemma 3.1, we have

$$\begin{aligned} &\| (s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s)) - (s, y_2^{[1]}(s), y_2^{[2]}(s), \dots, y_2^{[n]}(s)) \| \\ &= \max \{ |y_1^{[1]}(s) - y_2^{[1]}(s)|, |y_1^{[2]}(s) - y_2^{[2]}(s)|, \dots, |y_1^{[n]}(s) - y_2^{[n]}(s)| \} \\ &\leq \max \left\{ \|y_1 - y_2\|_{[a,b]}, \frac{1 - M_2^n}{1 - M_2} \|y_1 - y_2\|_{[a,b]}, \dots, \frac{1 - M_2^n}{1 - M_2} \|y_1 - y_2\|_{[a,b]} \right\} \\ &= \frac{1 - M_2^n}{1 - M_2} \|y_1 - y_2\|_{[a,b]} \\ &< \frac{1}{1 - M_2} \|y_1 - y_2\|_{[a,b]}. \end{aligned}$$

By the uniform continuity of f on $[a, b]^{n+1}$, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$. When $\|y_1 - y_2\|_{[a,b]} < \delta$, we obtain

$$|f(s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s)) - f(s, y_2^{[1]}(s), y_2^{[2]}(s), \dots, y_2^{[n]}(s))| < \varepsilon.$$

Consequently,

$$\begin{aligned} |\mathcal{T}y_1(t) - \mathcal{T}y_2(t)| &\leq \int_0^t |f(s, y_1^{[1]}(s), y_1^{[2]}(s), \dots, y_1^{[n]}(s)) \\ &\quad - f(s, y_2^{[1]}(s), y_2^{[2]}(s), \dots, y_2^{[n]}(s))| \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} ds, \quad t > s \\ &< \varepsilon(b-a), \end{aligned}$$

which means that \mathcal{T} is a continuous operator. It follows that φ_{M_2} is a convex, compact subset of the Banach space $C^0([a, b])$. By the Schauder fixed point theorem, \mathcal{T} has a fixed point $h \in \varphi_{M_2}$ and h is a solution of Eq. (1) on the interval $[a, b]$. This completes the proof. \square

5 Applications

In this section, our theorems are illustrated by the following example. First, we prove the existence of smooth solutions of (1) based on the results provided by Si and Wang [23]. In this paper, smooth function $g \in C^n$ means that the function g has a number of continuous derivatives, and its n th continuous derivative is Lipschitzian. We require the following result.

Lemma 5.1 *Let*

$$\begin{aligned} \Omega(N_1, \dots, N_{n+1}; I) = \{g \in C^n(I, I) : |g^{(i)}(t)| \leq N_i, i = 1, 2, \dots, n; \\ |g^{(n)}(t) - g^{(n)}(s)| \leq N_{n+1}|t - s|, t, s \in I\}. \end{aligned}$$

Then, for any $x(t) \in \Omega(N_1, \dots, N_{n+1}; I)$, there exist a function

$$y_{*ik}(t) = P_{ik}(y_{10}(t), \dots, y_{1,i-1}(t); \dots; y_{k0}(t), \dots, y_{k,i-1}(t))$$

and positive constants N_{uv}^{ik} such that

$$|P_{ik}(\bar{\lambda}_{10}, \dots, \bar{\lambda}_{k,i-1}) - P_{ik}(\tilde{\lambda}_{10}, \dots, \tilde{\lambda}_{k,i-1})| \leq \sum_{u=1}^k \sum_{v=0}^{i-1} N_{uv}^{ik} |\bar{\lambda}_{uv} - \tilde{\lambda}_{uv}|$$

*for $(\bar{\lambda}_{10}, \dots, \bar{\lambda}_{k,i-1})$, $(\tilde{\lambda}_{10}, \dots, \tilde{\lambda}_{k,i-1})$ belong to compact sets $[0, N_1]^i \times [0, N_2]^i \times \dots \times [0, N_k]^i$, where $y_{ri}(t) = y^{(r)}(y^{[i]}(t))$, $y_{*ik}(t) = (y^{[i]}(t))^{(k)}$ and P_{ik} is a uniquely defined multivariate polynomial with nonnegative coefficients and $1 \leq u \leq k$, $0 \leq v \leq i-1$.*

Theorem 5.1 *Consider the equation*

$$D^\gamma y(t) = \sum_{i=1}^m a_i(t) y^{[i]}(t) + F(t) \quad (14)$$

associated with $y(t_0) = y_0$, where $a_i(t), F(t) \in C^n$ are given smooth functions.

