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On the Cauchy problem for a linear harmonic oscillator with pure delay

Denys Ya Khusainov[1], Michael Pokojovy[2*] and Elvin I Azizbayov[3]

*Correspondence: michael.pokojovy@uni-konstanz.de
[2]Department of Mathematics and Statistics, University of Konstanz, Konstanz, Germany
Full list of author information is available at the end of the article

Abstract

In the present paper, we consider a Cauchy problem for a linear second order in time abstract differential equation with pure delay. In the absence of delay, this problem, known as the harmonic oscillator, has a two-dimensional eigenspace so that the solution of the homogeneous problem can be written as a linear combination of these two eigenfunctions. As opposed to that, in the presence even of a small delay, the spectrum is infinite and a finite sum representation is not possible. Using a special function referred to as the delay exponential function, we give an explicit solution representation for the Cauchy problem associated with the linear oscillator with pure delay. Finally, the solution asymptotics as the delay parameter goes to zero is studied. In contrast to earlier works, no positivity conditions are imposed.

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1 Introduction

Let X be a (real or complex) Banach space and let $x(t) \in X$ describe the state of a physical system at time $t \geq 0$. With $a(t) = \ddot{x}(t)$ denoting the acceleration of system, Newton's second law of motion states that

$$F(t) = Ma(t) \quad \text{for } t \geq 0, \tag{1}$$

where $M\colon D(M) \subset X \to X$ is a linear, continuously invertible, accretive operator representing the 'mass' of the system. When being displaced from its equilibrium situated in the origin, the system is affected by a restoring force $F(t)$. In classical mechanics, this force is postulated to be proportional to the instantaneous displacement, *i.e.*,

$$F(t) = Kx(t) \quad \text{for } t \geq 0 \tag{2}$$

for some closed, linear operator $K\colon D(K) \subset X \to X$. When $M^{-1}K$ is a bounded linear operator, plugging Equation (2) into (1), we arrive at the classical harmonic oscillator model

$$\ddot{x}(t) = M^{-1}Kx(t) \quad \text{for } t \geq 0. \tag{3}$$

Assuming now that the restoring force is proportional to the value of the system at some past time $t-\tau$, Equation (2) is replaced with the relation

$$F(t) = Kx(t-\tau) \quad \text{for } t \geq 0, \tag{4}$$

where $\tau > 0$ is a time delay. Plugging Equation (4) into (1) leads then to the linear harmonic oscillator equation with pure delay written as

$$\ddot{x}(t) = M^{-1}Kx(t-\tau) \quad \text{for } t \geq 0. \tag{5}$$

Problems similar to Equation (5) also arise when modeling systems with distributed parameters such as general wave phenomena (*cf.* [1]).

Equations similar to (5) are often referred to as delay or retarded differential equations. After being transformed to a first order in time system on a Banach space X, a general equation with constant delay can be written as

$$\dot{u}(t) = H\big(t, u(t), u_t\big) \quad \text{for } t > 0, \qquad u(0) = u^0, \qquad u_0 = \varphi. \tag{6}$$

Here, $\tau > 0$ is a fixed delay parameter, $u_t := u(t+\cdot) \in L^1(-\tau, 0; X)$, $t \geq 0$, denotes the history variable, H is an X-valued operator defined on a subset of $[0,\infty) \times X \times L^1(-\tau, 0; X)$ and $u^0 \in X$, $\varphi \in L^1(-\tau, 0; X)$ are appropriate initial data. Equations of type (6) have been intensively studied in the literature. We refer the reader to the monographs by Els'gol'ts and Norkin [2] and Hale and Lunel [3] for a detailed treatment of Equation (6) in finite-dimensional spaces X. In contrast to this, results on Equation (6) in infinite-dimensional spaces X are less numerous. A good overview can be found in the monograph of Bátkai and Piazzera [4].

Khusainov *et al.* considered in [5] Equation (6) in $\mathbb{R}^n$ with

$$\begin{aligned} H\big(t, u(t), u_t\big) &= A_1 u(t) + A_2 u(t-\tau) + \big(u(t) \otimes b_1\big) u(t) \\ &\quad + \big(u(t) \otimes b_2\big) u(t-\tau) + \big(u(t-\tau) \otimes b_3\big) u(t-\tau) \end{aligned}$$

for symmetric matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and column vectors $b_1, b_2, b_3 \in \mathbb{R}^n$ and proposed a rational Lyapunov function to study the asymptotic stability of solutions to this system.

In their work [6], Khusainov *et al.* studied a modal, or spectrum, control problem for a linear delay equation on $\mathbb{R}^n$ reading as

$$\dot{x}(t) = Ax(t) + bu(t) \quad \text{for } t > 0 \tag{7}$$

with a feedback control $u(t) = \sum_{j=0}^{m} c_j^T x(t-j\tau)$ for some delay time $\tau > 0$ and parameter vectors $c_j \in \mathbb{R}^n$. For canonical systems, they developed a method to compute the unknown parameters such that the closed-loop system possesses the spectrum prescribed beforehand. Under appropriate 'concordance' conditions, they were able to carry over their considerations for a rather broad class of non-canonical systems.

In the infinite-dimensional situation, a rather general particular case of (6) with $H(t, v, \psi) = Av + F(\psi)$, where A generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X and F is a nonlinear operator on $L^2(-\tau, 0; X)$, was studied by Travies and Webb in their work [7]. Under appropriate assumptions on F, they proved the integral equation corresponding to the weak formulation of the delay equation given by

$$u(t) = S(t)\varphi(0) + \int_0^t S(t-s)F(u_s)\,\mathrm{d}s \quad \text{for } t > 0$$

to possess a unique solution in $H^1_{\mathrm{loc}}(0,\infty;X)$.

Di Blasio *et al.* addressed in [8] a similar problem

$$\dot{u}(t) = (A+B)u(t) + L_1u(t-r) + L_2u_t \quad \text{for } t > 0, \qquad u(0) = u^0, \qquad u_0 = \varphi, \tag{8}$$

where A generates a holomorphic C_0-semigroup on a Hilbert space H, B is a perturbation of A and L_1, L_2 are appropriate linear operators. If u^0 and φ possess a certain regularity, they proved the existence of a unique strong solution in $H^1_{\mathrm{loc}}(0,\infty;X) \cap L^2_{\mathrm{loc}}(0,\infty;D(A))$ by analyzing the C_0-semigroup inducing the semiflow $t \mapsto (u(t), u_t)$. These results were elaborated on by Di Blasio *et al.* in [9] leading to a generalization for the case of weighted and interpolation spaces and including a description of the associated infinitesimal generator. Finally, the general L^p-case for $p \in (1,\infty)$ was investigated by Di Blasio in [10].

Diblík *et al.* [11] studied Equation (8) for the case that A and B are 2×2-second order and first order commuting differential operators, respectively, in a bounded interval $(0,l)$ of $\mathbb{R}$ and $L_1 \equiv 0$, $L_2 \equiv 0$. Additionally, they allowed for non-homogeneous Dirichlet boundary conditions. For this parabolic system, they proved the existence of solution in a class of classically differentiable functions both with respect to time and space under appropriate regularity conditions.

Recently, in their work [12], Khusainov *et al.* proposed an explicit L^2-solution theory for a non-homogeneous initial-boundary value problem for an isotropic heat equation with constant delay

$$\begin{aligned}
u_t(t,x) &= \partial_i\big(a_{ij}(x)\partial_j u(t,x)\big) + b_i(x)\partial_i u(t,x) + c(x)u(t,x) \\
&\quad + \partial_i\big(\tilde{a}_{ij}(x)\partial_j u(t-\tau,x)\big) + \tilde{b}_i(x)\partial_i u(t-\tau,x) + \tilde{c}(x)u(t-\tau,x) \\
&\quad + f(t,x) \quad \text{for } (t,x) \in (0,\infty)\times\Omega, \\
u(t,x) &= \gamma(t,x) \quad \text{for } (t,x) \in (0,\infty)\times\partial\Omega, \\
u(0,x) &= u^0(x) \quad \text{for } x \in \Omega, \\
u(t,x) &= \varphi(t,x) \quad \text{for } (t,x) \in (-\tau,0)\times\Omega,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a regular bounded domain and the coefficient functions are appropriate. Conditions assuring for exponential stability were also given.

