Gao and Chen *Advances in Difference Equations* (2016) 2016:314
DOI 10.1186/s13662-016-1034-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions for a class of nonlinear higher-order fractional differential equation with fractional nonlocal boundary condition

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Abstract

In this paper, we study the existence of solutions for a class of nonlinear higher-order fractional differential equation with fractional nonlocal boundary condition by using the monotone iterative technique based on the method of upper and lower solutions and give a specific iterative equation about its solutions.

MSC: 26A33; 34B10; 34B15

Keywords: nonlinear fractional differential equation; nonlocal boundary value problem; monotone iterative technique of upper and lower solutions

1 Introduction

We consider the existence of solutions for the following nonlinear fractional differential equation with nonlocal boundary value condition:

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0^+}^{\beta_1} u(t), {}^C D_{0^+}^{\beta_2} u(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)), \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \qquad u^{(n-1)}(0) = \rho I_{0^+}^{\gamma} u(1), \quad j = 0, 1, \ldots, n-2, \end{cases} \tag{1}$$

where $n-1 < \alpha < n$ is a real number, $n \geq 2$, ${}^C D_{0^+}^{\alpha}$, ${}^C D_{0^+}^{\beta_i}$, $i = 1, 2, \ldots, n-1$, $i-1 < \beta_i < i$ is the standard Caputo fractional derivative, $I_{0^+}^{\gamma}$ is the standard Riemann-Liouville integral, $0 < \gamma$, and $0 < \rho < \Gamma(n+\gamma)$. The nonlinear term $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

The boundary value problem of fractional equations *has* emerged as a new branch in the fields of differential equations for their deep backgrounds. In recent years, it is popular and important because the subject of fractional calculus frequently appears in various fields such as physics, chemistry, biology, economics, control theory, signal and image processing, and blood flow phenomenon. For more details about fractional calculus and fractional differential equations, we refer the reader to the monographs by Miller and Ross [1], Heikkila et al. [2], Podlubny [3], Hilfer [4], and Kilbas et al. [5], the survey by Agarwal et al. [6], and the papers [7–14]. Many scholars have studied the existence for nonlinear fractional differential equations with a variety of boundary conditions; see [15–23] and the references therein. However, sometimes it is better to impose integral conditions because they lead to more precise measures than those proposed by a local condition; then, it is greatly important to obtain specific solutions when a solution exists. For this reason, the aim of this paper is to study the existence of solutions for problem (1) by using the monotone iterative technique based on the method of upper and lower solutions, to obtain the existence of solutions for problem (1) by establishing a comparison theorem, and to give a specific iterative equation. For monotone iterative technique, which is based on the method of upper and lower solutions, see recent papers [24–28].

2 Preliminaries

Let $I = [0,1]$. We denote by $C(I)$ the Banach space of all continuous functions $u(t)$ on I with norm $\|u\|_C = \max_{t\in I}|u(t)|$. Generally, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all nth-order continuously differentiable functions on I with norm

$$\|u\|_{C^n} = \max\{\|u\|_C, \|u'\|_C, \dots, \|u^{(n)}\|_C\}.$$

Let $C^+(I)$ denote the cone of all nonnegative functions in $C(I)$. Let AC^n be the Banach space of all absolutely continuous functions $u(t)$ on I differentiable up to order n with norm

$$\|u\|_{\mathrm{AC}^n} = \max\Big\{\max_{t\in I}|u(t)|, \max_{t\in I}\big|{}^C D_{0^+}^{\beta_i} u(t)\big|, i = 1,2,\dots,n-1, i-1 < \beta_i < i\Big\}.$$

Definition 1 If $g \in C([a,b])$ and $q > 0$, then the Riemann-Liouville fractional integral is defined by

$$I_{a^+}^q g(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} g(s)\,ds,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 Let $q \geq 0$ and $n = [q]+1$. If $g \in \mathrm{AC}^n[a,b]$, then the Caputo fractional derivative of order q of g defined by

$${}^C D_{a^+}^q g(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} u^{(n)}(s)\,ds$$

exists almost everywhere on $[a,b]$ ($[q]$ is the integer part of q).

