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A coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions

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Abstract

In this paper, we investigate the existence and the uniqueness of solutions for coupled and uncoupled systems of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions. The existence and the uniqueness of the solutions are established by using the Banach contraction principle, while the existence of solutions is derived by applying Leray-Schauder's alternative. Examples illustrating our results are also presented.

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1 Introduction

In this paper, we investigate a coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions given by

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_r^\delta y(t)), & t \in [0, T], 1 < \alpha \le 2, \\ D_p^\beta y(t) = g(t, y(t), I_z^\varepsilon x(t)), & t \in [0, T], 1 < \beta \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu x(\tau), \end{cases} \quad (1.1)$$

where $0 < p, q, r, z, m, n, h, k < 1$ are quantum numbers, $\eta, \xi, \theta, \tau \in (0, T)$ are fixed points, $\delta, \varepsilon, \gamma, \kappa, \mu, \nu > 0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, D_ω^ρ is the fractional ω-derivative of Riemann-Liouville type of order ρ, when $\rho \in \{\alpha, \beta\}$ and $\omega \in \{p, q\}$, I_ϕ^ψ is the fractional ϕ-integral of order ψ with $\phi \in \{r, z, m, n, h, k\}$ and $\psi \in \{\delta, \varepsilon, \gamma, \kappa, \mu, \nu\}$ and $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The early work on q-difference calculus or *quantum calculus* dates back to Jackson's paper [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in [5–18], and the references cited therein, whereas the background material on q-fractional calculus can be found in a recent book [19].

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Recently in [20], we have studied the existence and the uniqueness of solutions of a class of boundary value problems for fractional q-integro-difference equations with nonlocal fractional q-integral conditions which have different quantum numbers. Here we extend the results of [20] to a coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions.

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. Some auxiliary lemmas are presented in Section 3. In Section 4, we establish an existence and a uniqueness result via the Banach contraction principle, and an existence result by applying Leray-Schauder's alternative. Results on the uncoupled integral boundary conditions case are contained in Section 5. Examples illustrating our results are also presented.

2 Preliminaries

To make this paper self-contained, below we recall some well-known facts on fractional q-calculus. The presentation here can be found in, for example, [6, 19].

For $q \in (0,1)$, define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \tag{2.1}$$

The q-analog of the power function $(a - b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is

$$(a - b)^{(0)} = 1, \qquad (a - b)^{(k)} = \prod_{i=0}^{k-1} (a - bq^i), \quad k \in \mathbb{N}, a, b \in \mathbb{R}. \tag{2.2}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(a - b)^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty} \frac{1 - (b/a)q^i}{1 - (b/a)q^{\gamma+i}}, \quad a \neq 0. \tag{2.3}$$

Note if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(t) = \frac{(1 - q)^{(t-1)}}{(1 - q)^{t-1}}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}. \tag{2.4}$$

Obviously, $\Gamma_q(t + 1) = [t]_q \Gamma_q(t)$.

The q-derivative of a function h is defined by

$$(D_q h)(t) = \frac{h(t) - h(qt)}{(1 - q)t} \quad \text{for } t \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{t \to 0} (D_q h)(t), \tag{2.5}$$

and q-derivatives of higher order are given by

$$(D_q^0 h)(t) = h(t) \quad \text{and} \quad (D_q^k h)(t) = D_q(D_q^{k-1} h)(t), \quad k \in \mathbb{N}. \tag{2.6}$$

The q-integral of a function h defined on the interval $[0, b]$ is given by

$$(I_q h)(t) = \int_0^t h(s)\, d_q s = t(1 - q) \sum_{i=0}^{\infty} h(tq^i) q^i, \quad t \in [0, b]. \tag{2.7}$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s)\,d_q s = \int_0^b h(s)\,d_q s - \int_0^a h(s)\,d_q s. \tag{2.8}$$

Similar to derivatives, an operator I_q^k is given by

$$\left(I_q^0 h\right)(t) = h(t) \quad \text{and} \quad \left(I_q^k h\right)(t) = I_q\left(I_q^{k-1} h\right)(t), \quad k \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators D_q and I_q, *i.e.*,

$$(D_q I_q h)(t) = h(t), \tag{2.10}$$

and if h is continuous at $t = 0$, then

$$(I_q D_q h)(t) = h(t) - h(0). \tag{2.11}$$

Definition 2.1 Let $\nu \geq 0$ and h be a function defined on $[0, T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(t) = h(t)$ and

$$\left(I_q^\nu h\right)(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{(\nu-1)} h(s)\,d_q s, \quad \nu > 0, t \in [0, T]. \tag{2.12}$$

Definition 2.2 The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(t) = h(t)$ and

$$\left(D_q^\nu h\right)(t) = \left(D_q^l I_q^{l-\nu} h\right)(t), \quad \nu > 0, \tag{2.13}$$

where l is the smallest integer greater than or equal to ν.

Definition 2.3 For any $t, s > 0$,

$$B_q(t, s) = \int_0^1 u^{(t-1)}(1 - qu)^{(s-1)}\,d_q u \tag{2.14}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t + s)}.$$

Lemma 2.4 [4] *Let $\alpha, \beta \geq 0$ and f be a function defined in $[0, T]$. Then the following formulas hold*:
  (1) $(I_q^\beta I_q^\alpha f)(t) = (I_q^{\alpha+\beta} f)(t)$,
  (2) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.5 [6] *Let $\alpha > 0$ and n be a positive integer. Then the following equality holds*:

$$\left(I_q^\alpha D_q^n f\right)(t) = \left(D_q^n I_q^\alpha f\right)(t) - \sum_{i=0}^{n-1} \frac{t^{\alpha-n+i}}{\Gamma_q(\alpha + i - n + 1)} \left(D_q^i f\right)(0). \tag{2.15}$$

3 Some auxiliary lemmas

The following formulas have been modified from Lemmas 3.2 and 7 in [21] and [20], respectively.

Lemma 3.1 *Let $x, y, z > 0$ and $0 < u, v, w < 1$. Then, for $\phi \in \mathbb{R}_+$, we have*

(i) $I_u^x I_v^y(1)(\phi) = \frac{\Gamma_u(y+1)}{\Gamma_u(x+y+1)\Gamma_v(y+1)} \phi^{x+y}$;

(ii) $I_u^x I_v^y I_w^z(1)(\phi) = \frac{\Gamma_u(y+z+1)\Gamma_v(z+1)}{\Gamma_u(x+y+z+1)\Gamma_v(y+z+1)\Gamma_w(z+1)} \phi^{x+y+z}$.

Lemma 3.2 *Given $u, v \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = u(t), & t \in [0, T], 1 < \alpha \leq 2, \\ D_p^\beta y(t) = v(t), & t \in [0, T], 1 < \beta \leq 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu x(\tau), \end{cases} \tag{3.1}$$

is

$$\begin{aligned} x(t) = {}& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s) \, d_q s + \frac{\lambda_2 \Omega_1}{\Omega} t^{\alpha-1} I_h^\mu I_p^\beta v(\theta) \\ & - \frac{\Omega_1}{\Omega} t^{\alpha-1} I_k^\nu I_q^\alpha u(\tau) + \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_m^\gamma I_q^\alpha u(\eta) \\ & - \frac{\lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_n^\kappa I_p^\beta v(\xi) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} y(t) = {}& \frac{1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} v(s) \, d_p s + \frac{\lambda_1 \Omega_2}{\Omega} t^{\beta-1} I_m^\gamma I_q^\alpha u(\eta) \\ & - \frac{\Omega_2}{\Omega} t^{\beta-1} I_n^\kappa I_p^\beta v(\xi) + \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} t^{\beta-1} I_h^\mu I_p^\beta v(\theta) \\ & - \frac{\lambda_1 \Omega_3}{\Omega} t^{\beta-1} I_k^\nu I_q^\alpha u(\tau), \end{aligned} \tag{3.3}$$

where

$$\Omega_1 = \frac{\Gamma_n(\beta)}{\Gamma_n(\beta + \kappa)} \xi^{\beta+\kappa-1},$$

$$\Omega_2 = \frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha + \nu)} \tau^{\alpha+\nu-1},$$

$$\Omega_3 = \frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha + \gamma)} \eta^{\alpha+\gamma-1},$$

$$\Omega_4 = \frac{\Gamma_h(\beta)}{\Gamma_h(\beta + \mu)} \theta^{\beta+\mu-1},$$

$$\Omega = \Omega_1 \Omega_2 - \lambda_1 \lambda_2 \Omega_3 \Omega_4 \neq 0.$$

Proof From $1 < \alpha \leq 2$, we let $n = 2$. Applying Lemma 2.5, the equations in (3.1) can be expressed as equivalent integral equations