Proof For $R > 0$, by the smoothness of the given functions, we have positive M_{a_i} and M_F such that

$$|a_i(t)| \leq M_{a_i}, \quad |F(t)| \leq M_F, \quad t \in [t_0 - R, t_0 + R], i = 1, 2, \dots, m.$$

Denote

$$M_a = \max_{1 \leq i \leq m} \{M_{a_i}\}, \quad N_1 = mM_a(|t_0| + R) + M_F.$$

If $(1 - N_1)R \geq |y_0 - t_0|$, then Eq. (14) has a solution in the function set

$$\Phi_{N_1} = \{y \in C^0([t_0 - l_1, t_0 + l_1]) : y(t_0) = y_0, |y(t) - y(s)| \leq N_1|t - s|, \forall t, s \in [t_0 - l_1, t_0 + l_1]\}$$

by Theorem 4.1, where arbitrary $l_1 \in [|y_0 - t_0|/(1 - N_1), R]$. In fact, for any $y \in \varphi_{N_1}$, we see that the function

$$f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t)) = \sum_{i=1}^m a_i(t)y^{[i]}(t) + F(t)$$

is continuous on $[t_0 - l_1, t_0 + l_1]$ and

$$\begin{aligned} & |f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t))| \\ &= \left| \sum_{i=1}^m a_i(t)y^{[i]}(t) + F(t) \right| \\ &\leq \sum_{i=1}^m M_{a_i}(|t_0| + R) + M_F \\ &= mM_a(|t_0| + R) + M_F = N_1. \end{aligned}$$

Since $(1 - N_1)R \geq |y_0 - t_0|$, the condition of Theorem 4.1 is satisfied, there exists a solution $y = \psi(t)$ of Eq. (14) in the functional set ψ_{N_1} . In the sequel, we show that $\psi^{(n+1)}(t)$ also is Lipschitzian on the compact interval $[t_0 - l_1, t_0 + l_1]$. In view of Lemma 5.1, we obtain

$$\begin{aligned} y_{*ik}(t) &= P_{ik}(y_{10}(t), \dots, y_{1,i-1}(t); \dots; y_{k0}(t), \dots, y_{k,i-1}(t)) \\ &= P_{ik}(y'(t), y'(y_1), \dots, y'(y_{i-1}); \dots; y^{(k)}(t), y^{(k)}(y_1), \dots, y^{(k)}(y_{i-1})), \end{aligned}$$

where $y_n = y^{[n]}(t)$, $n = 1, 2, \dots, i - 1$.

Denote

$$\begin{aligned} H_{ik} &= P_{ik}(\overbrace{N_1, \dots, N_1}^{i \text{ terms}}; \overbrace{N_2, \dots, N_2}^{i \text{ terms}}; \dots; \overbrace{N_k, \dots, N_k}^{i \text{ terms}}), \\ a_i(t) &\in \Omega(L_{i1}, \dots, L_{i(n+1)}; [t_0 - l_1, t_0 + l_1]), \\ F(t) &\in \Omega(M_1, \dots, M_{n+1}; [t_0 - l_1, t_0 + l_1]). \end{aligned}$$

