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Forced oscillation for solutions of boundary value problems of fractional partial difference equations

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Abstract
In this paper, we obtain the forced oscillation of solutions for certain fractional partial difference equations with two different types of boundary conditions. Our results are based on discrete Gaussian formula and some basic theories of discrete fractional calculus.

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Keywords: oscillation; fractional partial difference equation; discrete fractional calculus

1 Introduction

In 1974, Diaz and Osler [1] presented a discrete fractional difference operator based on an infinite series. In 1988, Miller and Ross [2] introduced the definitions of noninteger-order differences and sums. Since then, the theory of fractional difference equations has been studied by several scholars. In recent years, some papers [3–21] on discrete fractional calculus were published, which helped to build up the theory of fractional difference equations. For example, Atici and Eloe [3] discussed the properties of the generalized falling function, the corresponding power rule for fractional delta operators, and the commutativity of fractional sums.

Very recently, the oscillation theory as a part of the qualitative theory of fractional differential equations and fractional difference equations has been developed. We refer the reader to [20–28] and the references therein. In particular, we notice that a few papers [24–28] studied the oscillation of fractional partial differential equations that involve the Riemann-Liouville fractional partial derivatives.

Motivated by the papers [24–29], we investigate the forced oscillation of the fractional partial difference equation of the form

$$\Delta_n^\alpha u(m,n) = a(n)Lu(m,n) - q(m,n)u(m,n) + h(m,n), \quad (m,n) \in \Omega \times \mathbb{N}_a, \tag{1}$$

where $m = (m_1, m_2, \dots, m_\ell)$, Ω is a convex connected solid net (for the definition of a convex connected solid net, we refer to [29]), and

$$\Omega = \mathbb{N}(1,N_1) \times \mathbb{N}(1,N_2) \times \cdots \times \mathbb{N}(1,N_\ell), \tag{2}$$

$\mathbb{N}(1,N_i) = \{1,2,\ldots N_i\}$, $i = 1,2,\ldots,\ell$, L is the discrete Laplacian on Ω defined as

$$Lu(m,n) = \sum_{i=1}^{\ell} \Delta_{m_i}^2 u\big((m_1,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_\ell),n\big), \tag{3}$$

$\Delta_n^\alpha u(m,n)$ is the Riemann-Liouville fractional difference operator of order α of u with respect to n, $\alpha \in (0,1)$ is a constant, $\mathbb{N}_a = \{a,a+1,a+2,\ldots\}$, and $a \geq 0$ is a real number.

Throughout this paper, we always assume that

(A) $a(n) \geq 0, n \in \mathbb{N}_a$; $q(m,n) \geq 0$, $q(n) = \min_{m\in\Omega} q(m,n)$, $(m,n) \in \Omega \times \mathbb{N}_a$; and $h : \Omega \times \mathbb{N}_a \to \mathbb{R}$.

Consider one of the two following boundary conditions:

$$\text{(B1)} \qquad \Delta_N u(m-1,n) + g(m,n)u(m,n) = 0, \quad (m,n) \in \partial\Omega \times \mathbb{N}_a,$$

or

$$\text{(B2)} \qquad \Delta_N u(m-1,n) = \phi(m,n), \quad (m,n) \in \partial\Omega \times \mathbb{N}_a,$$

where

$$\begin{aligned}\partial\Omega = \bigcup_{i=1}^{\ell} \big\{&(m_1,\ldots,m_{i-1},0,m_{i+1},\ldots,m_\ell),(m_1,\ldots,m_{i-1},\\ &N_i+1,m_{i+1},\ldots,m_\ell)\big\}, \quad m_i \in \mathbb{N}(1,N_i), 1 \leq i \leq \ell,\end{aligned} \tag{4}$$

$\Delta_N u(m-1,n)$ is the normal difference at $(m,n) \in \partial\Omega \times \mathbb{N}_a$ defined by

$$\Delta_N u(m-1,n) = \sum_{\text{all } m\pm1\notin\Omega} \big(\Delta_m\big(u(m,n)\big) - \Delta_m u(m-1,n)\big) = \sum_{\text{all } m\pm1\notin\Omega} \Delta_m^2 u(m,n),$$