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s) \, d_q s, \tag{3.4}$$

$$y(t) = d_1 t^{\beta-1} + d_2 t^{\beta-2} + \frac{1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} v(s) \, d_p s \tag{3.5}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0$ and $y(0) = 0$ imply that $c_2 = 0$ and $d_2 = 0$, respectively. Taking the Riemann-Liouville fractional ϕ-integral of order $\psi > 0$ for (3.4) and (3.5), we have the system

$$I_\phi^\psi x(t) = c_1 \frac{\Gamma_\phi(\alpha)}{\Gamma_\phi(\alpha + \psi)} t^{\alpha+\psi-1}$$
$$+ \frac{1}{\Gamma_\phi(\psi)\Gamma_q(\alpha)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - qw)^{(\alpha-1)} u(w) \, d_q w \, d_\phi s, \tag{3.6}$$

$$I_\phi^\psi y(t) = d_1 \frac{\Gamma_\phi(\beta)}{\Gamma_\phi(\beta + \psi)} t^{\beta+\psi-1}$$
$$+ \frac{1}{\Gamma_\phi(\psi)\Gamma_p(\beta)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - pw)^{(\beta-1)} v(w) \, d_p w \, d_\phi s. \tag{3.7}$$

Substituting (ψ, ϕ, t) by (γ, m, η), (ν, k, τ) in (3.6), and (κ, n, ξ), (μ, h, θ) in (3.7) and using Lemma 2.4 with nonlocal conditions in (3.1), we have

$$c_1 = \frac{\lambda_2 \Omega_1}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\Omega_1}{\Omega} I_k^\nu I_q^\alpha u(\tau)$$
$$+ \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\lambda_2 \Omega_4}{\Omega} I_n^\kappa I_p^\beta v(\xi)$$

and

$$d_1 = \frac{\lambda_1 \Omega_2}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\Omega_2}{\Omega} I_n^\kappa I_p^\beta v(\xi)$$
$$+ \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\lambda_1 \Omega_3}{\Omega} I_k^\nu I_q^\alpha u(\tau).$$

Substituting the values of c_1, c_2, d_1, and d_2 in (3.4) and (3.5), we obtain the solutions (3.2) and (3.3) as required. $\quad\square$

4 Main results

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$. Let us introduce the space $X = \{x(t) | x(t) \in C([0, T], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t) | y(t) \in C([0, T], \mathbb{R})\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0, T]\}$. Obviously the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 3.2, we define an operator $\mathcal{K} : X \times Y \to X \times Y$ by

$$\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix},$$

where

$$\mathcal{K}_1(x,y)(t) = I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(t) + \frac{\lambda_2 \Omega_1}{\Omega} t^{\alpha-1} I_h^\mu I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\theta)$$

$$- \frac{\Omega_1}{\Omega} t^{\alpha-1} I_k^\nu I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\tau)$$

$$+ \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_m^\gamma I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\eta)$$

$$- \frac{\lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_n^\kappa I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\xi) \tag{4.1}$$

and

$$\mathcal{K}_2(x,y)(t) = I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(t) + \frac{\lambda_1 \Omega_2}{\Omega} t^{\beta-1} I_m^\gamma I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\eta)$$

$$- \frac{\Omega_2}{\Omega} t^{\beta-1} I_n^\kappa I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\xi)$$

$$+ \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} t^{\beta-1} I_h^\mu I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\theta)$$

$$- \frac{\lambda_1 \Omega_3}{\Omega} t^{\beta-1} I_k^\nu I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\tau). \tag{4.2}$$

For the sake of convenience, we set

$$A_1 = \frac{T^\alpha}{\Gamma_q(\alpha+1)}, \qquad A_2 = \frac{T^\beta}{\Gamma_p(\beta+1)},$$

$$A_3 = \frac{\Gamma_q(\delta+1)T^{\alpha+\delta}}{\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}, \qquad A_4 = \frac{\Gamma_p(\varepsilon+1)T^{\beta+\varepsilon}}{\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},$$

$$A_5 = \frac{\Gamma_m(\alpha+1)\eta^{\gamma+\alpha}}{\Gamma_m(\gamma+\alpha+1)\Gamma_q(\alpha+1)}, \qquad A_6 = \frac{\Gamma_n(\beta+1)\xi^{\kappa+\beta}}{\Gamma_n(\kappa+\beta+1)\Gamma_p(\beta+1)},$$

$$A_7 = \frac{\Gamma_h(\beta+1)\theta^{\mu+\beta}}{\Gamma_h(\mu+\beta+1)\Gamma_p(\beta+1)}, \qquad A_8 = \frac{\Gamma_k(\alpha+1)\tau^{\nu+\alpha}}{\Gamma_k(\nu+\alpha+1)\Gamma_q(\alpha+1)},$$

$$A_9 = \frac{\Gamma_m(\alpha+\delta+1)\Gamma_q(\delta+1)\eta^{\gamma+\alpha+\delta}}{\Gamma_m(\gamma+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)},$$

$$A_{10} = \frac{\Gamma_n(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\xi^{\kappa+\beta+\varepsilon}}{\Gamma_n(\kappa+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},$$

$$A_{11} = \frac{\Gamma_h(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\theta^{\mu+\beta+\varepsilon}}{\Gamma_h(\mu+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},$$

$$A_{12} = \frac{\Gamma_k(\alpha+\delta+1)\Gamma_q(\delta+1)\tau^{\nu+\alpha+\delta}}{\Gamma_k(\nu+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}.$$

Theorem 4.1 *Assume that $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants $M_i, N_i, i = 1, 2$, such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,*

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \leq M_1|u_1 - v_1| + M_2|u_2 - v_2|$$

and

$$\big|g(t, u_1, u_2) - g(t, v_1, v_2)\big| \leq N_1|u_1 - v_1| + N_2|u_2 - v_2|.$$

In addition, we suppose that

$$B_1 + B_2 + C_1 + C_2 < 1,$$

where

$$B_1 = M_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 N_2 A_{11} + |\lambda_1| \Omega_4 M_1 A_5 + \Omega_4 N_2 A_{10} \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_1 A_8,$$

$$B_2 = M_2 A_3 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 N_1 A_7 + |\lambda_1| \Omega_4 M_2 A_9 + \Omega_4 N_1 A_6 \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_2 A_{12},$$

$$C_1 = N_2 A_4 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 M_1 A_5 + |\lambda_2| \Omega_3 N_2 A_{11} + \Omega_3 M_1 A_8 \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_2 A_{10},$$

$$C_2 = N_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 M_2 A_9 + |\lambda_2| \Omega_3 N_1 A_7 + \Omega_3 M_2 A_{12} \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_1 A_6.$$

Then the system (1.1) *has a unique solution on* $[0, T]$.

Proof Firstly, we define $\sup_{t \in [0,T]} |f(t,0,0)| = G_1 < \infty$ and $\sup_{t \in [0,T]} |g(t,0,0)| = G_2 < \infty$ such that

$$r \geq \max\left\{ \frac{B_3}{1 - (B_1 + B_2)}, \frac{C_3}{1 - (C_1 + C_2)} \right\},$$

where

$$B_3 = G_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 G_2 A_7 + |\lambda_1| \Omega_4 G_1 A_5 + \Omega_4 G_2 A_6 \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} G_1 A_8,$$

$$C_3 = G_2 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 G_1 A_5 + |\lambda_2| \Omega_3 G_2 A_7 + \Omega_3 G_1 A_8 \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} G_2 A_6.$$

We will show that $\mathcal{K} B_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq r\}$.