Lemma 3 *Let $h \in C(I)$. Then the linear boundary value problem* (*LBVP*)

$$\begin{cases} {}^C D_{0^+}^\alpha u(t) = h(t), & 0 < t < 1, \\ u^{(j)}(0) = 0, \quad u^{(n-1)}(0) = \rho I_{0^+}^\gamma u(1), & j = 0,1,\dots,n-2, \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\,ds := Sh(t), \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{n-1}(1-s)^{\alpha+\gamma-1}\rho\Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma)-\rho)\Gamma(\alpha+\gamma)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{n-1}(1-s)^{\alpha+\gamma-1}\rho\Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma)-\rho)\Gamma(\alpha+\gamma)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Moreover, the solution operator $S : \mathrm{AC}(I) \to \mathrm{AC}^{(n-1)}(I)$ *is a completely continuous linear operator.*

Proof We may deduce equation (2) equivalent to an integral equation

$$u(t) = I_{0^+}^{\alpha} h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}. \tag{5}$$

Since $u^{(j)}(0) = 0$, we deduce that $c_j = 0$, $j = 0, 1, \ldots, n-2$. Therefore, taking the derivatives of equation (5) gives

$$u^{(n-1)}(t) = I_{0^+}^{\alpha-(n-1)} h(t) + \Gamma(n) c_{n-1},$$

and we have

$$I_{0^+}^{\gamma} u(t) = I_{0^+}^{\alpha+\gamma} h(t) + c_{n-1} I_{0^+}^{\alpha+\gamma} t^{n-1}.$$

Because of the integral boundary condition $u^{(n-1)}(0) = \rho I_{0^+}^{\gamma} u(1)$, we have

$$c_{n-1} = \frac{\rho \Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma) - \rho)} I_{0^+}^{\alpha+\gamma} h(1).$$

Substituting the values of c_j, c_{n-1}, $j = 0, 1, \ldots, n-2$, into (5), we obtain

$$u(t) = I_{0^+}^{\alpha} h(t) + \frac{t^{n-1} \rho \Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma) - \rho)} I_{0^+}^{\alpha+\gamma} h(1),$$

which can be written as

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds \\ &\quad + \frac{t^{n-1} \rho \Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma) - \rho)\Gamma(\alpha+\gamma)} \int_0^1 (1-s)^{\alpha+\gamma-1} h(s)\,ds \\ &= \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{n-1}(1-s)^{\alpha+\gamma-1} \rho \Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma) - \rho)\Gamma(\alpha+\gamma)} \right) h(s)\,ds \\ &\quad + \int_t^1 \frac{t^{n-1}(1-s)^{\alpha+\gamma-1} \rho \Gamma(n+\gamma)}{\Gamma(n)(\Gamma(n+\gamma) - \rho)\Gamma(\alpha+\gamma)} h(s)\,ds \\ &= \int_0^1 G(t,s) h(s)\,ds. \end{aligned}$$

From expression (3) we easily see that $S : \mathrm{AC}(I) \to \mathrm{AC}^{(n-1)}(I)$ is a completely continuous linear operator. This completes the proof. □

Lemma 4 *Let* $h \in C^+(I)$. *Then the unique solution* $u = Sh$ *of LBVP* (2) *has the following properties*:

$$u(t) \geq 0, \qquad {}^C D_{0^+}^{\beta_1} u(t) \geq 0, \qquad {}^C D_{0^+}^{\beta_2} u(t) \geq 0, \qquad \ldots, \qquad {}^C D_{0^+}^{\beta_{n-1}} u(t) \geq 0.$$

Proof By expression (3) of the solution of LBVP (2) we easily see that $u(t) \geq 0$. Next, we show that ${}^C D_{0^+}^{\beta_i} u(t) \geq 0$, $i = 1, 2, \ldots, n-1$.

From (3) we have

$$ {}^C D_{0^+}^{\beta_i} u(t) = \frac{1}{\Gamma(i-\beta_i)} \int_0^t (t-s)^{i-\beta_i-1} u^{(i)}(s)\, ds, \tag{6}$$

where

$$u^{(i)}(s) = \int_0^1 G_s^{(i)}(s,r) h(r)\, dr,$$

and $G_s^{(i)}(s,r)$ is the ith-order partial derivative of $G(s,r)$ to s, which is given by

$$G_s^{(i)}(s,r) = \begin{cases} \frac{(s-r)^{\alpha-1-i}}{\Gamma(\alpha-i)} + \frac{s^{n-1-i}(1-r)^{\alpha+\gamma-1}\rho\Gamma(n+\gamma)}{\Gamma(n-i)(\Gamma(n+\gamma)-\rho)\Gamma(\alpha+\gamma)}, & 0 \leq r \leq s \leq 1, \\ \frac{s^{n-1-i}(1-r)^{\alpha+\gamma-1}\rho\Gamma(n+\gamma)}{\Gamma(n-i)(\Gamma(n+\gamma)-\rho)\Gamma(\alpha+\gamma)}, & 0 \leq s \leq r \leq 1. \end{cases} \tag{7}$$