N is the unit exterior normal vector to $\partial\Omega$, $m+1 := \{m_1+1,m_2,\ldots,m_\ell\} \cup \cdots \cup \{m_1,\ldots,m_{\ell-1},m_\ell+1\}$, $m-1 := \{m_1-1,m_2,\ldots,m_\ell\} \cup \cdots \cup \{m_1,\ldots,m_{\ell-1},m_\ell-1\}$, and $g(m,n) \geq 0, \phi(m,n) \geq 0, (m,n) \in \partial\Omega \times \mathbb{N}_a$. For the details on $\partial\Omega$ and $\Delta_N u(m-1,n)$, we refer to the monograph [30] and paper [29], respectively.

The function $u(m,n)$ is said to be a solution of problem (1)-(B1) (or (1)-(B2)) if it satisfies (1) for $(m,n) \in \Omega \times \mathbb{N}_a$ and satisfies (B1) (or (B2)) for $(m,n) \in \partial\Omega \times \mathbb{N}_a$.

The solution $u(m,n)$ of problem (1)-(B1) (or (1)-(B2)) is said to be oscillatory in $\Omega \times \mathbb{N}_a$ if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory.

2 Preliminaries

In this section, we present some preliminary results of discrete fractional calculus and partial differences.

Definition 2.1 ([3]) Let $0 < \nu < 1$. The νth fractional sum of f is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s), \tag{5}$$

where f is defined for $s \in \mathbb{N}_a$, $\Delta_a^{-\nu} f$ is defined for $s \in \mathbb{N}_{a+\nu} = \{a+\nu, a+\nu+1, a+\nu+2, \ldots\}$, Γ is the gamma function, and

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}.$$

Definition 2.2 Let $0 < \nu < 1$. The νth fractional sum with respect to n of $u(m,n)$ is defined by

$$\Delta_n^{-\nu} u(m,n) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{n-\nu} (n-s-1)^{(\nu-1)} u(m,s). \tag{6}$$

Definition 2.3 ([3]) Let $\mu > 0$ and $k-1 < \mu < k$, where k denotes a positive integer, $k = \lceil \mu \rceil$. Set $\nu = k - \mu$. The μth fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{k-\nu} f(t) = \Delta^{k} \Delta^{-\nu} f(t), \tag{7}$$

where $\lceil \mu \rceil$ is the ceiling function of μ.

Definition 2.4 Let $0 < \mu < 1$ and $\nu = 1 - \mu$. The μth fractional partial difference with respect to n of a function $u(m,n)$ is defined as

$$\Delta_n^{\mu} u(m,n) = \Delta_n^{1-\nu} u(m,n) = \Delta_n \Delta_n^{-\nu} u(m,n). \tag{8}$$

Lemma 2.5 ([3]) *Let f be a real-valued function defined on $\mathbb{N}_a$, and let $\mu, \nu > 0$. Then the following equalities hold*:

$$\Delta^{-\nu}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}\left[\Delta^{-\nu} f(t)\right]; \tag{9}$$

$$\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a). \tag{10}$$

Lemma 2.6 *For $n_0 \in \mathbb{N}_a$, let*

$$E(n) = \sum_{s=n_0}^{n-1+\alpha} (n-s-1)^{(-\alpha)} x(n), \quad n \in \mathbb{N}_a, \alpha \in (0,1). \tag{11}$$

Then

$$\Delta E(n) = \Gamma(1-\alpha) \Delta^{\alpha} x(n). \tag{12}$$

Proof By Definition 2.1, from (11) we have

$$\begin{aligned} E(n) &= \sum_{s=n_0}^{n-1+\alpha} (n-s-1)^{(-\alpha)} x(s) = \sum_{s=n_0}^{n-(1-\alpha)} (n-s-1)^{((1-\alpha)-1)} x(s) \\ &= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(n). \end{aligned} \tag{13}$$

Using Definition 2.3, from (13) it follows that

$$\Delta E(n) = \Gamma(1-\alpha)\Delta\Delta^{-(1-\alpha)}x(n) = \Gamma(1-\alpha)\Delta^{\alpha}x(n).$$