For $(x,y) \in B_r$, taking into account Lemma 3.1, we have

$$\big| \mathcal{K}_1(x,y)(t) \big|$$

$$\leq \sup_{t \in T} \bigg\{ I_q^\alpha \big| f\big(s, x(s), I_r^\delta y(s)\big) \big|(t) + \frac{|\lambda_2| \Omega_1}{|\Omega|} t^{\alpha-1} I_h^\mu I_p^\beta \big| g\big(s, y(s), I_z^\varepsilon x(s)\big) \big|(\theta)$$

$$+ \frac{\Omega_1}{|\Omega|} t^{\alpha-1} I_k^\nu I_q^\alpha \big| f\big(s, x(s), I_r^\delta y(s)\big) \big|(\tau) + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} t^{\alpha-1} I_m^\gamma I_q^\alpha \big| f\big(s, x(s), I_r^\delta y(s)\big) \big|(\eta)$$

$$+ \frac{|\lambda_2|\Omega_4}{|\Omega|} t^{\alpha-1} I_n^\kappa I_p^\beta \big| g\big(s, y(s), I_z^\varepsilon x(s)\big) \big|(\xi) \bigg\}$$

$$\leq I_q^\alpha \big(\big| f\big(s, x(s), I_r^\delta y(s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(t)$$

$$+ \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta \big(\big| g\big(s, y(s), I_z^\varepsilon x(s)\big) - g(s,0,0) \big| + \big| g(s,0,0) \big| \big)(\theta)$$

$$+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha \big(\big| f\big(s, x(s), I_r^\delta y(s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(\tau)$$

$$+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha \big(\big| f\big(s, x(s), I_r^\delta y(s)\big) - f(s,0,0) \big| + \big| f(s,0,0) \big| \big)(\eta)$$

$$+ \frac{|\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_n^{\kappa} I_p^{\beta} \left(\left| g\left(s, y(s), I_z^{\varepsilon} x(s)\right) - g(t,0,0) \right| + \left| g(t,0,0) \right| \right)(\xi)$$

$$\leq M_1 \|x\| A_1 + M_2 \|y\| A_3 + G_1 A_1$$

$$+ \frac{|\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} \left(N_1 \|y\| A_7 + N_2 \|x\| A_{11} + G_2 A_7 \right)$$

$$+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} \left(M_1 \|x\| A_8 + M_2 \|y\| A_{12} + G_1 A_8 \right)$$

$$+ \frac{|\lambda_1||\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} \left(M_1 \|x\| A_5 + M_2 \|y\| A_9 + G_1 A_5 \right)$$

$$+ \frac{|\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} \left(N_1 \|y\| A_6 + N_2 \|x\| A_{10} + G_2 A_6 \right)$$

$$= B_1 \|x\| + B_2 \|y\| + B_3$$

$$\leq (B_1 + B_2) r + B_3 \leq r.$$

In a similar way, we get

$$\left| \mathcal{K}_2(x,y)(t) \right|$$

$$\leq I_p^{\beta} \left(\left| g\left(s, y(s), I_z^{\varepsilon} x(s)\right) - g(s,0,0) \right| + \left| g(s,0,0) \right| \right)(t)$$

$$+ \frac{|\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} I_m^{\gamma} I_q^{\alpha} \left(\left| f\left(s, x(s), I_r^{\delta} y(s)\right) - f(s,0,0) \right| + \left| f(s,0,0) \right| \right)(\eta)$$

$$+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^{\kappa} I_p^{\beta} \left(\left| g\left(s, y(s), I_z^{\varepsilon} x(s)\right) - g(s,0,0) \right| + \left| g(s,0,0) \right| \right)(\xi)$$

$$+ \frac{|\lambda_1||\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} I_h^{\mu} I_p^{\beta} \left(\left| g\left(s, y(s), I_z^{\varepsilon} x(s)\right) - g(s,0,0) \right| + \left| g(s,0,0) \right| \right)(\theta)$$

$$+ \frac{|\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} I_k^{\nu} I_q^{\alpha} \left(\left| f\left(s, x(s), I_r^{\delta} y(s)\right) - f(s,0,0) \right| + \left| f(s,0,0) \right| \right)(\tau)$$

$$\leq N_1 \|y\| A_2 + N_2 \|x\| A_4 + G_2 A_2$$

$$+ \frac{|\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} \left(M_1 \|x\| A_5 + M_2 \|y\| A_9 + G_1 A_5 \right)$$

$$+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} \left(N_1 \|y\| A_6 + N_2 \|x\| A_{10} + G_2 A_6 \right)$$

$$+ \frac{|\lambda_1||\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} \left(N_1 \|y\| A_7 + N_2 \|x\| A_{11} + G_2 A_7 \right)$$

$$+ \frac{|\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} \left(M_1 \|x\| A_8 + M_2 \|y\| A_{12} + G_1 A_8 \right)$$

$$= C_1 \|x\| + C_2 \|y\| + C_3$$

$$\leq (C_1 + C_2) r + B_3 \leq r.$$

Consequently, $\|\mathcal{K}(x,y)(t)\| \leq r$.

Next, for $(x_2, y_2), (x_1, y_1) \in X \times Y$, and for any $t \in [0, T]$, we have

$$\left| \mathcal{K}_1(x_2, y_2)(t) - \mathcal{K}_1(x_1, y_1)(t) \right|$$

$$\leq I_q^{\alpha} \left(\left| f\left(s, x_2(s), I_r^{\delta} y_2(s)\right) - f\left(s, x_1(s), I_r^{\delta} y_1(s)\right) \right| \right)(t)$$

$$+ \frac{|\lambda_2||\Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta \big(\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big) - g\big(s, y_1(s), I_z^\varepsilon x_1(s)\big)\big|\big)(\theta)$$

$$+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha \big(\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big) - f\big(s, x_1(s), I_r^\delta y_1(s)\big)\big|\big)(\tau)$$

$$+ \frac{|\lambda_1||\lambda_2||\Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha \big(\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big) - f\big(s, x_1(s), I_r^\delta y_1(s)\big)\big|\big)(\eta)$$

$$+ \frac{|\lambda_2||\Omega_4}{|\Omega|} T^{\alpha-1} I_n^\kappa I_p^\beta \big(\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big) - g\big(s, y_1(s), I_z^\varepsilon x_1(s)\big)\big|\big)(\xi)$$

$$\leq M_1 \|x_2 - x_1\| I_q^\alpha(1)(T) + M_2 \|y_2 - y_1\| I_q^\alpha I_r^\delta(1)(T)$$

$$+ \frac{|\lambda_2||\Omega_1}{|\Omega|} T^{\alpha-1} \big(N_1 \|y_2 - y_1\| I_h^\mu I_p^\beta(1)(\theta) + N_2 \|x_2 - x_1\| I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)\big)$$

$$+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} \big(M_1 \|x_2 - x_1\| I_k^\nu I_q^\alpha(1)(\tau) + M_2 \|y_2 - y_1\| I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)\big)$$

$$+ \frac{|\lambda_1||\lambda_2||\Omega_4}{|\Omega|} T^{\alpha-1} \big(M_1 \|x_2 - x_1\| I_m^\gamma I_q^\alpha(1)(\eta) + M_2 \|y_2 - y_1\| I_m^\gamma I_q^\alpha I_r^\delta(1)(\eta)\big)$$

$$+ \frac{|\lambda_2||\Omega_4}{|\Omega|} T^{\alpha-1} \big(N_1 \|y_2 - y_1\| I_n^\kappa I_p^\beta(1)(\xi) + N_2 \|x_2 - x_1\| I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)\big)$$

$$= B_1 \|x_2 - x_1\| + B_2 \|y_2 - y_1\|.$$

Therefore, we have

$$\big\|\mathcal{K}_1(x_2, y_2)(t) - \mathcal{K}_1(x_1, y_1)(t)\big\| \leq (B_1 + B_2)\big(\|x_2 - x_1\| + \|y_2 - y_1\|\big). \tag{4.3}$$