Over the past decade, hyperbolic partial differential equations have attracted a considerable amount of attention, too. In [13], Nicaise and Pignotti studied a homogeneous isotropic wave equation with an internal feedback with and without delay reading as

$$\begin{aligned}
&\partial_{tt}u(t,x) - \triangle u(t,x) + a_0\partial_t u(t,x) + a\partial_t u(t-\tau,x) = 0 \quad \text{for } (t,x) \in (0,\infty)\times\Omega, \\
&u(t,x) = 0 \quad \text{for } (t,x) \in (0,\infty)\times\Gamma_0, \\
&\frac{\partial u}{\partial \nu}(t,x) = 0 \quad \text{for } (t,x) \in (0,\infty)\times\Gamma_1
\end{aligned}$$

under the usual initial conditions where $\Gamma_0, \Gamma_1 \subset \partial\Omega$ are relatively open in $\partial\Omega$ with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and ν denotes the outer unit normal vector of a smooth bounded domain $\Omega \subset \mathbb{R}^d$. They showed the problem to possess a unique global classical solution and proved the latter to be exponentially stable if $a_0 > a > 0$ or instable, otherwise. These results have been carried over by Nicaise and Pignotti [14] and Nicaise *et al.* [15] to the case of time-varying internally distributed or boundary delays.

In [1], Khusainov *et al.* considered a non-homogeneous initial-boundary value problem for a one-dimensional wave equation with constant coefficients and a single constant delay

$$
\begin{aligned}
\partial_{tt}u(t,x) &= a^2\partial_{xx}u(t-\tau,x) + b\partial_x u(t-\tau,x) + cu(t-\tau,x) \\
&\quad + f(t,x) \quad \text{for } (t,x) \in (0,T)\times(0,l), \\
u(t,x) &= \gamma(t,x) \quad \text{for } (t,x) \in (0,T)\times\{0,1\}, \\
u(0,x) &= u^0(x) \quad \text{for } x \in (0,1), \\
u(t,x) &= \varphi(t,x) \quad \text{for } t \in (-\tau,0), x \in (0,1).
\end{aligned}
$$

Under appropriate regularity and compatibility assumptions, they proved the problem to possess a unique C^2-solution for any finite $T > 0$. Their proof was based on extrapolation methods for C_0-semigroups and an explicit solution representation formula.

Recently, Khusainov and Pokojovy presented in [16] a Hilbert-space treatment of the initial-boundary value problem for the equations of thermoelasticity with pure delay

$$
\begin{aligned}
&\partial_{tt}u(x,t) - a\partial_{xx}u(x,t-\tau) + b\partial_x\theta(x,t-\tau) = f(x,t) \quad \text{for } x \in \Omega, t > 0, \\
&\partial_t\theta(x,t) - c\partial_{xx}\theta(x,t-\tau) + d\partial_{tx}u(x,t-\tau) = g(x,t) \quad \text{for } x \in \Omega, t > 0, \\
&u(0,t) = u(l,t) = 0, \qquad \partial_x\theta(0,t) = \partial_x\theta(l,t) = 0 \quad \text{for } t > 0, \\
&u(x,0) = u^0(x), \qquad u(x,t) = u^0(x,t) \quad \text{for } x \in \Omega, t \in (-\tau,0), \\
&\partial_t u(x,0) = u^1(x), \qquad \partial_t u(x,t) = u^1(x,t) \quad \text{for } x \in \Omega, t \in (-\tau,0), \\
&\theta(x,0) = \theta^0(x), \qquad \theta(x,t) = \theta^0(x,t) \quad \text{for } x \in \Omega, t \in (-\tau,0).
\end{aligned}
$$

Their proof exploited extrapolation techniques for strongly continuous semigroups and an explicit solution representation formula.

In the present paper, we give a Banach space solution theory for Equation (5) subject to appropriate initial conditions. Our approach is solely based on the step method and does not incorporate any semigroup techniques. In contrast to earlier works by Khusainov *et al.* [1, 17, 18], we only require the invertibility and not the negativity of $M^{-1}K$ in Equation (5). In this sense, our framework is different from that employed by Diblík *et al.* in [19, 20], as they required the coefficient matrices to be negative definite. It should though be pointed out that their solution theory accounted for two and more delays, whereas we consider a single delay.

First, we briefly outline some seminal results on second order abstract Cauchy problems. Next, in our main section, we prove the existence and uniqueness of solutions to the Cauchy problem for the delay equation (5) as well as their continuous dependence on the data. Next, we give an explicit solution representation formula in a closed form based on the delayed exponential function introduced by Khusainov and Shuklin in [21]. Finally, we prove the solution of the delay equation to converge to the solution of the original second order abstract differential equation as the delay parameter τ goes to zero.

2 Classical harmonic oscillator

For the sake of completeness, we briefly discuss the initial value problem for the harmonic oscillator being a second order in time abstract differential equation

$$\ddot{x}(t) - \Omega^2 x(t) = f(t) \quad \text{for } t \geq 0 \tag{9}$$

subject to the initial conditions

$$x(0) = x_0 \in D(\Omega), \qquad \dot{x}(0) = x_1 \in X. \tag{10}$$

Here, we assume the linear operator $\Omega\colon D(\Omega) \subset X \to X$ to be continuously invertible and generate a C_0-group $(e^{t\Omega})_{t\in\mathbb{R}} \subset L(X)$ on a (real or complex) Banach space X with $L(X)$ denoting the space of bounded, linear operators on X equipped with the norm $\|A\|_{L(X)} := \sup\{\|Ax\|_X : x \in X, \|x\|_X \leq 1\}$. A more rigorous treatment of this problem can be found in [22], Section 3.14.

The general solution to the homogeneous equation is known to read as

$$x_h(t) = e^{\Omega t}c_1 + e^{-\Omega t}c_2 \quad \text{for } t \geq 0$$

with some $c_1, c_2 \in D(\Omega)$. Vectors c_1, c_2 can be computed using the initial conditions from Equation (10) leading to a system of linear operator equations

$$c_1 + c_2 = x_0, \qquad \Omega c_1 - \Omega c_2 = x_1.$$

The latter is uniquely solved by

$$c_1 = \frac{1}{2}\Omega^{-1}(\Omega x_0 + x_1), \qquad c_1 = \frac{1}{2}\Omega^{-1}(\Omega x_0 - x_1).$$

Thus, the unique solution of the homogeneous equation with the initial conditions (10) is given by

$$x_h(t) = \frac{1}{2}\Omega^{-1}e^{\Omega t}(\Omega x_0 + x_1) + \frac{1}{2}\Omega^{-1}e^{-\Omega t}(\Omega x_0 - x_1) \quad \text{for } t \geq 0 \tag{11}$$

or, equivalently,

$$x_h(t) = \frac{1}{2}\left(e^{\Omega t} + e^{-\Omega t}\right)x_0 + \frac{1}{2}\Omega^{-1}\left(e^{\Omega t} - e^{-\Omega t}\right)x_1 \quad \text{for } t \geq 0. \tag{12}$$

A particular solution to the non-homogeneous equation with zero initial conditions will be determined in the Cauchy form

$$x_p(t) = \int_0^t K(t,s)f(s)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{13}$$

We refer the reader to Chapter 1 in [22] for the definition of Bochner integrals for X-valued functions. In Equation (13), the function $K \in C^0([0,\infty) \times [0,\infty), L(X))$ is the Cauchy kernel, *i.e.*, for any fixed $s \geq 0$, the function $K(\cdot, s)$ is the solution of the homogeneous problem satisfying the initial conditions

$$K(t,s)|_{t=s} = 0_{L(X)}, \qquad \partial_t K(t,s)|_{t=s} = \mathrm{id}_X.$$

Using the ansatz

$$K(t,s) = e^{\Omega t} c_1(s) + e^{-\Omega t} c_2(s) \quad \text{for } t, s \geq 0$$

for some $c_1, c_2 \in C^1([0,\infty), L(X))$ and taking into account the initial conditions, we arrive at

$$K(t,s)|_{t=s} = e^{\Omega s} c_1(s) + e^{-\Omega s} c_2(s) = 0_{L(X)},$$
$$\partial_t K(t,s)|_{t=s} = \Omega e^{\Omega s} c_1(s) - \Omega e^{-\Omega s} c_2(s) = \mathrm{id}_X.$$