The proof of Lemma 2.6 is complete. □

Lemma 2.7 (Discrete Gaussian formula [29]) *Let Ω be a convex connected solid net. Then*

$$\sum_{m\in\Omega} Ly(m,n) = \sum_{m\in\partial\Omega} \Delta_N y(m-1,n). \tag{14}$$

Lemma 2.8 ([31]) *For $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{\Gamma(t)t^{\varepsilon}}{\Gamma(t+\varepsilon)} = 1. \tag{15}$$

For convenience, we introduce the following notations:

$$U(n) = \sum_{m\in\Omega} u(m,n), \qquad H(n) = \sum_{m\in\Omega} h(m,n), \qquad \Phi(n) = \sum_{m\in\partial\Omega} \phi(m,n). \tag{16}$$

3 Oscillation of problem (1)-(B1)

Theorem 3.1 *For $n_0 \in \mathbb{N}_a$, if*

$$\liminf_{n\to\infty} \sum_{s=n_0}^{n-1} H(s) = -\infty, \tag{17}$$

and

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} H(s) = +\infty, \tag{18}$$

where $H(n)$ is defined as in (16), *then every solution $u(m,n)$ of problem* (1)-(B1) *is oscillatory in $\Omega \times \mathbb{N}_a$.*

Proof Suppose to the contrary that there is a nonoscillatory solution $u(m,n)$ of problem (1)-(B1) that has no zero in $\Omega \times \mathbb{N}_a$ for some $n^* \geq a$. Then $u(m,n) > 0$ or $u(m,n) < 0$ for $n \geq n^*$.

Case 1. $u(m,n) > 0, n \geq n^*$. Summing equation (1) over Ω, we have

$$\begin{aligned}\sum_{m\in\Omega} \Delta_n^{\alpha} u(m,n) &= a(n)\sum_{m\in\Omega} Lu(m,n) - \sum_{m\in\Omega} q(m,n)u(m,n) \\ &\quad + \sum_{m\in\Omega} h(m,n), \quad n \in \mathbb{N}_a.\end{aligned} \tag{19}$$

The discrete Gaussian formula and (B1) yield

$$\sum_{m\in\Omega} Lu(m,n) = \sum_{m\in\partial\Omega} \Delta_N u(m-1,n) = \sum_{m\in\partial\Omega} -g(m,n)u(m,n) \leq 0, \quad n \in \mathbb{N}_a. \tag{20}$$

From assumption (A) we have

$$\sum_{m\in\Omega} q(m,n)u(m,n) \geq q(n)\sum_{m\in\Omega} u(m,n), \quad n\in\mathbb{N}_a. \tag{21}$$

Combining (19)-(21), we obtain

$$\Delta^\alpha U(n) + q(n)U(n) \leq H(n), \quad n\in\mathbb{N}_a, \tag{22}$$

where $U(n)$ is defined as in (16). It follows from (22) that

$$\Delta^\alpha U(n) \leq H(n), \quad n\in\mathbb{N}_a. \tag{23}$$

Using Lemma 2.6, from (23) we have

$$\Delta G(n) \leq \Gamma(1-\alpha)H(n), \tag{24}$$

where

$$G(n) = \sum_{s=n^*}^{n-1+\alpha} (n-s-1)^{(-\alpha)}U(n), \quad n\in\mathbb{N}_a.$$

Summing both sides of (24) from n^* to $n-1$, we obtain

$$G(n) \leq G(n^*) + \Gamma(1-\alpha)\sum_{s=n^*}^{n-1} H(s). \tag{25}$$

Taking $n\to\infty$ in (25), we have

$$\liminf_{n\to\infty} G(n) = -\infty,$$

which contradicts with $G(n) > 0$.