In the same way, we have

$$\big|\mathcal{K}_2(x_2, y_2)(t) - \mathcal{K}_2(x_1, y_1)(t)\big|$$

$$\leq I_p^\beta \big(\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big) - g\big(s, y_1(s), I_z^\varepsilon x_1(s)\big)\big|\big)(t)$$

$$+ \frac{|\lambda_1||\Omega_2}{|\Omega|} T^{\beta-1} I_m^\gamma I_q^\alpha \big(\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big) - f\big(s, x_1(s), I_r^\delta y_1(s)\big)\big|\big)(\eta)$$

$$+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^\kappa I_p^\beta \big(\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big) - g\big(s, y_1(s), I_z^\varepsilon x_1(s)\big)\big|\big)(\xi)$$

$$+ \frac{|\lambda_1||\lambda_2||\Omega_3}{|\Omega|} T^{\beta-1} I_h^\mu I_p^\beta \big(\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big) - g\big(s, y_1(s), I_z^\varepsilon x_1(s)\big)\big|\big)(\theta)$$

$$+ \frac{|\lambda_1||\Omega_3}{|\Omega|} T^{\beta-1} I_k^\nu I_q^\alpha \big(\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big) - f\big(s, x_1(s), I_r^\delta y_1(s)\big)\big|\big)(\tau)$$

$$\leq N_1 \|y_2 - y_1\| A_2 + N_2 \|x_2 - x_1\| A_4$$

$$+ \frac{|\lambda_1||\Omega_2}{|\Omega|} T^{\beta-1} \big(M_1 \|x_2 - x_1\| A_5 + M_2 \|y_2 - y_1\| A_9\big)$$

$$+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} \big(N_1 \|y_2 - y_1\| A_6 + N_2 \|x_2 - x_1\| A_{10}\big)$$

$$+ \frac{|\lambda_1||\lambda_2||\Omega_3}{|\Omega|} T^{\beta-1} \big(N_1 \|y_2 - y_1\| A_7 + N_2 \|x_2 - x_1\| A_{11}\big)$$

$$+ \frac{|\lambda_1||\Omega_3}{|\Omega|} T^{\beta-1} \big(M_1 \|x_2 - x_1\| A_8 + M_2 \|y_2 - y_1\| A_{12}\big)$$

$$= C_1 \|x_2 - x_1\| + C_2 \|y_2 - y_1\|,$$

which implies

$$\big\| \mathcal{K}_2(x_2, y_2)(t) - \mathcal{K}_2(x_1, y_1)(t) \big\| \le (C_1 + C_2)\big(\|x_2 - x_1\| + \|y_2 - y_1\| \big). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\big\| \mathcal{K}(x_2, y_2)(t) - \mathcal{K}(x_1, y_1)(t) \big\| \le (B_1 + B_2 + C_1 + C_2)\big(\|x_2 - x_1\| + \|y_2 - y_1\| \big).$$

Since $B_1 + B_2 + C_1 + C_2 < 1$, therefore, $\mathcal{K}$ is a contraction operator. So, by Banach's fixed point theorem, the operator $\mathcal{K}$ has a unique fixed point, which is the unique solution of problem (1.1). The proof is completed. □

In the next result, we prove the existence of solutions for the problem (1.1) by applying the Leray-Schauder alternative.

Lemma 4.2 (Leray-Schauder alternative, see [22], p.4) *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \big\{ x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1 \big\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

For convenience, we set constants

$$E_0 = P_0 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 Q_0 A_7 + |\lambda_1| \Omega_4 P_0 A_5 + \Omega_4 Q_0 A_6 \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_0 A_8,$$

$$E_1 = P_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 Q_2 A_{11} + |\lambda_1| \Omega_4 P_1 A_5 + \Omega_4 Q_2 A_{10} \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_1 A_8,$$

$$E_3 = P_2 A_3 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} \big(\Omega_1 Q_1 A_7 + |\lambda_1| \Omega_4 P_2 A_9 + \Omega_4 Q_1 A_6 \big) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_2 A_{12},$$

$$F_0 = Q_0 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 P_0 A_5 + |\lambda_2| \Omega_3 Q_0 A_7 + \Omega_3 P_0 A_8 \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_0 A_6,$$

$$F_1 = Q_2 A_4 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 P_1 A_5 + |\lambda_2| \Omega_3 Q_2 A_{11} + \Omega_3 P_1 A_8 \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_2 A_{10},$$

$$F_2 = Q_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 P_2 A_9 + |\lambda_2| \Omega_3 Q_1 A_7 + \Omega_3 P_2 A_{12} \big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_1 A_6$$

and

$$G^* = \max \big\{ 1 - (E_1 + F_1), 1 - (E_2 + F_2) \big\}.$$

Theorem 4.3 *Assume that there exist real constants $P_i, Q_i \ge 0$ $(i = 1, 2)$, and $P_0 > 0$, $Q_0 > 0$ such that for all $u_i, v_i \in \mathbb{R}$ $(i = 1, 2)$ we have*

$$\big| f(t, u_1, u_2) \big| \le P_0 + P_1 |u_1| + P_2 |u_2|,$$

$$\big| g(t, v_1, v_2) \big| \le Q_0 + Q_1 |v_1| + Q_2 |v_2|.$$

In addition it is assumed that

$$E_1 + F_1 < 1 \quad and \quad E_2 + F_2 < 1.$$

Then there exists at least one solution for the system (1.1).

Proof We first prove that the operator $\mathcal{K} : X \times Y \to X \times Y$ is completely continuous. The continuity of functions f and g imply that the operator $\mathcal{K}$ is continuous. Let $\Phi \subset X \times Y$ be a bounded set. Then there exist positive constants D_1 and D_2 such that

$$\left| f\big(t, u_1(t), u_2(t)\big) \right| \le D_1, \qquad \left| g\big(t, v_1(t), v_2(t)\big) \right| \le D_2, \quad \forall (u_1, u_2), (v_1, v_2) \in \Phi.$$

Then for any $(u_1, u_2), (v_1, v_2) \in \Phi$, and using Lemma 3.1, we have

$$
\begin{aligned}
\left\| \mathcal{K}_1(x, y) \right\| \le{}& I_q^\alpha \left| f\big(s, x(s), I_r^\delta y(s)\big) \right|(t) + \frac{|\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta \left| g\big(s, y(s), I_z^\varepsilon x(s)\big) \right|(\theta) \\
&+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha \left| f\big(s, x(s), I_r^\delta y(s)\big) \right|(\tau) \\
&+ \frac{|\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha \left| f\big(s, x(s), I_r^\delta y(s)\big) \right|(\eta) \\
&+ \frac{|\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_n^\kappa I_p^\beta \left| g\big(s, y(s), I_z^\varepsilon x(s)\big) \right|(\xi) \\
\le{}& D_1 A_1 + \frac{D_2 |\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} A_7 + \frac{D_1 \Omega_1}{|\Omega|} T^{\alpha-1} A_8 \\
&+ \frac{D_1 |\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} A_5 + \frac{D_2 |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} A_6.
\end{aligned}
$$

In the same way, we deduce that

$$
\begin{aligned}
\left\| \mathcal{K}_2(x, y) \right\| \le{}& I_p^\beta \left(\left| g\big(s, y(s), I_z^\varepsilon x(s)\big) \right| \right)(t) + \frac{|\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} I_m^\gamma I_q^\alpha \left(\left| f\big(s, x(s), I_r^\delta y(s)\big) \right| \right)(\eta) \\
&+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^\kappa I_p^\beta \left(\left| g\big(s, y(s), I_z^\varepsilon x(s)\big) \right| \right)(\xi) \\
&+ \frac{|\lambda_1| |\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} I_h^\mu I_p^\beta \left(\left| g\big(s, y(s), I_z^\varepsilon x(s)\big) \right| \right)(\theta) \\
&+ \frac{|\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} I_k^\nu I_q^\alpha \left(\left| f\big(s, x(s), I_r^\delta y(s)\big) \right| \right)(\tau) \\
\le{}& D_2 A_2 + \frac{D_1 |\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} A_5 + \frac{D_2 \Omega_2}{|\Omega|} T^{\beta-1} A_6 \\
&+ \frac{D_2 |\lambda_1| |\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} A_7 + \frac{D_1 |\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} A_8.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathcal{K}$ is uniformly bounded.