Then, for any $t_1, t_2 \in [t_0 - l_1, t_0 + l_1]$, we get

$$\begin{aligned}
 & \left| \psi^{(m+1)}(t_1) - \varphi^{(m+1)}(t_2) \right| \\
 & \leq \sum_{i=1}^n \sum_{s=0}^m C_m^s \left| a_i^{(m-s)}(t_1) (\psi^{[i]}(t_1))^{(s)} - a_i^{(m-s)}(t_2) (\psi^{[i]}(t_2))^{(s)} \right| \\
 & \quad + \left| F^{(n)}(t_1) - F^{(n)}(t_2) \right| \\
 & \leq \sum_{i=1}^n \left\{ \left| a_i^{(m)}(t_1) - a_i^{(m)}(t_2) \right| \cdot \left| \psi^{[i]}(t_1) \right| + \left| a_i^{(m)}(t_2) \right| \cdot \left| \psi^{[i]}(t_1) - \psi^{[i]}(t_2) \right| \right\} \\
 & \quad + \sum_{i=1}^n \sum_{s=1}^m C_m^s \left(\left| a_i^{(m-s)}(t_1) - a_i^{(m-s)}(t_2) \right| \cdot \left| (\psi^{[i]}(t_1))^{(s)} \right| \right. \\
 & \quad \left. + \left| a_i^{(m-s)}(t_2) \right| \cdot \left| \psi(\psi_{1,0}(t_1), \dots, \psi_{s,i-1}(t_1)) - \psi(\psi_{1,0}(t_2), \dots, \psi_{s,i-1}(t_2)) \right| \right) \\
 & \quad + M_{n+1} \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)} \\
 & \leq \sum_{i=1}^n \left(L_{i(m+1)}(|t_0| + l_1) + L_{im} N_1^i \right) \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)} \\
 & \quad + \sum_{i=1}^n \sum_{s=1}^m C_m^s \left(L_{i(m+1-s)} H_{is} |t_1 - t_2| + L_{i(m-s)} \sum_{u=1}^s \sum_{v=0}^{i-1} N_{uv}^{is} |\psi_{uv}(t_1) - \psi_{uv}(t_2)| \right) \\
 & \quad + M_{n+1} |t_1 - t_2|.
 \end{aligned}$$

Since

$$\left| \psi_{uv}(t_1) - \psi_{uv}(t_2) \right| \leq N_{u+1} \left| \psi^{[v]}(t_1) - \psi^{[v]}(t_2) \right| \leq N_{u+1} N_1^v \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)},$$

we attain

$$\begin{aligned}
 & \left| \psi^{(m+1)}(t_1) - \psi^{(m+1)}(t_2) \right| \\
 & \leq \sum_{i=1}^n \left(L_{i(m+1)}(|t_0| + l_1) + L_{im} N_1^i \right) \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)} + \sum_{i=1}^n \sum_{s=1}^m C_m^s \left(L_{i(m+1-s)} H_{is} |t_1 - t_2| \right. \\
 & \quad \left. + L_{i(m-s)} \sum_{u=1}^s \sum_{v=0}^{i-1} N_{uv}^{is} |\psi_{uv}(t_1) - \psi_{uv}(t_2)| \right) + M_{n+1} \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)} \\
 & = \left\{ \left(\sum_{i=1}^n L_{i(m+1)}(|t_0| + l_1) + L_{im} N_1^i \right) \right. \\
 & \quad \left. + \left(\sum_{i=1}^n \sum_{s=1}^m C_m^s \left(L_{i(m+1-s)} H_{is} + L_{i(m-s)} \sum_{u=1}^s \sum_{v=0}^{i-1} N_{uv}^{is} N_{u+1}^v N_1^v \right) \right) + M_{n+1} \right\} \frac{|t_1 - t_2|^\gamma}{\Gamma(\gamma + 1)},
 \end{aligned}$$