Consequently, (6) becomes

$$ {}^C D_{0^+}^{\beta_i} u(t) = \frac{1}{\Gamma(i-\beta_i)} \int_0^t \int_0^1 (t-s)^{i-\beta_i-1} G_s^{(i)}(s,r) h(r)\, dr\, ds. \tag{8}$$

From (7) we see that

$$G_s^{(i)}(s,r) \geq 0, \quad s, r \in I, i = 1, 2, \ldots, n-1.$$

Combining (8) and this inequality, we have

$$ {}^C D_{0^+}^{\beta_i} u(t) \geq 0, \quad i = 1, 2, \ldots, n-1,$$

and the proof is completed. □

Now, by expression (3) of the solution to LBVP (2) we easily see that problem (1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f\bigl(s, u(s), {}^C D_{0^+}^{\beta_1} u(s), {}^C D_{0^+}^{\beta_2} u(s), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(s)\bigr)\, ds := Tu(t). \tag{9}$$

Therefore, the solution of problem (1) is equivalent to the fixed point of operator T. Next, we give a comparison theorem.

Lemma 5 (Comparison result) *If* $u(t) \in \mathrm{AC}^n(I)$ *satisfies*

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) \geq 0, & 0 < t < 1, n-1 < \alpha < n, \\ u^{(j)}(0) = 0, & u^{(n-1)}(0) \geq \rho I_{0^+}^{\gamma} u(1), \quad j = 0, 1, \ldots, n-2, \end{cases}$$

then $u(t) \geq 0$, $t \in I$.

Proof By Lemma 3 we know that LBVP (2) has a unique solution $u(t) = \int_0^1 G(t,s)h(s)\,ds$. From (4) it is easy to verify that Green's function $G(t,s) \geq 0$, $t,s \in I$. Let $h(t) \in C^+(I)$. Then $u(t) \geq 0$, $t \in I$. □

According to the comparison result of Lemma 5, we give the definition of upper solution and lower solutions.

Definition 6 If $v \in \mathrm{AC}^n(I)$ satisfies

$$\begin{cases} {}^C D_{0^+}^{\alpha} v(t) \leq f(t, v(t), {}^C D_{0^+}^{\beta_1} v(t), {}^C D_{0^+}^{\beta_2} v(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} v(t)), & 0 < t < 1, \\ v^{(j)}(0) = 0, \qquad v^{(n-1)}(0) \leq \rho I_{0^+}^{\gamma} v(1), & j = 0,1,\ldots,n-2, \end{cases}$$

then we call v a lower solution of problem (1). If $w \in \mathrm{AC}^n(I)$ satisfies

$$\begin{cases} {}^C D_{0^+}^{\alpha} w(t) \geq f(t, v(t), {}^C D_{0^+}^{\beta_1} w(t), {}^C D_{0^+}^{\beta_2} w(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} w(t)), & 0 < t < 1, \\ w^{(j)}(0) = 0, \qquad w^{(n-1)}(0) \geq \rho I_{0^+}^{\gamma} w(1), & j = 0,1,\ldots,n-2, \end{cases}$$

then we call w an upper solution of problem (1).

3 Main results

Theorem 7 *Let v, w be lower solution and upper solutions of problem* (1) *such that $v(t) \leq w(t)$ for all $t \in I$. Assume that the nonlinear term $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies the following assumption*:

(H) *For all $t \in I$, $x_0, y_0 \in [v,w]$ and $x_i, y_i \in [{}^C D_{0^+}^{\beta_i} v, {}^C D_{0^+}^{\beta_i} w]$, $i = 1,2,\ldots,n-1$, such that $x_0 \geq y_0$, $x_i \geq y_i$, we have*

$$f(t, x_0, x_1, x_2, \ldots, x_{n-1}) \geq f(t, y_0, y_1, y_2, \ldots, y_{n-1}).$$

Then problem (1) *has a minimum solution $\underline{u}$ and maximum solution $\overline{u}$ between v and w.*

Proof Denote

$$D = \{u \in \mathrm{AC}^{n-1}(I) \mid v \leq u \leq w, {}^C D_{0^+}^{\beta_i} v \leq {}^C D_{0^+}^{\beta_i} u \leq {}^C D_{0^+}^{\beta_i} w, i = 1,2,\ldots,n-1\}.$$