Next, we show that $\mathcal{K}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$
\begin{aligned}
&\left| \mathcal{K}_1(x, y)(t_2) - \mathcal{K}_1(x, y)(t_1) \right| \\
&\quad \le \left| I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(t_2) - I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(t_1) \right|
\end{aligned}
$$

$$+ \frac{|\lambda_2|\Omega_1}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|\big|I_h^\mu I_p^\beta\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big)\big|(\theta)$$

$$+ \frac{\Omega_1}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|\big|I_k^\nu I_q^\alpha\big|f\big(s, x(s), I_r^\delta y(s)\big)\big|(\tau)$$

$$+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|\big|I_m^\gamma I_q^\alpha\big|f\big(s, x(s), I_r^\delta y(s)\big)\big|(\eta)$$

$$+ \frac{|\lambda_2|\Omega_4}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|\big|I_n^\kappa I_p^\beta\big|g\big(s, y(s), I_z^\varepsilon x(s)\big)\big|(\xi)$$

$$\leq \frac{D_1}{\Gamma_q(\alpha)}\int_0^{t_1}\big[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\big]\,d_q s$$

$$+ \frac{D_1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(t_2 - qs)^{(\alpha-1)}\,d_q s$$

$$+ \frac{|\lambda_2|\Omega_1 D_2}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|A_7 + \frac{\Omega_1 D_1}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|A_8$$

$$+ \frac{|\lambda_1||\lambda_2|\Omega_4 D_1}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|A_5 + \frac{|\lambda_2|\Omega_4 D_2}{|\Omega|}\big|t_2^{\alpha-1} - t_1^{\alpha-1}\big|A_6.$$

Analogously, we can get

$$\big|\mathcal{K}_2(x,y)(t_2) - \mathcal{K}_2(x,y)(t_1)\big|$$

$$\leq \big|I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(t_2) - I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(t_1)\big|$$

$$+ \frac{|\lambda_1|\Omega_2}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|\big|I_m^\gamma I_q^\alpha\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big)\big|(\eta)$$

$$+ \frac{\Omega_2}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|\big|I_n^\kappa I_p^\beta\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big)\big|(\xi)$$

$$+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|\big|I_h^\mu I_p^\beta\big|g\big(s, y_2(s), I_z^\varepsilon x_2(s)\big)\big|(\theta)$$

$$+ \frac{|\lambda_1|\Omega_3}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|\big|I_k^\nu I_q^\alpha\big|f\big(s, x_2(s), I_r^\delta y_2(s)\big)\big|(\tau)$$

$$\leq \frac{D_2}{\Gamma_p(\beta)}\int_0^{t_1}\big[(t_2 - ps)^{(\beta-1)} - (t_1 - ps)^{(\beta-1)}\big]\,d_p s$$

$$+ \frac{D_2}{\Gamma_p(\beta)}\int_{t_1}^{t_2}(t_2 - ps)^{(\beta-1)}\,d_p s$$

$$+ \frac{|\lambda_1|\Omega_2 D_1}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|A_5 + \frac{\Omega_2 D_2}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|A_6$$

$$+ \frac{|\lambda_1||\lambda_2|\Omega_3 D_2}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|A_7 + \frac{|\lambda_1|\Omega_3 D_1}{|\Omega|}\big|t_2^{\beta-1} - t_1^{\beta-1}\big|A_8.$$

Therefore, the operator $\mathcal{K}(x,y)$ is equicontinuous, and thus the operator $\mathcal{K}(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x,y) \in X \times Y : (x,y) = \lambda\mathcal{K}(x,y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x,y) \in \mathcal{E}$, then $(x,y) = \lambda\mathcal{K}(x,y)$. For any $t \in [0,T]$, we have

$$x(t) = \lambda\mathcal{K}_1(x,y)(t), \qquad y(t) = \lambda\mathcal{K}_2(x,y)(t).$$

Then we have

$$
\begin{aligned}
\left|x(t)\right| \leq{}& P_0 I_q^\alpha(1)(t) + P_1\|x\|I_q^\alpha(1)(t) + P_2\|y\|I_q^\alpha I_r^\delta(1)(t) \\
&+ \frac{|\lambda_2|\Omega_1}{|\Omega|}T^{\alpha-1}\big(Q_0 I_h^\mu I_p^\beta(1)(\theta) + Q_1\|y\|I_h^\mu I_p^\beta(1)(\theta) + Q_2\|x\|I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)\big) \\
&+ \frac{\Omega_1}{|\Omega|}T^{\alpha-1}\big(P_0 I_k^\nu I_q^\alpha(1)(\tau) + P_1\|x\|I_k^\nu I_q^\alpha(1)(\tau) + P_2\|y\|I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)\big) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|}T^{\alpha-1}\big(P_0 I_m^\nu I_q^\alpha(1)(\eta) + P_1\|x\|I_m^\nu I_q^\alpha(1)(\eta) + P_2\|y\|I_m^\nu I_q^\alpha I_r^\delta(1)(\eta)\big) \\
&+ \frac{|\lambda_2|\Omega_4}{|\Omega|}T^{\alpha-1}\big(Q_0 I_n^\kappa I_p^\beta(1)(\xi) + Q_1\|y\|I_n^\kappa I_p^\beta(1)(\xi) + Q_2\|x\|I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)\big) \\
\leq{}& E_0 + E_1\|x\| + E_2\|y\|
\end{aligned}
$$

and

$$
\begin{aligned}
\left|y(t)\right| \leq{}& Q_0 I_p^\beta(1)(t) + Q_1\|y\|I_p^\beta(1)(t) + Q_1\|x\|I_p^\beta I_z^\varepsilon(1)(t) \\
&+ \frac{|\lambda_1|\Omega_2}{|\Omega|}T^{\beta-1}\big(P_0 I_m^\nu I_q^\alpha(1)(\eta) + P_1\|x\|I_m^\nu I_q^\alpha(1)(\eta) + P_2\|y\|I_m^\nu I_q^\alpha I_r^\delta(1)(\eta)\big) \\
&+ \frac{\Omega_2}{|\Omega|}T^{\beta-1}\big(Q_0 I_n^\kappa I_p^\beta(1)(\xi) + Q_1\|y\|I_n^\kappa I_p^\beta(1)(\xi) + Q_2\|x\|I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)\big) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|}T^{\beta-1}\big(Q_0 I_h^\mu I_p^\beta(1)(\theta) + Q_1\|y\|I_h^\mu I_p^\beta(1)(\theta) + Q_2\|x\|I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)\big) \\
&+ \frac{|\lambda_1|\Omega_3}{|\Omega|}T^{\beta-1}\big(P_0 I_k^\nu I_q^\alpha(1)(\tau) + P_1\|x\|I_k^\nu I_q^\alpha(1)(\tau) + P_2\|y\|I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)\big) \\
\leq{}& F_0 + F_1\|x\| + F_2\|y\|,
\end{aligned}
$$

which yields

$$
\|x\| \leq E_0 + E_1\|x\| + E_2\|y\|
$$

and

$$
\|y\| \leq F_0 + F_1\|x\| + F_2\|y\|.
$$

Therefore, we have

$$
\|x\| + \|y\| \leq (E_0 + F_0) + (E_1 + F_1)\|x\| + (E_2 + F_2)\|y\|,
$$

and, consequently,

$$
\left\|(x,y)\right\| \leq \frac{E_0 + F_0}{G^*}
$$

for any $t \in [0,T]$, which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 4.2, the operator $\mathcal{K}$ has at least one fixed point. Hence the system (1.1) has at least one solution. The proof is complete. □

4.1 Examples
In this subsection, we present some examples to illustrate our results.

Example 4.4 Consider the following coupled system of fractional q-integro-difference equations:

$$
\begin{cases}
D_{1/2}^{3/2}x(t) = \frac{\cos^2 \pi t}{(e^t+4)^2} \cdot \frac{|x(t)|}{4+|x(t)|} + \frac{e^{-t^2}}{(t+8)^2} \cdot I_{1/4}^{\pi}y(t) + \frac{\sqrt{2}}{2}, & 0 < t < 2, \\
D_{1/3}^{4/3}y(t) = \frac{\sin^2 \pi t}{(11+t)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+8)^2} \cdot I_{1/5}^{\pi/2}x(t) + \sqrt{3}, & \\
x(0) = 0, & \sqrt{2}I_{1/8}^{7/6}x(\tfrac{3}{2}) = I_{1/6}^{\sqrt{2}}y(\tfrac{1}{2}), \\
y(0) = 0, & \frac{\sqrt{3}}{2}I_{1/9}^{e}y(\tfrac{1}{3}) = I_{1/7}^{\sqrt{3}}x(\tfrac{5}{3}).
\end{cases}
\tag{4.5}
$$