Solving this system with generalized Cramer's rule, we obtain, for $s \geq 0$,

$$c_1(s) = \left(\det_{L(X)} \begin{pmatrix} e^{\Omega s} & e^{-\Omega s} \\ \Omega e^{\Omega s} & -\Omega e^{-\Omega s} \end{pmatrix}\right)^{-1} \det_{L(X)} \begin{pmatrix} 0_{L(X)} & e^{-\Omega s} \\ \mathrm{id}_X & -\Omega e^{-\Omega s} \end{pmatrix} = \frac{1}{2} \Omega^{-1} e^{-\Omega s},$$
$$c_2(s) = \left(\det_{L(X)} \begin{pmatrix} e^{\Omega s} & e^{-\Omega s} \\ \Omega e^{\Omega s} & -\Omega e^{-\Omega s} \end{pmatrix}\right)^{-1} \det_{L(X)} \begin{pmatrix} e^{\Omega s} & 0_{L(X)} \\ \Omega e^{\Omega s} & \mathrm{id}_X \end{pmatrix} = \frac{1}{2} \Omega^{-1} e^{\Omega s}.$$

Thus, the Cauchy kernel is given by

$$K(t,s) = \frac{1}{2} \Omega^{-1} \left(e^{\Omega(t-s)} - e^{-\Omega(t-s)}\right) \quad \text{for } t, s \geq 0,$$

whereas the particular solution satisfying zero initial conditions reads as

$$x_p(t) = \frac{1}{2} \Omega^{-1} \int_0^t \left(e^{\Omega(t-s)} - e^{-\Omega(t-s)}\right) f(s) \, \mathrm{d}s \quad \text{for } t \geq 0.$$

Hence, for $x_0 \in D(\Omega)$, $x_1 \in X$ and $f \in L^1_{\mathrm{loc}}(0,\infty;X)$, the unique mild solution $x \in W^{1,1}_{\mathrm{loc}}(0,\infty;X)$ to the Cauchy problem (9)-(10) can be written as

$$x(t) = \frac{1}{2}\left(e^{\Omega t} + e^{-\Omega t}\right) x_0 + \frac{1}{2} \Omega^{-1} \left(e^{\Omega t} - e^{-\Omega t}\right) x_1 + \frac{1}{2} \Omega^{-1} \int_0^t \left(e^{\Omega(t-s)} - e^{-\Omega(t-s)}\right) f(s) \, \mathrm{d}s \quad \text{for } t \geq 0. \tag{14}$$

If the data additionally satisfy $x_0 \in D(\Omega^2)$, $x_1 \in D(\Omega)$ and $f \in W^{1,1}_{\mathrm{loc}}(0,\infty;X) \cup C^0([0,\infty), D(\Omega^2))$, then the mild solution x given in Equation (14) is a classical solution satisfying $x \in C^2([0,\infty), X) \cap C^1([0,\infty), D(\Omega)) \cap C^0([0,\infty), D(\Omega^2))$.

3 The linear oscillator with pure delay

In this section, we consider a Cauchy problem for the linear oscillator with a single pure delay

$$\ddot{x}(t) - \Omega^2 x(t - 2\tau) = f(t) \quad \text{for } t \geq 0 \tag{15}$$

subject to the initial condition

$$x(t) = \varphi(t) \quad \text{for } t \in [-2\tau, 0]. \tag{16}$$

Here, X is a Banach space, $\Omega \in L(X)$ is a bounded, linear operator and $\varphi \in C^1([-2\tau, 0], X)$, $f \in L^1_{\text{loc}}(0, \infty; X)$ are given functions. In contrast to the previous section, the boundedness of Ω is indispensable here. Indeed, Dreher *et al.* proved in [23] that Equations (15)-(16) are ill-posed even if X is a Hilbert space and Ω possesses a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\lambda_n \to \infty$ or $\lambda_n \to -\infty$ as $n \to \infty$. The necessity for Ω being bounded has also been pointed out by Rodrigues *et al.* in [24] when treating a linear heat equation with pure delay.

Definition 1 A function $x \in C^1([-2\tau, \infty), X) \cap C^2([-2\tau, 0], X) \cap C^2([0, \infty), X)$ satisfying Equations (15)-(16) pointwise is called a classical solution to the Cauchy problem (15)-(16).

A mild formulation of (15)-(16) is given by

$$\dot{x}(t) = \dot{x}(0) + \Omega^2 \int_0^t x(s - 2\tau)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \quad \text{for } t \geq 0, \tag{17}$$

$$x(t) = \varphi(t) \quad \text{for } t \in [-2\tau, 0]. \tag{18}$$

Definition 2 A function $x \in C^1([-2\tau, \infty), X)$ satisfying Equations (17)-(18) is called a mild solution to the Cauchy problem (15)-(16).

By the virtue of fundamental theorem of calculus, any mild solution x to (15)-(16) with $x \in C^1([-2\tau, \infty), X) \cap C^2([-2\tau, 0], X) \cap C^2([0, \infty), X)$ is also a classical solution. Obviously, for the problem (15)-(16) to possess a classical solution, one necessarily requires $\varphi \in C^2([-2\tau, 0], X)$.

In the following subsection, we want to study the existence and uniqueness of mild and classical solutions to the Cauchy problem (15)-(16) as well as their continuous dependence on the data.

3.1 Existence and uniqueness

Rather than using the semigroup approach (*cf.* [3], Chapter 2), we decided to use the more straightforward step method here reducing (17)-(18) to a difference equation on the functional vector space $\hat{C}^1_{2\tau}(\mathbb{N}_0, X)$ defined as follows.

Definition 3 Let X be a Banach space, $\tau > 0$ and $s \in \mathbb{N}_0$. We introduce the metric vector space

$$\begin{aligned}
\hat{C}^s_\tau(\mathbb{N}_0, X) &:= l^\infty_{\text{loc}}\big(\mathbb{N}_0, C^s\big([-\tau, 0], X\big)\big) \\
&:= \bigg\{ x = (x_n)_{n\in\mathbb{N}_0} \,\bigg|\, x_n \in C^s\big([-\tau, 0], X\big) \text{ for } n \in \mathbb{N}_0, \\
&\quad \frac{\mathrm{d}^j}{\mathrm{d}t^j} x_n(-\tau) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} x_{n-1}(0) \text{ for } j = 0, \ldots, s-1, n \in \mathbb{N} \bigg\}
\end{aligned}$$

equipped with the distance function

$$d_{\hat{C}^s_\tau(\mathbb{N}_0,X)}(x,y) := \sum_{n\in N} 2^{-n} \frac{\max_{k=0,\dots,n} \|x_k - y_k\|_{C^s([-\tau,0],X)}}{1 + \max_{k=0,\dots,n} \|x_k - y_k\|_{C^s([-\tau,0],X)}} \quad \text{for } x,y \in \hat{C}^s_\tau(\mathbb{N}_0,X).$$

Obviously, $\hat{C}^s_\tau(\mathbb{N}_0,X)$ is a complete metric space which is isometrically isomorphic to the metric space $C^s_\tau([-\tau,\infty),X) := C^s([-\tau,\infty),X)$ equipped with the distance

$$d_{C^s_\tau([0,\infty),X)}(x,y) := \sum_{n\in N} 2^{-n} \frac{\|x-y\|_{C^s([-\tau,\tau n],X)}}{1 + \|x-y\|_{C^s([-\tau,\tau n],X)}} \quad \text{for } x,y \in C^s\big([-\tau,\infty),X\big).$$

For any $x\colon [-\tau,\infty) \to X$, we define for $n \in \mathbb{N}_0$ the nth segment of x via

$$x_n := x(n\tau + s) \quad \text{for } s \in [-\tau, 0].$$

By induction, x is a mild solution of (15)-(16) if and only if $(x_n)_{n\in\mathbb{N}_0} \in \hat{C}^1_{2\tau}(\mathbb{N}_0,X)$ solves

$$\begin{aligned} \dot{x}_n(s) &= \dot{x}_{n-1}(0) + \Omega^2 \int_{-2\tau}^{s} x_{n-1}(\sigma)\,\mathrm{d}\sigma \\ &\quad + \int_{2(n-1)\tau}^{2n\tau+s} f(\sigma)\,\mathrm{d}\sigma \quad \text{for } s \in [-2\tau,0] \text{ and } n \in \mathbb{N}, \qquad (19) \\ x_0(s) &= \varphi(s) \quad \text{for } s \in [-2\tau, 0]. \end{aligned}$$

Theorem 4 *Equation* (19) *has a unique solution* $(x_n)_{n\in\mathbb{N}_0} \in \hat{C}^1_{2\tau}(\mathbb{N}_0,X)$. *Moreover, x continuously depends on the data in sense of the estimate*

$$\|x_n\|_{C^1([-2\tau,0],X)} \le \kappa^n \big(\|\varphi\|_{C^1([-2\tau,0],X)} + \|f\|_{L^1(0,2\tau n;X)}\big) \quad \textit{for any } n \in \mathbb{N}$$

with $\kappa := 1 + 2\tau(1+2\tau)(1 + \|\Omega\|^2_{L(X)})$.