Then $D \subset \mathrm{AC}^{n-1}(I)$ is a nonempty, convex, and closed set. Define the operator $F : D \to \mathrm{AC}(I)$ as follows:

$$F(u)(t) = f(t, u(t), {}^C D_{0^+}^{\beta_1} u(t), {}^C D_{0^+}^{\beta_2} u(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)), \quad t \in I, u \in D. \tag{10}$$

From the continuity of f we easily see that $F : D \to \mathrm{AC}(I)$ is a continuous operator that maps bounded sets into bounded sets. By Lemma 3 we know that the composite mapping $S \circ F : D \to \mathrm{AC}^{n-1}(I)$ is a completely continuous operator. Therefore, by (9), for every $u \in D$, we have $Tu = (S \circ F)(u)$, and $T : D \to \mathrm{AC}^{n-1}(I)$ is a completely continuous operator. Then the solution of problem (1) is equivalent to the fixed point of operator T defined by (9). We the proof in three steps.

Step 1: $T : D \to D$ is an increasing operator.

For $u \in D$, suppose that $x = Tu = (S \circ F)(u)$. Letting $h = F(u)$, we know that $x = Sh$ is a solution of LBVP (2). Then $x \in \mathrm{AC}^n(I)$ satisfies

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) = f(t, u(t), {}^C D_{0^+}^{\beta_1} u(t), {}^C D_{0^+}^{\beta_2} u(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)), \quad t \in I, \\ x^{(j)}(0) = 0, \qquad x^{(n-1)}(0) = \rho I_{0^+}^{\gamma} x(1), \quad j = 0, 1, \ldots, n-2. \end{cases} \tag{11}$$

Thus, using the definition of upper and lower solutions and condition (H), we have

$$\begin{aligned} {}^C D_{0^+}^{\alpha} (w - x)(t) \geq{}& f\bigl(t, w(t), {}^C D_{0^+}^{\beta_1} w(t), {}^C D_{0^+}^{\beta_2} w(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} w(t)\bigr) \\ &- f\bigl(t, u(t), {}^C D_{0^+}^{\beta_1} u(t), {}^C D_{0^+}^{\beta_2} u(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)\bigr) \geq 0; \\ (w - x)^{(j)}(0) = 0, \qquad & (w - x)^{(n-1)}(0) \geq \rho I_{0^+}^{\gamma} (w - x)(1), \quad j = 0, 1, \ldots, n-2. \end{aligned}$$

Then, by Lemma 5 we have

$$(w - x) \geq 0, \qquad {}^C D_{0^+}^{\beta_i} (w - x) \geq 0, \quad i = 1, 2, \ldots, n-1.$$

Further, we have

$$x \leq w, \qquad {}^C D_{0^+}^{\beta_i} x \leq {}^C D_{0^+}^{\beta_i} w, \quad i = 1, 2, \ldots, n-1.$$

Similarly,

$$\begin{aligned} {}^C D_{0^+}^{\alpha} (x - v)(t) \geq{}& f\bigl(t, u(t), {}^C D_{0^+}^{\beta_1} u(t), {}^C D_{0^+}^{\beta_2} u(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)\bigr) \\ &- f\bigl(t, v(t), {}^C D_{0^+}^{\beta_1} v(t), {}^C D_{0^+}^{\beta_2} v(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} v(t)\bigr) \geq 0; \\ (x - v)^{(j)}(0) = 0, \qquad & (x - v)^{(n-1)}(0) \geq \rho I_{0^+}^{\gamma} (x - v)(1), \quad j = 0, 1, \ldots, n-2. \end{aligned}$$

From Lemma 5 we have

$$(x - v) \geq 0, \quad {}^C D_{0^+}^{\beta_i} (x - v) \geq 0, \quad i = 1, 2, \ldots, n-1.$$

Namely,

$$v \leq x, \quad {}^C D_{0^+}^{\beta_i} v \leq {}^C D_{0^+}^{\beta_i} x, \quad i = 1, 2, \ldots, n-1.$$

Hence,

$$v \leq Tu \leq w, \quad {}^C D_{0^+}^{\beta_i} v \leq {}^C D_{0^+}^{\beta_i} (Tu) \leq {}^C D_{0^+}^{\beta_i} w, \quad i = 1, 2, \ldots, n-1. \tag{12}$$

This implies that $T : D \to D$.