Here $\alpha = 3/2$, $\delta = \pi$, $\beta = 4/3$, $\varepsilon = \pi/2$, $\gamma = 7/6$, $\kappa = \sqrt{2}$, $\mu = e$, $\nu = \sqrt{3}$, $q = 1/2$, $r = 1/4$, $p = 1/3$, $z = 1/5$, $m = 1/8$, $n = 1/6$, $h = 1/9$, $k = 1/7$, $\eta = 3/2$, $\xi = 1/2$, $\theta = 1/3$, $\tau = 5/3$, $\lambda_1 = \sqrt{2}$, $\lambda_2 = \sqrt{3}/2$, $T = 2$, $f(t,x,I_r^{\delta}y) = (|x|\cos^2 \pi t)/((e^t+4)^2(4+|x|)) + (e^{-t^2}/((t+8)^2))I_{1/4}^{\pi}y + \sqrt{2}/2$, and $g(t,y,I_r^{\delta}x) = (|y|\sin^2 \pi t)/((11+t)^2(1+|y|)) + (1/(e^t+8)^2)I_{1/5}^{\pi/2}x + \sqrt{3}$. Since

$$
\left| f(t,u_1,u_2) - f(t,v_1,v_2) \right| \le \frac{1}{100}|u_1 - v_1| + \frac{1}{64}|u_2 - v_2|
$$

and

$$
\left| g(t,u_1,u_2) - g(t,v_1,v_2) \right| \le \frac{1}{121}|u_1 - v_1| + \frac{1}{81}|u_2 - v_2|,
$$

then the assumptions of Theorem 4.1 are satisfied with $M_1 = 1/100$, $M_2 = 1/64$, $N_1 = 1/121$, and $N_2 = 1/81$. By using the Maple program, we can find that

$$
\Omega = \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \approx 0.61154471 \ne 0
$$

and

$$
B_1 = M_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\big(\Omega_1 N_2 A_{11} + |\lambda_1|\Omega_4 M_1 A_5 + \Omega_4 N_2 A_{10}\big) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_1 A_8
$$

$$
\approx 0.0577709,
$$

$$
B_2 = M_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\big(\Omega_1 N_1 A_7 + |\lambda_1|\Omega_4 M_2 A_9 + \Omega_4 N_1 A_6\big) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_2 A_{12}
$$

$$
\approx 0.1489994,
$$

$$
C_1 = N_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2 M_1 A_5 + |\lambda_2|\Omega_3 N_2 A_{11} + \Omega_3 M_1 A_8\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_2 A_{10}
$$

$$
\approx 0.22629179,
$$

$$
C_2 = N_1A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2 M_2 A_9 + |\lambda_2|\Omega_3 N_1 A_7 + \Omega_3 M_2 A_{12}\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_1 A_6
$$

$$
\approx 0.28656994.
$$

Therefore, we get

$$
B_1 + B_2 + C_1 + C_2 \approx 0.71963204 < 1.
$$

Hence, by Theorem 4.1, the problem (4.5) has a unique solution on $[0,2]$.

Example 4.5 Consider the following coupled system of fractional q-integro-difference equations with fractional q-integral conditions:

$$\begin{cases} D_{1/3}^{4/3}x(t) = \frac{4e^{-t}}{(t+11)^2} \cdot \frac{|x(t)|}{2+|x(t)|} + \frac{1}{(e^{-t^2}+8)^2} \cdot I_{1/5}^{\sqrt{3}}y(t) + \sqrt{2}, & 0 < t < \pi, \\ D_{1/4}^{7/5}y(t) = \frac{\cos^2 2\pi t}{(10+t)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+7)^2} \cdot I_{1/6}^{\sqrt{2}}x(t) + \frac{1}{2}, \\ x(0) = 0, \qquad I_{1/3}^{1/3}y(\frac{2\pi}{5}) + \sqrt{5}I_{1/2}^{\sqrt{5}}x(\frac{\pi}{5}) = 0, \\ y(0) = 0, \qquad I_{1/8}^{\sqrt{2}/2}x(\frac{4\pi}{5}) + \sqrt{2}I_{1/7}^{\sqrt{\pi}}y(\frac{3\pi}{5}) = 0. \end{cases} \tag{4.6}$$

Here $\alpha = 4/3$, $\delta = \sqrt{3}$, $\beta = 7/5$, $\varepsilon = \sqrt{2}$, $\gamma = \sqrt{5}$, $\kappa = 1/3$, $\mu = \sqrt{\pi}$, $\nu = \sqrt{2}/2$, $q = 1/3$, $r = 1/5$, $p = 1/4$, $z = 1/6$, $m = 1/2$, $n = 1/3$, $h = 1/7$, $k = 1/8$, $\eta = \pi/5$, $\xi = 2\pi/5$, $\theta = 3\pi/5$, $\tau = 4\pi/5$, $\lambda_1 = -\sqrt{5}$, $\lambda_2 = -\sqrt{2}$, $T = \pi$, $f(t,x,I_r^\delta y) = (4e^{-t}/(t+11)^2)(|x|/(2+|x|)) + (1/(e^{-t^2}+8)^2)I_{1/5}^{\sqrt{3}}y + \sqrt{2}$, and $g(t,y,I_r^\delta x) = (\cos^2 2\pi t/(10+t)^2)(|y|/(1+|y|)) + (1/(e^t+7)^2)I_{1/6}^{\sqrt{2}}x + (1/2)$. Since

$$\left|f(t,u_1,u_2)\right| \le \sqrt{2} + \frac{1}{72}|u_1| + \frac{1}{81}|u_2|$$

and

$$\left|g(t,v_1,v_2)\right| \le \frac{1}{2} + \frac{1}{100}|v_1| + \frac{1}{81}|v_2|,$$

then the assumptions of Theorem 4.3 are satisfied with $P_0 = \sqrt{2}$, $P_1 = 1/72$, $P_2 = 1/81$, $Q_0 = 1/2$, $Q_1 = 1/100$, and $Q_2 = 1/81$. By using the Maple program, we can find that

$$\Omega = \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \approx -4.18385985 \ne 0$$

and

$$E_0 = P_0A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\left(\Omega_1Q_0A_7 + |\lambda_1|\Omega_4P_0A_5 + \Omega_4Q_0A_6\right) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_0A_8$$

$$\approx 11.08984581,$$

$$E_1 = P_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\left(\Omega_1Q_2A_{11} + |\lambda_1|\Omega_4P_1A_5 + \Omega_4Q_2A_{10}\right) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_1A_8$$

$$\approx 0.1629615871,$$

$$E_2 = P_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\left(\Omega_1Q_1A_7 + |\lambda_1|\Omega_4P_2A_9 + \Omega_4Q_1A_6\right) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_2A_{12}$$

$$\approx 0.34091157,$$

$$F_0 = Q_0A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\left(\Omega_2P_0A_5 + |\lambda_2|\Omega_3Q_0A_7 + \Omega_3P_0A_8\right) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_0A_6$$

$$\approx 25.68580671,$$

$$F_1 = Q_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\left(\Omega_2P_1A_5 + |\lambda_2|\Omega_3Q_2A_{11} + \Omega_3P_1A_8\right) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_2A_{10}$$

$$\approx 0.68153261,$$

$$F_2 = Q_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} \big(\Omega_2 P_2 A_9 + |\lambda_2| \Omega_3 Q_1 A_7 + \Omega_3 P_2 A_{12}\big) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_1 A_6$$

$$\approx 0.5902944$$

and

$$G^* = \max\big\{1 - (E_1 + F_1), 1 - (E_2 + F_2)\big\} = \max\{0.1555058, 0.06879403\} = 0.1555058.$$

Therefore, we get

$$E_1 + F_1 \approx 0.8444942 < 1 \quad \text{and} \quad E_2 + F_2 \approx 0.93120597 < 1.$$

Hence, by Theorem 4.3, the problem (4.6) has at least one solution on $[0, \pi]$.