Case 2. $u(m,n) < 0, n \geq n^*$. As in the proof of Case 1, we obtain (19). The discrete Gaussian formula and (B1) yield

$$\sum_{m\in\Omega} Lu(m,n) = \sum_{m\in\partial\Omega} \Delta_N u(m-1,n) = \sum_{m\in\partial\Omega} -g(m,n)u(m,n) \geq 0, \quad n\in\mathbb{N}_a. \tag{26}$$

From assumption (A) we have

$$\sum_{m\in\Omega} q(m,n)u(m,n) \leq q(n)\sum_{m\in\Omega} u(m,n), \quad n\in\mathbb{N}_a. \tag{27}$$

Combining (19), (26), and (27), we obtain

$$\Delta^\alpha U(n) + q(n)U(n) \geq H(n), \quad n\in\mathbb{N}_a. \tag{28}$$

Then we have

$$\Delta^{\alpha} U(n) \geq H(n), \quad n \in \mathbb{N}_a. \tag{29}$$

Using the above-mentioned method in Case 1, we easily obtain a contradiction. This completes the proof of Theorem 3.1. □

Theorem 3.2 *If*

$$\liminf_{n\to\infty} (n-a)^{1-\alpha} \left\{ \sum_{s=a}^{n-\alpha} (n-s-1)^{(\alpha-1)} H(s) \right\} = -\infty \tag{30}$$

and

$$\limsup_{n\to\infty} (n-a)^{1-\alpha} \left\{ \sum_{s=a}^{n-\alpha} (n-s-1)^{(\alpha-1)} H(s) \right\} = +\infty, \tag{31}$$

where $H(n)$ *is defined as in* (16), *then every solution* $u(m,n)$ *of problem* (1)-(B1) *is oscillatory in* $\Omega \times \mathbb{N}_a$.

Proof Suppose to the contrary that there is a nonoscillatory solution $u(m,n)$ of problem (1)-(B1) that has no zero in $\Omega \times \mathbb{N}_a$ for some $n^* \geq a$. Then $u(m,n) > 0$ or $u(m,n) < 0$ for $n \geq n^*$.

Case 1. $u(m,n) > 0, n \geq n^*$. As in the proof of Theorem 3.1, we obtain (22). Applying the operator $\Delta^{-\alpha}$ to inequality (22), we have

$$\Delta^{-\alpha} \Delta_n^{\alpha} U(n) \leq \Delta^{-\alpha} H(n). \tag{32}$$

By Lemma 2.5 it follows from the left-hand side of (32) that

$$\begin{aligned} \Delta^{-\alpha} \Delta_n^{\alpha} U(n) &= \Delta^{-\alpha} \Delta \Delta^{-(1-\alpha)} U(n) \\ &= \Delta \Delta^{-\alpha} \Delta^{-(1-\alpha)} U(n) - \frac{(n-a)^{(\alpha-1)}}{\Gamma(\alpha)} \Delta^{-(1-\alpha)} U(a) \\ &= U(n) - \frac{C_0}{\Gamma(\alpha)} (n-a)^{(\alpha-1)}, \end{aligned} \tag{33}$$

where $\Delta^{-(1-\alpha)} U(a) = \Delta^{-(1-\alpha)} U(n)|_{n=a} = C_0$ is a constant. Applying Definition 2.1 to the right-hand side of (32), we have

$$\Delta^{-\alpha} H(n) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{n-\alpha} (n-s-1)^{(\alpha-1)} H(s). \tag{34}$$

Combining (32)-(34), we get

$$U(n) \leq \frac{C_0}{\Gamma(\alpha)} (n-a)^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{n-\alpha} (n-s-1)^{(\alpha-1)} H(s). \tag{35}$$

It follows from (35) that

$$\begin{aligned}\Gamma(\alpha)(n-a)^{1-\alpha}U(n) \le{}& C_0(n-a)^{(\alpha-1)}(n-a)^{1-\alpha}\\ &+ (n-a)^{1-\alpha}\sum_{s=a}^{n-\alpha}(n-s-1)^{(\alpha-1)}H(s).\end{aligned} \tag{36}$$

Using Lemma 2.8, we obtain

$$\begin{aligned}&\lim_{n\to\infty}(n-a)^{1-\alpha}(n-a)^{(\alpha-1)}\\ &\quad=\lim_{n\to\infty}(n-a)^{1-\alpha}\frac{\Gamma(n-a+1)}{\Gamma(n-a+1+(1-\alpha))}\\ &\quad=\lim_{n\to\infty}(n-a)^{1-\alpha}\frac{(n-a)\Gamma(n-a)}{(n-a+1-\alpha)\Gamma(n-a+(1-\alpha))}\\ &\quad=\lim_{n\to\infty}\frac{n-a}{n-a+1-\alpha}\frac{\Gamma(n-a)(n-a)^{1-\alpha}}{\Gamma(n-a+(1-\alpha))}\\ &\quad=1.\end{aligned} \tag{37}$$

Noting (37) and taking $n\to\infty$ in (36), we have

$$\liminf_{n\to\infty}\bigl\{(n-a)^{1-\alpha}U(n)\bigr\}\le-\infty,$$

which contradicts with $U(n)>0$.