Proof By the virtue of fundamental theorem of calculus, Equation (19) is satisfied if and only if

$$\begin{aligned} x_n(s) &= x_{n-1}(0) + (s+2\tau)\dot{x}_{n-1}(0) + \Omega^2 \int_{-2\tau}^{s} \int_{-2\tau}^{\sigma} x_{n-1}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ &\quad + \int_{-2\tau}^{s} \int_{2(n-1)\tau}^{2n\tau+\sigma} f(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \quad \text{for } s \in [-2\tau,0], n \in \mathbb{N}, \qquad (20) \\ x_n(-2\tau) &= x_{n-1}(0), \qquad \dot{x}_n(-2\tau) = \dot{x}_{n-1}(0) \quad \text{for } n \in \mathbb{N}, \qquad (21) \\ x_0(s) &= \varphi(s) \quad \text{for } s \in [-2\tau, 0]. \qquad (22) \end{aligned}$$

By induction, we can easily show that for any $n \in \mathbb{N}$ there exists a unique local solution $(x_0, x_1, \dots, x_n) \in (C^1([-2\tau,0],X))^{n+1}$ to (20)-(22) up to the index n. Here, we used the Sobolev embedding theorem stating

$$W^{1,1}(0,T;X) \hookrightarrow C^0\big([0,T],X\big) \quad \text{for any } T > 0.$$

Further, we can estimate

$$\|x_n\|_{C^0([-2\tau,0],X)} \leq \big(1 + 2\tau + 4\tau^2\|\Omega\|^2_{L(X)}\big)\|x_{n-1}\|_{C^1([-2\tau,0],X)} + 2\tau\|f\|_{L^1(2(n-1)\tau,2n\tau;X)}. \quad (23)$$

Similarly, Equation (19) yields

$$\|\dot{x}_n\|_{C^0([-2\tau,0],X)} \leq \big(1 + 2\tau\|\Omega\|^2_{L(X)}\big)\|x_{n-1}\|_{C^0([-2\tau,0],X)} + \|f\|_{L^1(2(n-1)\tau,2n\tau;X)}. \quad (24)$$

Equations (23) and (24) imply together

$$\|x_n\|_{C^1([-2\tau,0],X)} \leq \kappa\big(\|\varphi\|_{C^1([-2\tau,0],X)} + \|f\|_{L^1(2(n-1)\tau,n\tau:X)}\big).$$

By induction, we then get, for any $n \in \mathbb{N}$,

$$\|x_n\|_{C^1([-2\tau,0],X)} \leq \kappa^n\big(\|\varphi\|_{C^0([-2\tau,0],X)} + \|f\|_{L^1(0,2\tau n,X)}\big),$$

which finishes the proof. □

Letting $x(t) := x_k(t - 2k\tau)$ for $t \geq 0$ and $k := \lfloor \frac{t}{2\tau} \rfloor \in \mathbb{N}_0$, we obtain the unique mild solution x of Equations (15)-(16).

Corollary 5 *Equations* (15)-(16) *possess a unique mild solution x satisfying, for any* $T := 2n\tau$, $n \in \mathbb{N}$,

$$\|x\|_{C^1([-2\tau,T],X)} \leq \kappa^n\big(\|\varphi\|_{C^1([-2\tau,T],X)} + \|f\|_{L^1(0,T;X)}\big) \quad \textit{for any } n \in \mathbb{N}$$

with $\kappa := 1 + (1 + 2\tau)(1 + \|\Omega\|^2_{L(X)})$.

Theorem 6 *Under an additional condition that* $\varphi \in C^2([-2\tau,0],X)$ *as well as* $f \in C^0([0,\infty),X)$, *the unique mild solution given in Corollary* 5 *is a classical solution.*

Proof Differentiating Equation (19) with respect to t, using the assumptions and the fact that $x \in C^1([-2\tau,\infty),X)$, we deduce that $x|_{[-2\tau,0]} \equiv \varphi \in C^2([-2\tau,0],X)$ and

$$\ddot{x} = \Omega^2 x(\cdot - 2\tau) + f \in C^0\big([0,\infty),X\big).$$

Hence, $x \in C^1([-2\tau,\infty),X) \cap C^2([-2\tau,0],X) \cap C^2([0,\infty),X)$ and is thus a classical solution of Equations (15)-(16). □

3.2 Explicit representation of solutions

Following Khusainov and Shuklin [21] and Khusainov *et al.* [12], we define for $t \in \mathbb{R}$ the operator-valued delayed exponential function

$$\exp_\tau(t;\Omega) := \begin{cases} 0_{L(X)}, & -\infty < t < -\tau, \\ \mathrm{id}_X, & -\tau \leq t < 0, \\ \mathrm{id}_X + \Omega\frac{t}{1!}, & 0 \leq t < \tau, \\ \mathrm{id}_X + \Omega\frac{t}{1!} + \Omega^2\frac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \ldots, & \ldots, \\ \mathrm{id}_X + \Omega\frac{t}{1!} + \Omega^2\frac{(t-\tau)^2}{2!} + \cdots + \Omega^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leq t < k\tau, \\ \ldots, & \ldots. \end{cases} \quad (25)$$

Throughout this section, we additionally assume that $\Omega: X \to X$ is an isomorphism from the Banach space X onto itself.

Theorem 7 *The delayed exponential function* $\exp_\tau(\cdot;\Omega)$ *lies in* $C^0([-\tau,\infty),X) \cap C^1([0,\infty),X) \cap C^2([\tau,\infty),X)$ *and solves the Cauchy problem*

$$\ddot{x}(t) - \Omega^2 x(t-2\tau) = 0_X \quad \textit{for } t \geq \tau, \tag{26}$$

$$x(t) = \varphi(t) \quad \textit{for } t \in [-\tau,\tau], \tag{27}$$

where

$$\varphi(t) = \begin{cases} \mathrm{id}_X, & -\tau \leq t < 0, \\ \mathrm{id}_X + \Omega t, & 0 \leq t \leq \tau. \end{cases}$$

Proof To prove the smoothness of x, we first note that x is an operator-valued polynomial and thus analytic on each of the intervals $[(k-1)\tau, k\tau]$ for $k \in \mathbb{Z}$. By the definition of $\exp_\tau(\cdot;\Omega)$, we further find

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} x(k\tau - 0) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} x(k\tau + 0) \quad \text{for } j = 0,\ldots,k, k \in \mathbb{N}_0.$$

Hence, $x \in C^0([-\tau,\infty),X) \cap C^1([0,\infty),X) \cap C^2([\tau,\infty),X)$.

For $k \in \mathbb{N}$, $k \geq 2$, we have

$$\begin{aligned} x(t) &= \mathrm{id}_X + \Omega \frac{t}{1!} + \Omega^2 \frac{(t-\tau)^2}{2!} + \Omega^3 \frac{(t-3\tau)^3}{4!} \\ &\quad + \Omega^4 \frac{(t-3\tau)^4}{4!} + \cdots + \Omega^k \frac{(t-(k-1)\tau)^k}{k!}. \end{aligned}$$

For $t \geq \tau$, differentiation yields

$$\begin{aligned} \dot{x}(t) &= \Omega + \Omega^2 \frac{t-\tau}{2!} + \Omega^3 \frac{(t-2\tau)^2}{4!} + \Omega^4 \frac{(t-3\tau)^3}{3!} + \cdots + \Omega^k \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!} \\ &= \Omega \left(\mathrm{id}_X + \Omega \frac{t-\tau}{2!} + \Omega^2 \frac{(t-2\tau)^2}{4!} + \Omega^3 \frac{(t-3\tau)^3}{3!} + \cdots + \Omega^{k-1} \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!} \right) \\ &= \Omega \exp_\tau(t-\tau;\Omega) = \Omega x(t-\tau) \end{aligned}$$

and, therefore,

$$\begin{aligned} \ddot{x}(t) &= \Omega^2 + \Omega^3 \frac{t-2\tau}{1!} + \Omega^4 \frac{(t-3\tau)^2}{2!} + \cdots + \Omega^k \frac{(t-(k-1)\tau)^{k-2}}{(k-2)!} \\ &= \Omega^2 \left(\mathrm{id}_X + \Omega \frac{t-2\tau}{1!} + \Omega^2 \frac{(t-3\tau)^2}{2!} + \cdots + \Omega^{k-2} \frac{(t-(k-1)\tau)^{k-2}}{(k-2)!} \right) \\ &= \Omega^2 \exp_\tau(t-2\tau;\Omega) = \Omega^2 x(t-2\tau). \end{aligned}$$