5 Uncoupled integral boundary conditions case

In this section we consider the following system:

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_r^\delta y(t)), & t \in [0, T], 1 < \alpha \le 2, \\ D_p^\beta y(t) = g(t, y(t), I_z^\varepsilon x(t)), & t \in [0, T], 1 < \beta \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\upsilon y(\tau). \end{cases} \tag{5.1}$$

Lemma 5.1 (Auxiliary lemma, see [20]) *For $h \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = h(t), & t \in [0, T], 1 < \alpha \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \end{cases} \tag{5.2}$$

is given by

$$x(t) = I_q^\alpha h(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha h(\eta) - \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha h(\xi), \tag{5.3}$$

where

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa + \alpha)} \xi^{\kappa+\alpha-1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma + \alpha)} \eta^{\gamma+\alpha-1} \ne 0.$$

5.1 Existence results for uncoupled case

In view of Lemma 5.1, we define an operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(x, y)(t) = \begin{pmatrix} \mathcal{T}_1(x, y)(t) \\ \mathcal{T}_2(x, y)(t) \end{pmatrix},$$

where

$$\mathcal{T}_1(x, y)(t) = I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\eta)$$

$$- \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\xi)$$

and

$$\mathcal{T}_2(x,y)(t) = I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(t) + \frac{\lambda_2 t^{\beta-1}}{\Psi} I_h^\mu I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\theta)$$

$$- \frac{t^{\beta-1}}{\Psi} I_k^\nu I_p^\beta g\big(s, y(s), I_z^\varepsilon x(s)\big)(\tau),$$

where

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu+\beta)} \tau^{\nu+\beta-1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu+\beta)} \theta^{\mu+\beta-1} \neq 0.$$

In the sequel, we set constants

$$A_{13} = \frac{\Gamma_n(\alpha+1)\xi^{\kappa+\alpha}}{\Gamma_n(\kappa+\alpha+1)\Gamma_q(\alpha+1)},$$

$$A_{14} = \frac{\Gamma_n(\alpha+\delta+1)\Gamma_q(\delta+1)\xi^{\kappa+\alpha+\delta}}{\Gamma_n(\kappa+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)},$$

$$A_{15} = \frac{\Gamma_k(\beta+1)\tau^{\nu+\beta}}{\Gamma_k(\nu+\beta+1)\Gamma_p(\beta+1)},$$

$$A_{16} = \frac{\Gamma_k(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\tau^{\nu+\beta+\varepsilon}}{\Gamma_k(\nu+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},$$

$$H_1 = \overline{M}_1 A_1 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_{13},$$

$$H_2 = \overline{M}_2 A_3 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_{14},$$

$$L_1 = \overline{N}_2 A_4 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{16},$$

$$L_2 = \overline{N}_1 A_2 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_1 A_{15}.$$

Now we present the existence and the uniqueness result for the problem (5.1). We do not provide the proof of this result as it is similar to the one for Theorem 4.1.

Theorem 5.2 *Assume that $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\overline{K}_i, \overline{L}_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2,$*

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le \overline{M}_1|u_1 - v_1| + \overline{M}_2|u_2 - v_2|$$

and

$$\big|g(t, u_1, u_2) - g(t, v_1, v_2)\big| \le \overline{N}_1|u_1 - v_1| + \overline{N}_2|u_2 - v_2|.$$

In addition, assume that

$$H_1 + H_2 + L_1 + L_2 < 1.$$

Then the boundary value problem (5.1) *has a unique solution on* $[0, T]$.

The second result deals with the existence of solutions for the problem (5.1), is analogous to Theorem 4.3 and is given below.

Theorem 5.3 *Assume that there exist real constants* $\bar{m}_i, \bar{n}_i \geq 0$ ($i = 1, 2$), *and* $\bar{m}_0 > 0, \bar{n}_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$ ($i = 1, 2$) *we have*

$$\left| f(t, x_1, x_2) \right| \leq \bar{m}_0 + \bar{m}_1 |x_1| + \bar{m}_2 |x_2|,$$

$$\left| g(t, x_1, x_2) \right| \leq \bar{n}_0 + \bar{n}_1 |x_1| + \bar{n}_2 |x_2|.$$

In addition it is assumed that

$$U_1 + V_1 < 1 \quad and \quad U_2 + V_2 < 1,$$

where $U_i, V_i, i = 1, 2$, *are given by*

$$U_1 = \bar{m}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_{13},$$

$$U_2 = \bar{m}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_{14},$$

$$V_1 = \bar{n}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{16},$$

$$V_2 = \bar{n}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_1 A_{15}.$$

Then the boundary value problem (5.1) *has at least one solution on* $[0, T]$.

Proof By setting

$$G^* = \min \left\{ 1 - (U_1 + V_1), 1 - (U_2 + V_2) \right\},$$

the proof is similar to that of Theorem 4.3. So we omit it. □

5.2 Examples
In this subsection, we present two examples of uncoupled case of nonlocal conditions.

Example 5.4 Consider the following system of fractional q-integro-difference equations with q-integral conditions:

$$\begin{cases} D_{1/9}^{3/2} x(t) = \frac{e^{-t \sin \pi t}}{(t+3)^2} \cdot \frac{|x(t)|}{9 + |x(t)|} + \frac{\cos^2 \pi t}{\pi (t+7)^2} \cdot I_{1/8}^{\sqrt{2}} y(t) - \frac{1}{3}, & 0 < t < 3, \\ D_{1/7}^{5/4} y(t) = \frac{2\pi e^{-2t}}{(7\pi + t)^2} \cdot \frac{|y(t)|}{2 + |y(t)|} + \frac{\sin 2\pi t}{(3e^t + 5)^2} \cdot I_{1/6}^{\sqrt{3}} x(t) + \frac{1}{3}, & \\ x(0) = 0, \quad \frac{1}{2} I_{1/3}^{\frac{\sqrt{2}}{2}} x(\frac{3}{4}) = I_{1/4}^{\sqrt{3}} x(\frac{9}{4}), & \\ y(0) = 0, \quad I_{1/6}^{\sqrt{\pi}} y(3) + \frac{1}{3} I_{1/5}^{\pi} y(\frac{3}{2}) = 0. & \end{cases} \tag{5.4}$$

Here $\alpha = 3/2$, $\delta = \sqrt{2}$, $\beta = 5/4$, $\varepsilon = \sqrt{3}$, $\gamma = \sqrt{2}/2$, $\kappa = \sqrt{3}/2$, $\mu = \pi$, $\nu = \sqrt{\pi}$, $q = 1/9$, $r = 1/8$, $p = 1/7$, $z = 1/6$, $m = 1/3$, $n = 1/4$, $h = 1/5$, $k = 1/6$, $\eta = 3/4$, $\xi = 9/4$, $\theta = 3/2$, $\tau = 3$, $\lambda_1 =$

$1/2$, $\lambda_2 = -1/3$, $T = 3$, $f(t, x, I_r^\delta y) = (e^{-t \sin \pi t}/(t+3)^2)(|x|/(9+|x|)) + (\cos^2 \pi t/\pi (t+7)^2) I_{1/8}^{\sqrt{2}} y - (1/3)$, and $g(t, y, I_r^\delta x) = (2\pi e^{-2t}/(7\pi + t)^2)(|y|/(2+|y|)) + (\sin 2\pi t/(3e^t + 5)^2) I_{1/6}^{\sqrt{3}} x + (1/3)$. Since

$$\left| f(t, u_1, u_2) - f(t, v_1, v_2) \right| \leq \frac{1}{81} |u_1 - v_1| + \frac{1}{49\pi} |u_2 - v_2|$$

and

$$\left| g(t, u_1, u_2) - g(t, v_1, v_2) \right| \leq \frac{1}{36\pi} |u_1 - v_1| + \frac{1}{64} |u_2 - v_2|,$$

then the assumptions of Theorem 5.2 are satisfied with $\overline{M}_1 = 1/81$, $\overline{M}_2 = 1/49\pi$, $\overline{N}_1 = 1/36\pi$, and $\overline{N}_2 = 1/64$. By using the Maple program, we can find that

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa + \alpha)} \xi^{\kappa + \alpha - 1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma + \alpha)} \eta^{\gamma + \alpha - 1} \approx 1.9245172 \neq 0,$$

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu + \beta)} \tau^{\nu + \beta - 1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu + \beta)} \theta^{\mu + \beta - 1} \approx 8.37494759 \neq 0$$

and

$$H_1 = \overline{M}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_{13} \approx 0.11268247,$$

$$H_2 = \overline{M}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_{14} \approx 0.19713212,$$

$$L_1 = \overline{N}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{16} \approx 0.58490031,$$

$$L_2 = \overline{N}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_1 A_{15} \approx 0.06286768.$$

Therefore, we get

$$H_1 + H_2 + L_1 + L_2 \approx 0.95758259 < 1.$$

Hence, by Theorem 5.2, the problem (5.4) has a unique solution on $[0, 3]$.