which implies that $\psi^{(m+1)}(t)$ is Lipschitzian. \square

Example 5.1 Consider the equation

$$D^{\frac{1}{2}} y(t) = \frac{1}{5} y(y(t)) - \frac{1}{4} \tag{15}$$

associated with

$$\gamma(-1) = -\frac{1}{2}. \quad (16)$$

Now we check the conditions of Theorem 4.2. For the compact interval $[-1, 0]$, we see that $\gamma_0 = -\frac{1}{2}$, $t_0 = -1$, $\gamma = \frac{1}{2}$, $\Gamma(\frac{3}{2}) = 0.88$ for all $t \in [-1, 0]$. Since $M_2 = \|y\|_{[-1, 0]^{n+1}} = 1/5 + 1/4 = 9/20 < 0.5$, also $A_{t_0} = \max\{t_0 - a, b - t_0\} = \max\{-1 + 1, 0 + 1\} = \max\{0, 1\} = 1$, $B_{\gamma_0} = \min\{\gamma_0 - a, b - \gamma_0\} = \min\{-\frac{1}{2} + 1, 0 + \frac{1}{2}\} = \min\{\frac{1}{2}, \frac{1}{2}\} = 1/2$, and $\Gamma(\frac{3}{2}) = 0.88$, then $\frac{M_2}{\Gamma(\frac{3}{2})} = \frac{9}{17.6}$, which satisfies the conditions of Theorem 4.2. Then Eq. (15) associated with Eq. (16) has a solution.

Remark 5.1 In the proof of an invariant set, they require the inequalities

$$\begin{aligned} |(Fx)(t)| &\leq |x_0| + \left| \int_0^t f(s, x(s), x(x(s))) \frac{(t-s)^\gamma}{\Gamma(\gamma+1)} ds \right| \leq |x_0| + M \cdot \frac{|t-x_0|^\gamma}{\Gamma(\gamma+1)} \leq b, \\ |(Fx)(t)| &\geq |x_0| - \left| \int_0^t f(s, x(s), x(x(s))) \frac{(t-s)^\gamma}{\Gamma(\gamma+1)} ds \right| \geq x_0 - C_{x_0} \geq a. \end{aligned}$$

The right-most inequality of Eq. (15) contradicts the definition of C_{x_0} . We overcome this difficulty by defining B_{γ_0} .

6 Burton stability

In this section, we aim to discuss the stability of solutions of a class of iterative fractional differential equations. We need the following observations.

Definition 6.1 Let $g : [0, \infty[\mapsto [1, \infty[$ be an arbitrary continuous strictly increasing function with $g(t) \mapsto \infty$ as $t \mapsto \infty$ and define a Banach space $(W, |\cdot|_g)$ of continuous functions $\Phi : [0, \infty[\mapsto \mathbb{R}^n$ with the property that

$$|\Phi|_g = \sup_{0 \leq t < \infty} \frac{|\Phi(t)|}{g(t)} < \infty. \quad (17)$$

Definition 6.2 [24] Let B be a closed and densely defined operator on Y and $l \in \mathcal{C}(\mathbb{R}^+)$ be a scalar kernel. A family $\mathfrak{I}_{t \geq 0}$ is called a k -regularized resolvent family if the following conditions are satisfied:

- (a) \mathfrak{I} is strongly continuous on \mathbb{R}^+ and $\mathfrak{I}(0) = k(0)I$;
- (b) $\mathfrak{I}y \in D(B)$ and $B\mathfrak{I}(t)(y) = \mathfrak{I}(t)By$ for all $y \in D(B)$ and $t \geq 0$;
- (c) the k -regularized resolvent equation holds

$$\mathfrak{I}(t)(y) = K(t)(y) - \int_0^t BK(t-s)\mathfrak{I}(s)y ds.$$

Lemma 6.1 [25] Let N be a closed ball in AB , let $P : N \rightarrow N$ be continuous, and let PN be locally equicontinuous. Then there exists a point $\Psi \in N$ with $P\Psi = \Psi$.

Lemma 6.2 [25] Let N be a convex, nonempty, bounded subset of AB , let $P : N \mapsto N$, and let PN be locally equicontinuous. If, in addition, the g -norm closure of PN is in N , then there exists a point $\Psi \in M$ with $P\Psi = \Psi$.