Case 2. $u(m,n)<0, n\ge n_0$. As in the proof of Theorem 3.1, we obtain the fractional difference inequality (29). Then using the above-mentioned method, we easily obtain a contradiction. This completes the proof of Theorem 3.2. □

4 Oscillation of problem (1)-(B2)

Theorem 4.1 *For $n_0\in\mathbb{N}_a$, if*

$$\liminf_{n\to\infty}\sum_{s=n_0}^{n-1}\bigl(\Phi(s)+H(s)\bigr)=-\infty \tag{38}$$

and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\bigl(\Phi(s)+H(s)\bigr)=+\infty, \tag{39}$$

where $\Phi(n)$ and $H(n)$ are defined as in (16), *then every solution $u(m,n)$ of problem* (1)-(B2) *is oscillatory in $\Omega\times\mathbb{N}_a$.*

Proof Suppose to the contrary that there is a nonoscillatory solution $u(m,n)$ of problem (1)-(B2) that has no zero in $\Omega\times\mathbb{N}_a$ for some $n^*\ge a$. Then $u(m,n)>0$ or $u(m,n)<0$ for $n\ge n^*$.

Case 1. $u(m,n)>0, n\ge n^*$. As in the proof of Theorem 3.1, we obtain (19). Using the discrete Gaussian formula and noting the boundary condition (B2), it follows from (19) that

$$\sum_{m\in\Omega} Lu(m,n) = \sum_{m\in\partial\Omega} \Delta_N u(m-1,n) = \sum_{m\in\partial\Omega} \phi(m,n), \quad n\in\mathbb{N}_a. \tag{40}$$

Combing (19), (21), and (40), we have

$$\Delta^\alpha U(n) + q(n)U(n) \le \Phi(n) + H(n), \quad n\in\mathbb{N}_a. \tag{41}$$

The remainder of the proof is similar to that of Case 1 in Theorem 3.1. We omit it here.

Case 2. $u(m,n) < 0, n \ge n^*$. In this case, we easily obtain (19), (27), and (40). Then we have

$$\Delta^\alpha U(n) + q(n)U(n) \ge \Phi(n) + H(n), \quad n\in\mathbb{N}_a. \tag{42}$$

The remainder of the proof is similar to that of Case 2 in Theorem 3.1. We omit it here, too. The proof of Theorem 4.1 is complete. □

Theorem 4.2 *If*

$$\liminf_{n\to\infty}(n-a)^{1-\alpha}\left\{\sum_{s=a}^{n-\alpha}(n-s-1)^{(\alpha-1)}\big(\Phi(s)+H(s)\big)\right\} = -\infty \tag{43}$$

and

$$\limsup_{n\to\infty}(n-a)^{1-\alpha}\left\{\sum_{s=a}^{n-\alpha}(n-s-1)^{(\alpha-1)}\big(\Phi(s)+H(s)\big)\right\} = +\infty, \tag{44}$$

where $\Phi(n)$ *and* $H(n)$ *are defined as in* (16), *then every solution* $u(m,n)$ *of problem* (1)-(B2) *is oscillatory in* $\Omega\times\mathbb{N}_a$.