Example 5.5 Consider the following system of fractional q-integro-difference equations:

$$\begin{cases} D_{\sqrt{2}/2}^{\sqrt{\pi}} x(t) = \frac{25e^{-t}}{(e^{-t}+4)^2} \cdot \frac{|x(t)|}{5+|x(t)|} + \frac{3\pi^2}{(t+3\pi)^2} \cdot I_{\sqrt{3}/2}^{1/2} y(t) + \frac{1}{\sqrt{5}}, & 0 < t < 1, \\ D_{\pi/4}^{\pi/2} y(t) = \frac{9e^{-t \cos^2 \pi t}}{(t+6)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{6}{(t+6)^2} \cdot I_{\pi/5}^{3/2} x(t) + \frac{\sqrt{2}}{3}, \\ x(0) = 0, & \frac{\sqrt{3}}{2} I_{\pi/6}^{4/5} x(1) + I_{\pi/7}^{2/3} x(\frac{3}{4}) = 0, \\ y(0) = 0, & 5 I_{\pi/8}^{\sqrt{3}} y(\frac{1}{2}) = I_{\pi/9}^{1/3} y(\frac{1}{4}). \end{cases} \quad (5.5)$$

Here $\alpha = \sqrt{\pi}$, $\delta = 1/2$, $\beta = \pi/2$, $\varepsilon = 3/2$, $\gamma = 4/5$, $\kappa = 2/3$, $\mu = \sqrt{3}$, $\nu = 1/3$, $q = \sqrt{2}/2$, $r = \sqrt{3}/2$, $p = \pi/4$, $z = \pi/5$, $m = \pi/6$, $n = \pi/7$, $h = \pi/8$, $k = \pi/9$, $\eta = 1$, $\xi = 3/4$, $\theta = 1/2$, $\tau = 1/4$, $\lambda_1 = -\sqrt{3}/2$, $\lambda_2 = 5$, $T = 1$, $f(t, x, I_r^\delta y) = (25e^{-t}/(e^{-t}+4)^2)(|x|/(5+|x|)) + (3\pi^2/(t+3\pi)^2) I_{\sqrt{3}/2}^{1/2} y + (1/\sqrt{5})$, and $g(t, y, I_r^\delta x) = (9e^{-t \cos^2 \pi t}/(t+6)^2)(|y|/(1+|y|)) + (6/(t+6)^2) I_{\pi/5}^{3/2} x +$

($\sqrt{2}/3$). Since

$$\left|f(t,x_1,x_2)\right| \leq \frac{1}{\sqrt{5}} + \frac{1}{5}|x_1| + \frac{1}{3}|x_2|$$

and

$$\left|g(t,x_1,x_2)\right| \leq \frac{\sqrt{2}}{3} + \frac{1}{4}|x_1| + \frac{1}{6}|x_2|,$$

then the assumptions of Theorem 5.3 are satisfied with $\bar{m}_0 = 1/\sqrt{5}$, $\bar{m}_1 = 1/5$, $\bar{m}_2 = 1/3$, $\bar{n}_0 = \sqrt{2}/3$, $\bar{n}_1 = 1/4$, and $\bar{n}_2 = 1/6$. By using the Maple program, we can find that

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa+\alpha)}\xi^{\kappa+\alpha-1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma+\alpha)}\eta^{\gamma+\alpha-1} \approx 1.21235918 \neq 0,$$

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu+\beta)}\tau^{\nu+\beta-1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu+\beta)}\theta^{\mu+\beta-1} \approx -0.32647283 \neq 0$$

and

$$U_1 = \bar{m}_1 A_1 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\bar{m}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|}\bar{m}_1 A_{13} \approx 0.23965603,$$

$$U_2 = \bar{m}_2 A_3 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\bar{m}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|}\bar{m}_2 A_{14} \approx 0.27471434,$$

$$V_1 = \bar{n}_2 A_4 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\bar{n}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|}\bar{n}_2 A_{16} \approx 0.04758258,$$

$$V_2 = \bar{n}_1 A_2 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\bar{n}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|}\bar{n}_1 A_{15} \approx 0.36424461.$$

Therefore, we get

$$U_1 + V_1 \approx 0.28723861 < 1 \quad \text{and} \quad U_2 + V_2 \approx 0.63895895 < 1.$$

Hence, by Theorem 5.3, the problem (5.5) has at least one solution on $[0,1]$.

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References

1. Jackson, FH: *q*-Difference equations. Am. J. Math. **32**, 305-314 (1910)
2. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
3. Al-Salam, WA: Some fractional *q*-integrals and *q*-derivatives. Proc. Edinb. Math. Soc. **15**(2), 135-140 (1966/1967)
4. Agarwal, RP: Certain fractional *q*-integrals and *q*-derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
5. Ernst, T: The history of *q*-calculus and a new method. UUDM Report 2000:16, Department of Mathematics, Uppsala University (2000)
6. Ferreira, RAC: Nontrivial solutions for fractional *q*-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
7. Goodrich, CS: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. **61**, 191-202 (2011)
8. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
9. Srivastava, HM: Some generalizations and basic (or *q*-) extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inf. Sci. **5**, 390-444 (2011)
10. Ma, J, Yang, J: Existence of solutions for multi-point boundary value problem of fractional *q*-difference equation. Electron. J. Qual. Theory Differ. Equ. **2011**, 92 (2011)
11. Graef, JR, Kong, L: Positive solutions for a class of higher order boundary value problems with fractional *q*-derivatives. Appl. Math. Comput. **218**, 9682-9689 (2012)
12. Ahmad, B, Ntouyas, SK, Purnaras, IK: Existence results for nonlocal boundary value problems of nonlinear fractional *q*-difference equations. Adv. Differ. Equ. **2012**, 140 (2012)
13. Ahmad, B, Ntouyas, SK: Existence of solutions for nonlinear fractional *q*-difference inclusions with nonlocal Robin (separated) conditions. Mediterr. J. Math. **10**, 1333-1351 (2013)
14. Ahmad, B, Ntouyas, SK, Alsaedi, A, Al-Hutami, H: Nonlinear *q*-fractional differential equations with nonlocal and sub-strip type boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2014**, 26 (2014)
15. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Existence of solutions for nonlinear fractional *q*-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890-2909 (2014)
16. Alsaedi, A, Ahmad, B, Al-Hutami, H: A study of nonlinear fractional *q*-difference equations with nonlocal integral boundary conditions. Abstr. Appl. Anal. **2013**, Article ID 410505 (2013)
17. Li, X, Han, Z, Sun, S: Existence of positive solutions of nonlinear fractional *q*-difference equation with parameter. Adv. Differ. Equ. **2013**, 260 (2013)
18. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Boundary value problems of nonlinear fractional *q*-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **28**(8), 1719-1736 (2014)
19. Annaby, MH, Mansour, ZS: *q*-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012)
20. Asawasamrit, S, Tariboon, J, Ntouyas, SK: Existence of solutions for fractional *q*-integrodifference equations with nonlocal fractional *q*-integral conditions. Abstr. Appl. Anal. **2014**, Article ID 474138 (2014)
21. Pongarm, N, Asawasamrit, S, Tariboon, J, Ntouyas, SK: Multi-strip fractional *q*-integral boundary value problems for nonlinear fractional *q*-difference equations. Adv. Differ. Equ. **2014**, 193 (2014)
22. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)