Hence, x satisfies Equation (26). Finally, by definition of $\exp_\tau(\cdot;\Omega)$, x satisfies Equation (27), too. □

Corollary 8 *The delayed exponential function* $\exp_\tau(\cdot;-\Omega)$ *lies in* $C^0([-\tau,\infty),X)\cap C^1([0,\infty),X)\cap C^2([\tau,\infty),X)$ *and solves the Cauchy problem* (26)-(27) *with the initial data*

$$\varphi(t)=\begin{cases}\mathrm{id}_X, & -\tau\le t<0,\\ \mathrm{id}_X-\Omega t, & 0\le t\le\tau.\end{cases}$$

We define the functions

$$\begin{aligned}x_\tau^1(t;\Omega)&:=\frac{1}{2}\bigl(\exp_\tau(t;\Omega)+\exp_\tau(t;-\Omega)\bigr)\quad\text{for } t\ge-\tau,\\ x_\tau^2(t;\Omega)&:=\frac{1}{2}\Omega^{-1}\bigl(\exp_\tau(t;\Omega)-\exp_\tau(t;-\Omega)\bigr)\quad\text{for } t\ge-\tau.\end{aligned}\tag{28}$$

As we already pointed out in the introduction section, in contrast to earlier works by Khusainov *et al.* [1, 17, 18], only the invertibility and not the negativity of Ω is necessary for our purposes.

From Equation (25), we explicitly obtain

$$x_\tau^1(t;\Omega)=\begin{cases}\mathrm{id}_X, & -\infty<t<\tau,\\ \mathrm{id}_X+\Omega^2\frac{(t-\tau)^2}{2!}, & \tau\le t<3\tau,\\ \mathrm{id}_X+\Omega^2\frac{(t-\tau)^2}{2!}+\Omega^4\frac{(t-3\tau)^4}{4!}, & 3\tau\le t<5\tau,\\ \ldots, & \ldots,\\ \mathrm{id}_X+\Omega^2\frac{(t-\tau)^2}{2!}+\cdots+\Omega^{2k}\frac{(t-(2k-1)\tau)^{2k}}{(2k)!}, & (2k-1)\tau\le t<(2k+1)\tau\end{cases}$$

and

$$x_\tau^2(t;\Omega)=\begin{cases}0_{L(X)}, & -\infty<t<0,\\ \mathrm{id}_X\frac{t}{1!}, & 0\le t<2\tau,\\ \mathrm{id}_X\frac{t}{1!}+\Omega^2\frac{(t-2\tau)^3}{3!}, & 2\tau\le t<4\tau,\\ \mathrm{id}_X\frac{t}{1!}+\Omega^2\frac{(t-2\tau)^3}{3!}+\Omega^4\frac{(t-4\tau)^5}{5!}, & 4\tau\le t<6\tau,\\ \ldots, & \ldots,\\ \mathrm{id}_X\frac{t}{1!}+\Omega^2\frac{(t-2\tau)^3}{3!}+\cdots+\Omega^{2k}\frac{(t-(2k)\tau)^{2k+1}}{(2k+1)!}, & 2k\tau\le t<2(k+1)\tau.\end{cases}$$

Obviously, x_τ^1 and x_τ^2 are even functions with respect to Ω. Figures 1 and 2 display the functions $x_\tau^1(\cdot;\Omega)$ and $x_\tau^2(\cdot;\Omega)$ for various values of τ and Ω.

Theorem 9 *The functions* $x_\tau^1(\cdot;\Omega)$, $x_\tau^2(\cdot;\Omega)$ *have the following regularity properties*: $x_\tau^1(\cdot;\Omega), x_\tau^2(\cdot;\Omega)\in C^1([-\tau,\infty),X)\cap C^2([-\tau,0],X)\cap C^2([\tau,\infty),X)$. *Further,* $x_\tau^1(\cdot;\Omega)$ *and* $x_\tau^2(\cdot;\Omega)$ *are solutions to the Cauchy problem* (26)-(27) *with the initial data* $\varphi(t)=\mathrm{id}_X$, $-\tau\le t\le\tau$, *and* $\varphi(t)=0_{L(X)}$, $-\tau\le t\le\tau$, *respectively.*

First, assuming $f\equiv 0_X$, Equations (15)-(16) reduce to

$$\ddot{x}(t)-\Omega^2x(t-2\tau)=0\quad\text{for } t\ge 0,\tag{29}$$

$$x(t)=\varphi(t)\quad\text{for } t\in[-2\tau,0].\tag{30}$$

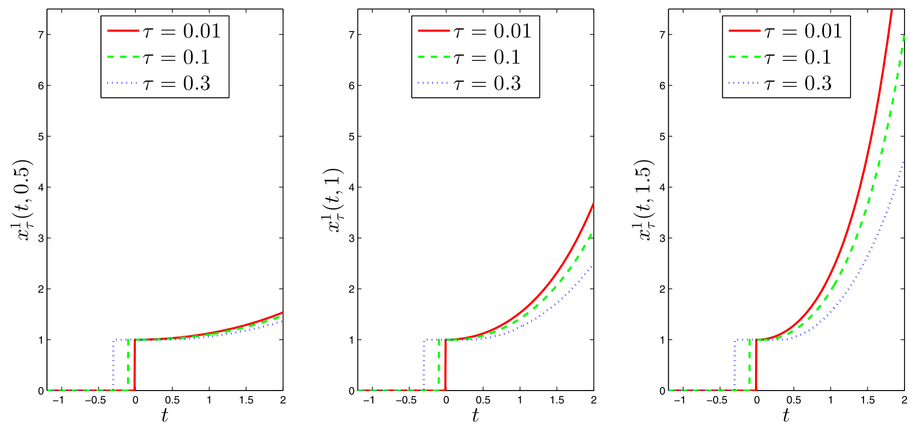

Figure 1 Plot of $x_\tau^1(\cdot;\Omega)$ function.

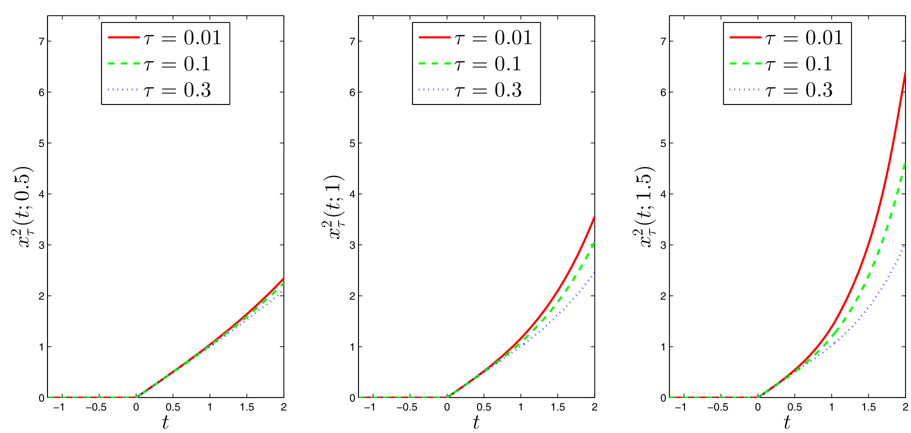

Figure 2 Plot of $x_\tau^2(\cdot;\Omega)$ function.

Theorem 10 *Let $\varphi \in C^2([-2\tau, 0], X)$. Then the unique classical solution x to the Cauchy problem* (29)-(30) *is given by*

$$x(t) = x_\tau^1(t+\tau;\Omega)\varphi(-2\tau) + x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) + \int_{-2\tau}^{0} x_\tau^2(t-s;\Omega)\ddot{\varphi}(s)\,\mathrm{d}s.$$

Proof To solve Equations (29)-(30), we use the ansatz

$$x(t) = x_\tau^1(t+\tau;\Omega)c_1 + x_\tau^2(t+2\tau;\Omega)c_2 + \int_{-2\tau}^{0} x_\tau^2(t-s;\Omega)\ddot{c}(s)\,\mathrm{d}s \tag{31}$$

for some $c_1, c_2 \in X$ and $c \in C^2([-2\tau, 0], X)$.