Next, we discuss the stability of Eq. (14) for a special case by putting

$$F(t) = \sum_{i=1}^m B_i(t)y^{[i]}(t-r(t)).$$

Theorem 6.1 *Consider the fractional differential equation*

$$D^\gamma y(t) = \sum_{i=1}^m [-A_i(t)y^{[i]}(t) + B_i(t)y^{[i]}(t-r_i(t))], \quad y(0) = y_0, \quad 0 < \gamma < 1, \quad (18)$$

where $A_i, B_i, r_i : [0, \infty) \mapsto [0, \infty)$ are continuous. Moreover, suppose that

- (a) $a_i := -A_i(t)$, $A_i(t)$ is bounded on $[0, \infty[$;
- (b) $A_i(t) - B_i(t) \geq \delta$ for all $t_0 \geq 0$, and a constant $\delta > 0$;
- (c) $r_i(t) \geq 0$ for all $t \geq 0$.

Then the zero solution of Eq. (18) is stable.

Proof Consider problem (18). Assume that $\exists \zeta > 0$ and $\max_i [\sup_{t \geq 0} A_i(t)] < \zeta$ for all $t \geq 0$ such that

$$K(t) = \frac{\zeta}{\Gamma(\gamma)} t^{\gamma-1}.$$

Then for the Lizama resolvent (\mathfrak{I}_γ) , which satisfies

$$\mathfrak{I}(t) = K(t) - \int_0^t K(t-s)\mathfrak{I}(s) ds,$$

we have

$$0 \leq \mathfrak{I}(t) \leq K(t), \quad t\mathfrak{I}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{and} \quad \int_0^\infty \mathfrak{I}(s) ds = 1.$$

Rewrite Eq. (18) as follows:

$$\begin{aligned} D^\gamma y(t) &= \sum_{i=1}^m [-A_i(t)y^{[i]}(t) + B_i(t)y^{[i]}(t-r_i(t))] \\ &= \sum_{i=1}^m [-\zeta y(t) + \zeta[y-y^{[i]}] + [\zeta - A_i(t)]y^{[i]} + B_i(t)y^{[i]}(t-r_i(t))]. \end{aligned}$$

Therefore, the solution $y(t)$ of Eq. (18) achieves

$$\begin{aligned} y(t) &= \sum_{i=1}^m \left[w(t) + \int_0^t \mathfrak{I}_\gamma(t-s)[y-y^{[i]}](s) ds \right. \\ &\quad \left. + \int_0^t \mathfrak{I}_\gamma(t-s) \left(1 - \frac{A_i(t)}{\zeta} \right) y^{[i]}(s) ds + \int_0^t \mathfrak{I}_\gamma(t-s) \frac{B_i(t)}{\zeta} y^{[i]}(s-r_i(s)) ds \right] =: (Py)(t), \end{aligned}$$

where $w(t) = y_0(1 - \int_0^\infty \mathfrak{I}(s) ds)$.

Let $0 < \epsilon < \frac{\sqrt{3}}{3}$, we may assume that $\zeta \geq 1$ and $0 < \delta < 1$ so that $\frac{\delta\epsilon^3}{\zeta} < \epsilon$. Now set

$$n_0 = \min_i \left[\inf \{s - r_i(s); s \geq 0\} \right].$$

Let $\phi : [n_0, 0] \rightarrow \mathbb{R}$ be a given continuous initial function with $\|\phi\| < \frac{\delta\epsilon^3}{\zeta} < \epsilon$. Define

$$N = \{\Psi \in AB \mid \|\Psi\| \leq \epsilon\}.$$

For the natural mapping defined above with $y(s) = \phi(s)$ for $s \leq 0$, we can show that $P : N \mapsto N$. We observe that $\sum_{i=1}^m [r - r_i]$ is increasing on $[0, \epsilon]$ and we obtain

$$\begin{aligned} |(Py)(t)| &\leq \sum_{i=1}^m \left[|w(t)| + (\epsilon - \epsilon^{[i]}) + \epsilon^{[i]} \int_0^t \Im_{\gamma}(t-s) \left(1 - \frac{A_i(t)}{\zeta} + \frac{|B_i(t)|}{\zeta} \right) ds \right] \\ &\leq \sum_{i=1}^m \left[|\phi(0)| + (\epsilon - \epsilon^{[i]}) + \epsilon^{[i]} \left(1 - \frac{\delta}{\zeta} \right) \right] < \epsilon. \end{aligned}$$