5 Examples

Example 5.1 Consider the fractional partial difference equation

$$\begin{aligned}\Delta_n^{\frac{1}{2}} u(m,n) = {} & 2nLu(m,n) - \frac{2n}{m}u(m,n) \\ & + \left\{\frac{m}{3} + \frac{1}{3}\big[(-1)^{n+1}e^{n+1} - (-1)^n e^n - 2\big]\right\}, \quad (m,n)\in\mathbb{N}(1,3)\times\mathbb{N}_0,\end{aligned} \tag{45}$$

with boundary condition

$$\Delta_N u(0,n) = \Delta_N u(4,n) = 0, \quad n\in\mathbb{N}_0. \tag{46}$$

Here $\alpha = \frac{1}{2}, a(n) = 2n$, $q(m,n) = \frac{2n}{m}$, $h(m,n) = \frac{m}{3} + \frac{1}{3}[(-1)^{n+1}e^{n+1} - (-1)^n e^n - 2]$. It is easy to see that $q(n) = \frac{2}{3}n$ and

$$H(n) = \sum_{m\in\mathbb{N}(1,3)} h(m,n) = (-1)^{n+1}e^{n+1} - (-1)^n e^n.$$

Therefore,

$$\sum_{s=n_0}^{n-1} H(s) = \sum_{s=n_0}^{n-1} \left\{ (-1)^{s+1} e^{s+1} - (-1)^s e^s \right\} = (-1)^n e^n - (-1)^{n_0} e^{n_0}, \quad n_0 \in \mathbb{N}_0. \tag{47}$$

It follows from (47) that

$$\liminf_{n\to\infty} \sum_{s=n_0}^{n-1} H(s) = -\infty$$

and

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} H(s) = +\infty.$$

Using Theorem 3.1, we obtain that every solution of problem (45)-(46) is oscillatory in $\mathbb{N}(1,3) \times \mathbb{N}_0$.

Example 5.2 Consider the fractional partial difference equation

$$\begin{aligned} \Delta_n^{\frac{1}{4}} u(m,n) = {} & 2\Gamma(n) Lu(m,n) - \frac{\Gamma(n+\frac{3}{4})}{2m\Gamma(n)} u(m,n) \\ & + \frac{1}{4}\Gamma\left(\frac{1}{4}\right) m + \frac{n}{2}, \quad (m,n) \in \mathbb{N}(1,2) \times \mathbb{N}_0, \end{aligned} \tag{48}$$

with boundary condition

$$\Delta_N u(0,n) = \Delta_N u(3,n) = 0, \quad n \in \mathbb{N}_0. \tag{49}$$

Here $\alpha = \frac{1}{4}, a(n) = 2\Gamma(n)$, $q(m,n) = \frac{\Gamma(n+\frac{3}{4})}{2m\Gamma(n)}$, $h(m,n) = \frac{1}{4}\Gamma(\frac{1}{4})m + \frac{n}{2}$. It is easy to see that

$$q(n) = \frac{\Gamma(n+\frac{3}{4})}{4\Gamma(n)}, \qquad H(n) = \sum_{m\in\mathbb{N}(1,2)} h(m,n) = \frac{3}{4}\Gamma\left(\frac{1}{4}\right) + n.$$

Therefore,

$$\sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} H(s) = \sum_{s=0}^{n-\frac{1}{4}} (n-s-1)^{(-\frac{3}{4})} \left(\frac{3}{4}\Gamma\left(\frac{1}{4}\right) + s \right) > 0, \quad n \in \mathbb{N}_0, \tag{50}$$

which shows that condition (30) of Theorem 3.2 does not hold. Indeed, $u(m,n) = mn^{(\frac{1}{4})}$ is a nonoscillatory solution of problem (48)-(49).

Example 5.3 Consider the fractional partial difference equation

$$\begin{aligned} \Delta_n^{\frac{1}{3}} u(m,n) = {} & \frac{1}{2} Lu(m,n) - \frac{2\Gamma(\frac{1}{3})\Gamma(n+\frac{2}{3})}{3n\Gamma(n)} u(m,n) \\ & + \Gamma\left(\frac{1}{3}\right) m^2 - \frac{n\Gamma(n)}{\Gamma(n+\frac{2}{3})}, \quad (m,n) \in \mathbb{N}(1,2) \times \mathbb{N}_0, \end{aligned} \tag{51}$$

with boundary condition

$$\Delta_N u(0,n) = \Delta_N u(3,n) = \frac{2n\Gamma(n)}{\Gamma(n+\frac{2}{3})}, \quad n \in \mathbb{N}_0. \tag{52}$$