Plugging the ansatz from Equation (31) into Equation (29), we obtain, for $t \geq 0$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(x_\tau^1(t+\tau;\Omega)c_1 + x_\tau^2(t+2\tau;\Omega)c_2 + \int_{-2\tau}^0 x_\tau^2(t-s;\Omega)\ddot{c}(s)\,\mathrm{d}s\right)$$
$$- \Omega^2\bigg(x_\tau^1\big((t+\tau)-2\tau;\Omega\big)c_1 + x_\tau^2\big((t+2\tau)-2\tau;\Omega\big)c_2$$
$$+ \int_{-2\tau}^0 x_\tau^2\big((t-2\tau)-s;\Omega\big)\ddot{c}(s)\,\mathrm{d}s\bigg) = 0$$

or, equivalently,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_\tau^1(t+\tau;\Omega) - \Omega^2 x_\tau^1\big((t+\tau)-2\tau;\Omega\big)\right)c_1$$
$$+ \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_\tau^2(t+2\tau;\Omega) - \Omega^2 x_\tau^2\big((t+2\tau)-2\tau;\Omega\big)\right)c_2$$
$$+ \int_{-2\tau}^0 \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_\tau^2(t-s;\Omega) - \Omega^2 x_\tau^2\big((t-2\tau)-s;\Omega\big)\right)\ddot{c}(s)\,\mathrm{d}s \equiv 0_X.$$

Since $x_\tau^1(\cdot;\Omega)$ and $x_\tau^2(\cdot;\Omega)$ solve the homogeneous equation, all three coefficients at c_1, c_2 and $\ddot{c}$ vanish implying that the function x in Equation (31) is a solution of Equation (29).

Now, we show that selecting $c_1 := \varphi(-2\tau)$, $c_2 := \dot{\varphi}(-2\tau)$ and $c := \varphi$, the function x in Equation (31) satisfies the initial condition (30). Letting, for $t \in [-2\tau, 0]$,

$$[I\varphi](t) := \int_{-2\tau}^0 x_\tau^2(t-s;\Omega)\ddot{\varphi}(s)\,\mathrm{d}s$$

and performing a change of variables $\sigma := t - s$, we find

$$[I\varphi](t) = -\int_{t+2\tau}^t x_\tau^2(\sigma;\Omega)\ddot{\varphi}(t-\sigma)\,\mathrm{d}\sigma = \int_t^{t+2\tau} x_\tau^2(\sigma;\Omega)\ddot{\varphi}(t-\sigma)\,\mathrm{d}\sigma.$$

Exploiting the fact that x_τ^2 vanishes on $[-2\tau, 0]$, we get

$$[I\varphi](t) = \int_0^{t+2\tau} x_\tau^2(\sigma;\Omega)\ddot{\varphi}(t-\sigma)\,\mathrm{d}\sigma.$$

Integrating by parts, we further get

$$\begin{aligned}
[I\varphi](t) &= \int_0^{t+2\tau} x_\tau^2(\sigma;\Omega)\ddot{\varphi}(t-\sigma)\,\mathrm{d}\sigma \\
&= -\int_0^{t+2\tau} x_\tau^2(\sigma;\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\varphi}(t-\sigma)\big)\,\mathrm{d}\sigma \\
&= -x_\tau^2(\sigma;\Omega)\dot{\varphi}(t-\sigma)\big|_{\sigma=0}^{\sigma=t+2\tau} + \int_0^{t+2\tau} \dot{x}_\tau^2(\sigma;\Omega)\dot{\varphi}(t-\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

Now, taking into account

$$x_\tau^2(t;\Omega) = t\,\mathrm{id}_X, \quad 0 \leq t \leq 2\tau, \tag{32}$$

we obtain

$$[I\varphi](t) = -x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) + \int_0^{t+2\tau} \dot{x}_\tau^2(\sigma;\Omega)\dot{\varphi}(t-\sigma)\,\mathrm{d}\sigma.$$

Again, using Equation (32) and

$$x_\tau^1(t;\Omega) = \mathrm{id}_X, \quad -\tau \le t \le \tau,$$

we compute

$$\begin{aligned}
[I\varphi](t) &= -x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) + \int_0^{t+2\tau} \dot{\varphi}(t-\sigma)\,\mathrm{d}\sigma \\
&= -x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) - \varphi(t-\sigma)|_{\sigma=0}^{\sigma=t+2\tau} \\
&= -x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) - x_\tau^1(t+\tau;\Omega)\varphi(-2\tau) + \varphi(t).
\end{aligned}$$

Hence, for $t \in [-2\tau, 0]$, we have

$$x(t) = x_\tau^1(t+\tau;\Omega)\varphi(-2\tau) + x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) + \int_{-2\tau}^0 x_\tau^2(t-s;\Omega)\ddot{\varphi}(s)\,\mathrm{d}s = \varphi(t)$$

as claimed. □

Next, we consider Equations (15)-(16) for the trivial initial data, *i.e.*,

$$\ddot{x}(t) - \Omega^2 x(t-2\tau) = f(t) \quad \text{for } t \ge 0, \tag{33}$$

$$x(t) = 0 \quad \text{for } t \in [-2\tau, 0]. \tag{34}$$

Theorem 11 *Let $f \in C^0([0,\infty), X)$. The unique classical solution x to the Cauchy problem* (33)-(34) *is given by*

$$x(t) = \int_0^t x_\tau^2(t-s;\Omega)f(s)\,\mathrm{d}s.$$

Proof To find an explicit solution representation, we use the ansatz

$$x(t) = \int_0^t x_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s \quad \text{for } t \ge \tau$$

for some function $c \in C^0([0,\infty), X)$. Differentiating this expression with respect to t and exploiting the initial conditions for $x_\tau^2(\cdot;\Omega)$, we get

$$\begin{aligned}
\dot{x}(t) &= \int_0^t \dot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s + x_\tau^2(t-s;\Omega)c(s)|_{s=t} \\
&= \int_0^t \dot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s + x_\tau^2(0)c(t) \\
&= \int_0^t \dot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s.
\end{aligned}$$

Differentiating again, we find

$$\begin{aligned}\ddot{x}(t) &= \int_0^t \ddot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s + \dot{x}_\tau^2(t-s;\Omega)c(s)|_{s=t} \\ &= \int_0^t \ddot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s + \dot{x}_\tau^2(0+;\Omega)c(t) \\ &= \int_0^t \ddot{x}_\tau^2(t-s;\Omega)c(s)\,\mathrm{d}s + c(t).\end{aligned}$$

Plugging this into Equation (33) and recalling that $x_\tau^2(\cdot;\Omega)$ is a solution of the homogeneous equation, we get

$$c(t) + \int_0^t \bigl(\ddot{x}_\tau^2(t-s;\Omega) - \Omega^2 x_\tau^2(t-2\tau-s;\Omega)\bigr)c(s)\,\mathrm{d}s = f(t)$$

and therefore $c \equiv f$. □

As a consequence from Theorems 10 and 11, we obtain using the linearity property of Equations (15)-(16) the following.

Theorem 12 *Let* $\varphi \in C^2([-2\tau,0],X)$ *and* $f \in C^0([0,\infty),X)$. *The unique classical solution to Equations* (15)-(16) *is given by*

$$\begin{aligned}x(t) &= x_\tau^1(t+\tau;\Omega)\varphi(-2\tau) + x_\tau^2(t+2\tau;\Omega)\dot{\varphi}(-2\tau) + \int_{-2\tau}^0 x_\tau^2(t-s;\Omega)\ddot{\varphi}(s)\,\mathrm{d}s \\ &\quad + \begin{cases} 0, & t \in [-2\tau,0), \\ \int_0^t x_\tau^2(t-s;\Omega)f(s)\,\mathrm{d}s, & t \geq 0\end{cases}\end{aligned}$$

for $t \in [-2\tau,\infty)$.

Finally, after a partial integration, we get the following.

Theorem 13 *Let* $\varphi \in C^1([-2\tau,0],X)$ *and* $f \in L^1_{\mathrm{loc}}(0,\infty;X)$. *The unique mild solution to Equations* (15)-(16) *is given by*

$$\begin{aligned}x(t) &= x_\tau^1(t+\tau;\Omega)\varphi(-2\tau) + x_\tau^2(t;\Omega)\dot{\varphi}(0) - \int_{-2\tau}^0 \dot{x}_\tau^2(t-s;\Omega)\dot{\varphi}(s)\,\mathrm{d}s \\ &\quad + \begin{cases} 0, & t \in [-2\tau,0), \\ \int_0^t x_\tau^2(t-s;\Omega)f(s)\,\mathrm{d}s, & t \geq 0\end{cases}\end{aligned}$$

for $t \in [-2\tau,\infty)$.