It is clear that PN is equicontinuous on $[0, \infty)$ and the g -norm closure of PN is in N (Lemma 6.1). Consequently, in view of Lemma 6.2, P has a fixed point $y \in N$ which is a solution of Eq. (18). Thus, the zero solution of Eq. (18) is stable. This completes the proof. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on deriving the results and approved the final manuscript.

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References

1. Srivastava, HM, Owa, S: Univalent Functions, Fractional Calculus, and Their Applications. Halsted, New York (1989)
2. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
3. Kiryakova, V: Generalized Fractional Calculus and Applications. Longman, Harlow (1994); copublished in the United States with John Wiley & Sons, Inc., New York
4. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
5. Hilfer, R: Application of Fractional Calculus in Physics. World Scientific, Singapore (2000)
6. West, BJ, Bologna, M, Grigolini, P: Physics of Fractal Operators. Springer, New York (2003)
7. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
8. Sabatier, J, Agrawal, OP, Machado, JA: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Berlin (2007)
9. Lakshmikantham, V, Leela, S, Vasundhara, DJ: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
10. Klafter, J, Lim, SC, Metzler, R (eds.): Fractional Dynamics: Recent Advances. World Scientific, Singapore (2011)
11. Agarwal, RP, Ntouyas, SK, Ahmad, B, Alhothuali, MS: Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions. *Adv. Differ. Equ.* **2013**, 128 (2013)
12. Agarwal, RP, Arshad, S, O'Regan, S, Lupulescu, V: A Schauder fixed point theorem in semilinear spaces and applications. *Fixed Point Theory Appl.* **2013**, 306 (2013)
13. Li, X, Han, Z, Sun, S, Zhao, P: Existence of solutions for fractional q -difference equation with mixed nonlinear boundary conditions. *Adv. Differ. Equ.* **2014**, 326 (2014)

14. Ibrahim, RW: Extremal solutions for certain type of fractional differential equations with maxima. *Adv. Differ. Equ.* **2012**, 7 (2012)
15. Ibrahim, RW: Modified fractional Cauchy problem in a complex domain. *Adv. Differ. Equ.* **2013**, 149 (2013)
16. Ibrahim, RW: On holomorphic solution for space- and time-fractional telegraph equations in complex domain. *J. Funct. Spaces Appl.* **2012**, Article ID 703681 (2012)
17. Eder, E: The functional differential equation $x'(t) = x(x(t))$. *J. Differ. Equ.* **54**(3), 390-400 (1984)
18. Fečkan, M: On a certain type of functional differential equations. *Math. Slovaca* **43**(1), 39-43 (1993)
19. Zhang, P, Gong, X: Existence of solutions for iterative differential equations. *Electron. J. Differ. Equ.* **2014**, 7 (2014)
20. Ibrahim, RW: Existence of deviating fractional differential equation. *CUBO* **14**, 127-140 (2012)
21. Ibrahim, RW: Existence of iterative Cauchy fractional differential equation. *J. Math.* **2013**, Article ID 838230 (2013)
22. Ibrahim, RW, Darus, M: Infective disease processes based on fractional differential equation. In: *Proceedings of the 3rd International Conference on Mathematical Sciences*, vol. 1602, pp. 696-703 (2014)
23. Si, J-G, Wang, X-P: Smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients. *J. Math. Anal. Appl.* **226**(2), 377-392 (1998)
24. Lizama, C, Vergara, V: Uniform stability of resolvent families. *Proc. Am. Math. Soc.* **132**(1), 175-181 (2004)
25. Burton, TA, Zhang, B: A Schauder-type fixed point theorem. *J. Math. Anal. Appl.* **417**(2), 552-558 (2014)

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