Here $\alpha = \frac{1}{3}, a(n) = \frac{1}{2}, q(m,n) = \frac{2\Gamma(\frac{1}{3})\Gamma(n+\frac{2}{3})}{3n\Gamma(n)}, h(m,n) = \Gamma(\frac{1}{3})m^2 - \frac{n\Gamma(n)}{\Gamma(n+\frac{2}{3})}, \phi(m,n) = \frac{2n\Gamma(n)}{\Gamma(n+\frac{2}{3})}$. Therefore,

$$q(n) = \frac{2\Gamma(\frac{1}{3})\Gamma(n+\frac{2}{3})}{3n\Gamma(n)}, \qquad H(n) = \sum_{m\in\mathbb{N}(1,2)} h(m,n) = 5\Gamma\left(\frac{1}{3}\right) - \frac{2n\Gamma(n)}{\Gamma(n+\frac{2}{3})},$$

$$\Phi(n) = \sum_{m\in\{0,3\}} \phi(m,n) = \frac{4n\Gamma(n)}{\Gamma(n+\frac{2}{3})}.$$

It is easy to see that

$$\sum_{s=0}^{n-1}\bigl[\Phi(s) + H(s)\bigr] = \sum_{s=0}^{n-1}\left[5\Gamma\left(\frac{1}{3}\right) + \frac{2s\Gamma(s)}{\Gamma(s+\frac{2}{3})}\right] > 0, \quad n \in \mathbb{N}_0. \tag{53}$$

Thus, this time, condition (38) of Theorem 4.1 is false. Indeed, we easily see that $u(m,n) = m^2 n^{(\frac{1}{3})}$ is a nonoscillatory solution of the problem (51)-(52).

Example 5.4 Consider the fractional partial difference equation

$$\begin{aligned}\Delta_n^{\frac{2}{3}} u(m,n) &= 3nLu(m,n) - \frac{n}{m}u(m,n) \\ &\quad + \left\{\frac{m}{3} + \frac{1}{2}\bigl[(-1)^{n+1}e^{n+1} - (-1)^n e^n - 1\bigr]\right\}, \quad (m,n) \in \mathbb{N}(1,2) \times \mathbb{N}_0,\end{aligned} \tag{54}$$

with boundary condition

$$\Delta_N u(0,n) = \Delta_N u(3,n) = \frac{1}{4}\bigl[(-1)^{n+1}e^{n+1} - (-1)^n e^n\bigr], \quad n \in \mathbb{N}_0. \tag{55}$$

Here $\alpha = \frac{2}{3}, a(n) = 3n$, $q(m,n) = \frac{n}{m}$, $h(m,n) = \frac{m}{3} + \frac{1}{2}[(-1)^{n+1}e^{n+1} - (-1)^n e^n - 1]$, and $\phi(m,n) = \frac{1}{4}[(-1)^{n+1}e^{n+1} - (-1)^n e^n]$. It is easy to see that $q(n) = \frac{n}{2}$,

$$H(n) = \sum_{m\in\mathbb{N}(1,2)} h(m,n) = (-1)^{n+1}e^{n+1} - (-1)^n e^n,$$

and

$$\Phi(n) = \sum_{m\in\{0,3\}} = \frac{1}{2}\bigl[(-1)^{n+1}e^{n+1} - (-1)^n e^n\bigr].$$

Therefore,

$$\begin{aligned}\sum_{s=n_0}^{n-1}\bigl(H(s) + \Phi(s)\bigr) &= \frac{3}{2}\sum_{s=n_0}^{n-1}\bigl\{(-1)^{s+1}e^{s+1} - (-1)^s e^s\bigr\} \\ &= \frac{3}{2}\bigl\{(-1)^n e^n - (-1)^{n_0} e^{n_0}\bigr\}, \quad n_0 \in \mathbb{N}_0.\end{aligned} \tag{56}$$

It follows from (56) that

$$\liminf_{n\to\infty}\sum_{s=n_0}^{n-1}\bigl(H(s)+\Phi(s)\bigr)=-\infty$$

and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\bigl(H(s)+\Phi(s)\bigr)=+\infty.$$

We easily see that the conditions of Theorem 4.1 are satisfied. Then every solution of problem (54)-(55) is oscillatory in $\mathbb{N}(1,2)\times\mathbb{N}_0$.

Competing interests
The authors declare that there are no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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