Proof Approximating φ in $C^1([-2\tau,0],X)$ with $(\varphi_n)_{n\in\mathbb{N}} \subset C^2([-2\tau,0],X)$ and f in $L^1_{\mathrm{loc}}(0,\infty;X)$ with $(f_n)_{n\in\mathbb{N}} \subset C^0([0,\infty),X)$, applying Theorem 12 to solve the Cauchy problem (15)-(16) for the right-hand side f and the initial data φ_n, performing a partial integration for the integral involving $\ddot{\varphi}_n$ and passing to the limit as $n \to \infty$, the claim follows. □

3.3 Asymptotic behavior as $\tau \to 0$

Again, we assume X to be a Banach space and prove the following generalization of Lemma 4 in [16].

Lemma 14 *Let* $\Omega \in L(X)$, $T > 0$, $\tau_0 > 0$ *and let*

$$\alpha := 1 + 2\exp\bigl(\tau_0 \|\Omega\|_{L(X)}\bigr)$$

Then, for any $\tau \in (0, \tau_0]$,

$$\bigl\|\exp_\tau(t-\tau;\Omega) - \exp(\Omega t)\bigr\|_{L(X)} \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha(T+\tau_0)\|\Omega\|_{L(X)}\bigr) \quad \textit{for } t \in [0, T].$$

Proof First, we want to exploit the mathematical induction to show, for any $k \in \mathbb{N}$,

$$\bigl\|\exp_\tau(t-\tau;\Omega) - \exp(t\Omega)\bigr\|_{L(X)} \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha k\tau \|\Omega\|_{L(X)}\bigr) \tag{35}$$

for $t \in [(k-1)\tau, k\tau]$. Let $\tau \in (0, \tau_0]$. For $t \in [0, \tau]$, the claim easily follows from the mean value theorem for Bochner integration since

$$\begin{aligned}
&\bigl\|\exp_\tau(t-\tau;\Omega) - \exp(\Omega t)\bigr\|_{L(X)} \\
&\quad = \bigl\|\exp(\Omega t) - \mathrm{id}_X\bigr\|_{L(X)} \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\tau\|\Omega\|_{L(X)}\bigr) \\
&\quad \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha\tau \|\Omega\|_{L(X)}\bigr),
\end{aligned}$$

where we used the fact $\alpha \geq 1$. Assuming now that inequality (35) is valid up to some $k \in \mathbb{N}$, we use the fundamental theorem of calculus to estimate, for $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}
&\bigl\|\exp_\tau(t-\tau;\Omega) - \exp(t\Omega)\bigr\|_{L(X)} \\
&\quad = \left\|\exp_\tau\bigl((k-1)\tau;\Omega\bigr) - \exp(k\tau\Omega) + \int_{k\tau}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\bigl(\exp_\tau(s-\tau;\Omega) - \exp(s\Omega)\bigr)\,\mathrm{d}s\right\|_{L(X)} \\
&\quad \leq \bigl\|\exp_\tau\bigl((k-1)\tau;\Omega\bigr) - \exp(k\tau\Omega)\bigr\|_{L(X)} \\
&\qquad + \int_{k\tau}^{t} \left\|\frac{\mathrm{d}}{\mathrm{d}s}\bigl(\exp_\tau(s-\tau;\Omega) - \exp(s\Omega)\bigr)\right\|_{L(X)} \mathrm{d}s \\
&\quad \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha k\tau \|\Omega\|_{L(X)}\bigr) \\
&\qquad + \int_{k\tau}^{(k+1)\tau} \left\|\frac{\mathrm{d}}{\mathrm{d}s}\exp_\tau(s-\tau;\Omega) - \frac{\mathrm{d}}{\mathrm{d}s}\exp(s\Omega)\right\|_{L(X)} \mathrm{d}s \\
&\quad \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha k\tau \|\Omega\|_{L(X)}\bigr) \\
&\qquad + \|\Omega\|_{L(X)} \int_{k\tau}^{(k+1)\tau} \bigl\|\exp_\tau(s-2\tau;\Omega) - \exp(s\Omega)\bigr\|_{L(X)}\,\mathrm{d}s \\
&\quad \leq \tau \|\Omega\|_{L(X)} \exp\bigl(\alpha k\tau \|\Omega\|_{L(X)}\bigr) \\
&\qquad + \|\Omega\|_{L(X)} \int_{k\tau}^{(k+1)\tau} \bigl\|\exp_\tau(s-2\tau;\Omega) - \exp\bigl((s-\tau)\Omega\bigr)\bigr\|_{L(X)}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|\Omega\|_{L(X)} \int_{k\tau}^{(k+1)\tau} \big\| \exp(s\Omega) - \exp\big((s-\tau)\Omega\big) \big\|_{L(X)} \,\mathrm{d}s \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big) \\
&\quad + \|\Omega\|_{L(X)} \int_{(k-1)\tau}^{k\tau} \big\| \exp_\tau(s-\tau;\Omega) - \exp(s\Omega) \big\|_{L(X)} \,\mathrm{d}s \\
&\quad + \|\Omega\|_{L(X)} \int_{k\tau}^{(k+1)\tau} \int_{s-\tau}^{s} \bigg\| \frac{\mathrm{d}}{\mathrm{d}\sigma} \exp(\sigma\Omega) \bigg\|_{L(X)} \,\mathrm{d}\sigma \,\mathrm{d}s \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big) + \tau^2 \|\Omega\|_{L(X)}^2 \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big) \\
&\quad + \tau^2 \|\Omega\|_{L(X)}^2 \exp\big((k+1)\tau \|\Omega\|_{L(X)}\big) \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big)\big(1 + \tau \|\Omega\|_{L(X)} + \tau \|\Omega\|_{L(X)} \exp\big(\tau \|\Omega\|_{L(X)}\big)\big) \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big)\big(1 + 2\tau \|\Omega\|_{L(X)} \exp\big(\tau \|\Omega\|_{L(X)}\big)\big) \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big) \exp\big(2\tau \|\Omega\|_{L(X)} \exp\big(\tau \|\Omega\|_{L(X)}\big)\big) \\
&\leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha (k+1)\tau \|\Omega\|_{L(X)}\big).
\end{aligned}
$$

By induction, we obtain, for any $k \in \mathbb{N}$,

$$
\big\| \exp_\tau(t-\tau;\Omega) - \exp(t\Omega) \big\|_{L(X)} \leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha k\tau \|\Omega\|_{L(X)}\big) \tag{36}
$$

for $t \in ((k-1)\tau, k\tau]$. Now, taking into account that for any $t \in [0,T]$, $\tau \in (0,\tau_0]$ and $k \in \mathbb{N}$ such that $t \in [(k-1)\tau, k\tau]$, we have $k\tau \leq T + \tau_0$. This together with (36) yields the claim. □

Corollary 15 *Let the assumptions of Lemma* 14 *be satisfied and let* $\gamma \geq 0$. *Then, for* $t \in [0,T]$ *and* $\tau \in (0,\tau_0]$, *we have*

$$
\big\| \exp_\tau(t+\gamma;\Omega) - e^{\Omega t} \big\|_{L(X)} \leq (\gamma + \tau)\|\Omega\|_{L(X)} \exp\big(\alpha(T + \gamma + \tau_0)\|\Omega\|_{L(X)}\big).
$$

Proof Lemma 14 and the mean value theorem for Bochner integration yield

$$
\begin{aligned}
&\big\| \exp_\tau(t+\gamma;\Omega) - e^{\Omega t} \big\|_{L(X)} \\
&\quad \leq \big\| \exp_\tau(t+\gamma;\Omega) - e^{\Omega(t+\gamma+\tau)} \big\|_{L(X)} + \big\| e^{\Omega(t+\gamma+\tau)} - e^{\Omega t} \big\|_{L(X)} \\
&\quad \leq \tau \|\Omega\|_{L(X)} \exp\big(\alpha(T+\gamma+\tau_0)\|\Omega\|_{L(X)}\big) \\
&\quad\quad + (\gamma+\tau)\|\Omega\|_{L(X)} \exp\big((T+\gamma+\tau)\|\Omega\|_{L(X)}\big) \\
&\quad \leq (\gamma+\tau)\|\Omega\|_{L(X)} \exp\big(\alpha(T+\gamma+\tau_0)\|\Omega\|_{L(X)}\big)
\end{aligned}
$$

as we claimed. □

Let $T > 0$, $\tau_0 > 0$, $x_0, x_1 \in X$ and $f \in L^1_{\mathrm{loc}}(0,\infty;X)$ be fixed and let $\bar{x} \in C^1([0,\infty),X)$ denote the unique mild solution to the Cauchy problem (9)-(10) from the section on classical harmonic oscillator.

Theorem 16 *Let $\tau_0 > 0$. For any $\tau \in (0, \tau_0)$, let $x(\cdot;\tau)$ denote the unique mild solution of* (15)-(16) *for the initial data $\varphi(\cdot;\tau) \in C^1([-2\tau, 0], X)$. Then we have*

$$\begin{aligned}\|x(\cdot;\tau) - \bar{x}\|_{C^0([0,T],X)} &\le 3\beta\big(\|\varphi(-2\tau;\tau) - x_0\|_X + \|\dot{\varphi}(0;\tau) - x_1\|_X\big)\\ &\quad + 3\beta\tau\big(\|\varphi(\cdot;\tau)\|_{C^1([-2\tau,0],X)} + \|f\|_{L^1(0,T;X)}\big)\end{aligned}$$

with $\beta(T) := 2(1 + \|\Omega\|_{L(X)})(1 + \|\Omega^{-1}\|_{L(X)})\exp(\alpha(T + 2\tau_0)\|\Omega\|_{L(X)})$.

Proof Using the explicit representation of the mild solution $\bar{x}$ and $x(\cdot;\tau)$, respectively, we can estimate

$$\|x(t;\tau) - \bar{x}(t)\|_X \le I_{0,1}(t) + I_{0,2}(t) + I_{0,3}(t) \quad \text{for } t \in [0, T]$$

with

$$\begin{aligned}I_{0,1}(t) &:= \left\|x_\tau^1(t+\tau;\Omega)\varphi(-2\tau;\tau) - \frac{1}{2}\big(e^{\Omega t} + e^{-\Omega t}\big)x_0\right\|_X\\ &\quad + \left\|x_\tau^2(t;\Omega)\dot{\varphi}(0;\tau) + \frac{1}{2}\Omega^{-1}\big(e^{\Omega t} - e^{-\Omega t}\big)x_1\right\|_X,\\ I_{0,2}(t) &:= \int_0^t \left\|x_\tau^2(t-s;\Omega) - \frac{1}{2}\Omega^{-1}\big(e^{\Omega(t-s)} - e^{-\Omega(t-s)}\big)\right\|_{L(X)} \|f(s)\|_X \,\mathrm{d}s,\\ I_{0,3}(t) &:= \int_{-2\tau}^0 \|\dot{x}_\tau^2(t-s;\Omega)\|_{L(X)} \|\dot{\varphi}(s;\tau)\|_X \,\mathrm{d}s.\end{aligned}$$

Corollary 15 yields

$$\left\|x_\tau^1(t+\tau;\Omega) - \frac{1}{2}\big(e^{\Omega t} + e^{-\Omega t}\big)\right\|_{L(X)} \le \beta\tau, \qquad \left\|x_\tau^2(t;\Omega) - \frac{1}{2}\Omega^{-1}\big(e^{\Omega t} - e^{-\Omega t}\big)\right\|_{L(X)} \le \beta\tau$$

and, therefore,

$$\begin{aligned}I_{0,1}(t) &\le \beta\tau\big(\|\varphi(-2\tau;\tau)\|_X + \|\dot{\varphi}(0;\tau)\|_X\big) + \beta\big(\|\varphi(-2\tau;\tau) - x_0\|_X + \|\dot{\varphi}(0;\tau) - x_1\|_X\big)\\ &\le \beta\tau\|\varphi\|_{C^1([-2\tau,0],X)} + \beta\big(\|\varphi(-2\tau;\tau) - x_0\|_X + \|\dot{\varphi}(0;\tau) - x_1\|_X\big).\end{aligned}$$

Similarly,

$$I_{0,2}(t) \le 2\beta\tau\|f\|_{L^1(0,T;X)} \quad \text{and} \quad I_{0,3}(t) \le 2\beta\tau\|\varphi\|_{C^1([0,T],X)}.$$

Hence, the claim follows. □

Corollary 17 *Under conditions of Theorem* 16, *we additionally have*

$$\begin{aligned}&\|x(\cdot;\tau) - \bar{x}\|_{C^1([0,T],X)}\\ &\quad \le 3\big(1 + \beta(T)\big)\big(1 + \delta(T)\big)(1 + T)\big(\|\varphi(-2\tau;\tau) - x_0\|_X + \|\dot{\varphi}(0;\tau) - x_1\|_X\\ &\qquad + \tau\big(\|\varphi(\cdot;\tau)\|_{C^1([-2\tau,0],X)} + \|f\|_{L^1(0,T;X)} + \|x_0\|_X + \|x_1\|_X\big)\big)\end{aligned}$$

with $\delta(T) := 2\|\Omega\|_{L(X)}^2(1 + \|\Omega^{-1}\|_{L(X)})e^{\|\Omega\|_{L(X)}T}$.

Proof Integrating Equation (9) and using Equation (10) as well as exploiting Equations (17)-(18) yields

$$\begin{aligned}\|\dot{x}(t;\tau)-\dot{\bar{x}}(t)\|_X &\le \|\dot{\varphi}(0;\tau)-x_1\|_X+\int_0^t \|\Omega^2 x(s-2\tau;\tau)-\Omega^2\bar{x}(s)\|_X\,\mathrm{d}s \\ &\le I_{1,1}(t)+I_{1,2}(t)+I_{1,3}(t) \quad \text{for } t\in[0,T]\end{aligned}$$

with

$$\begin{aligned} I_{1,1}(t) &:= \|\dot{\varphi}(0;\tau)-x_1\|_X, \qquad I_{1,2} := \|\Omega\|_{L(X)}^2\int_{-2\tau}^0 \|\varphi(s)-\bar{x}(s+2\tau)\|_X\,\mathrm{d}s, \\ I_{1,3}(t) &:= \|\Omega\|_{L(X)}^2\int_{2\tau}^t \|x(s-2\tau;\tau)-\bar{x}(s)\|_X\,\mathrm{d}s. \end{aligned}$$

Taking into account Equation (14), we can estimate

$$\begin{aligned}\|\bar{x}\|_{C^0([0,2\tau],X)} &\le \left(\|x_0\|+\|\Omega^{-1}\|_{L(X)}\|x_1\|\right)e^{\|\Omega\|_{L(X)}T} \\ &\quad + \|\Omega^{-1}\|_{L(X)}e^{\|\Omega\|_{L(X)}T}\|f\|_{L^1(0,T;X)}.\end{aligned}$$

Hence,

$$I_{1,2}(t)\le \delta\tau\left(\|\varphi\|_{C^0([0,T],X)}+\|x_0\|_X+\|x_1\|_X+\|f\|_{L^1(0,T;X)}\right).$$

Applying Theorem 16, we further get

$$\begin{aligned} I_{1,3}(t) &\le 3\|\Omega\|_{L(X)}^2 T\beta\left(\|\varphi(-2\tau;\tau)-x_0\|_X+\|\dot{\varphi}(0;\tau)-x_1\|_X \right.\\ &\quad \left.+\tau\left(\|\varphi(\cdot;\tau)\|_{C^1([-2\tau,0],X)}+\|f\|_{L^1(0,T;X)}\right)\right).\end{aligned}$$

Combining these inequalities and using again Theorem 16, we deduce the estimate asserted. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally contributed to the problem discussion and writing the introduction section. DK proved the explicit solution representation formula for the harmonic oscillator with pure delay. MP showed the abstract well-posedness for the harmonic oscillator with pure delay and studied its asymptotics as the delay parameter goes to zero. EA presented a study on the classical harmonic oscillator without delay as well as checked the proofs and verified the calculations. All the authors read and approved the final manuscript.

Author details
[1]Faculty of Cybernetics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine. [2]Department of Mathematics and Statistics, University of Konstanz, Konstanz, Germany. [3]Faculty of Mechanics and Mathematics, Baku State University, Baku, Azerbaijan.

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