For every $u_1, u_2 \in D$, and

$$v \leq u_1 \leq u_2 \leq w, \qquad {}^C D_{0^+}^{\beta_i} v \leq {}^C D_{0^+}^{\beta_i} u_1 \leq {}^C D_{0^+}^{\beta_i} u_2 \leq {}^C D_{0^+}^{\beta_i} w.$$

Assume that $x_1 = Tu_1$ and $x_2 = Tu_2$, this implies that x_1 and x_2 satisfy (11), respectively. Then, from condition (H) we have

$$\begin{aligned}
&{}^C D_{0^+}^{\alpha}(x_2 - x_1)(t) = f\bigl(t, u_2(t), {}^C D_{0^+}^{\beta_1} u_2(t), {}^C D_{0^+}^{\beta_2} u_2(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u_2(t)\bigr) \\
&\qquad\qquad - f\bigl(t, u_1(t), {}^C D_{0^+}^{\beta_1} u_1(t), {}^C D_{0^+}^{\beta_2} u_1(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u_1(t)\bigr) \geq 0; \\
&(x_2 - x_1)^{(j)}(0) = 0, \qquad (x_2 - x_1)^{(n-1)}(0) \geq \rho I_{0^+}^{\gamma}(x_2 - x_1)(1), \quad j = 0, 1, \ldots, n-2.
\end{aligned}$$

By Lemma 5 we have

$$x_2 - x_1 \geq 0, \qquad {}^C D_{0^+}^{\beta_i}(x_2 - x_1) \geq 0, \quad i = 1, \ldots, n-1,$$

namely,

$$Tu_1 \leq Tu_2, \qquad {}^C D_{0^+}^{\beta_i}(Tu_1) \leq {}^C D_{0^+}^{\beta_i}(Tu_2), \quad i = 1, \ldots, n-1.$$

Therefore, T is an increasing operator.

Step 2: Problem (1) has solutions between v and w.

Define two iterative sequences $\{v_n\}$ and $\{w_n\}$ starting from $v_0 = v$ and $w_0 = w$, respectively, by the following procedure

$$v_n = Tv_{n-1}, \qquad w_n = Tw_{n-1}, \quad n = 1, 2, \ldots. \tag{13}$$

This implies that $\{v_n\}$, $\{w_n\}$ satisfy the following monotonous conditions

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0, \tag{14}$$

$${}^C D_{0^+}^{\beta_i} v_0 \leq {}^C D_{0^+}^{\beta_i} v_1 \leq \cdots \leq {}^C D_{0^+}^{\beta_i} v_n \leq {}^C D_{0^+}^{\beta_i} w_n \leq \cdots \leq {}^C D_{0^+}^{\beta_i} w_1 \leq {}^C D_{0^+}^{\beta_i} w_0, \tag{15}$$

where $i = 1, \ldots, n-1$. Namely, $\{v_n\}$, $\{{}^C D_{0^+}^{\beta_i} v_n\}$ are increasing sequences in $[v, w]$, $[{}^C D_{0^+}^{\beta_i} v, {}^C D_{0^+}^{\beta_i} w]$, and $\{w_n\}$, $\{{}^C D_{0^+}^{\beta_i} w_n\}$ are decreasing sequences in $[v, w]$, $[{}^C D_{0^+}^{\beta_i} v, {}^C D_{0^+}^{\beta_i} w]$, respectively. By the compactness of T we easily see that $\{v_n\}, \{w_n\} \subset T(D)$ are relatively compact in $\mathrm{AC}^{n-1}(I)$, which means that they have at least one uniformly convergent subsequence, respectively. From the monotonicity of $\{v_n\}$, $\{w_n\}$ we obtain that $\{v_n\}$, $\{w_n\}$ are convergent in $\mathrm{AC}^{n-1}(I)$, which implies that there exist $\underline{u}, \overline{u} \in \mathrm{AC}^{n-1}$ such that $v_n \to \underline{u}$, $w_n \to \overline{u}$. Since D is a convex closed set, we also obtain $\underline{u}, \overline{u} \in D$. Further, by the continuity of T we know that $\underline{u} = T\underline{u}$, $\overline{u} = T\overline{u}$. Therefore, $\underline{u}$ and $\overline{u}$ are solutions of problem (1).

Step 3: We show that $\underline{u}$ and $\overline{u}$ are minimum and maximum solutions between v and w, respectively.

Suppose that $u \in D$ is an arbitrary solution of problem (1). Then u satisfies

$$v \leq u \leq w, \qquad {}^C D_{0^+}^{\beta_i} v \leq {}^C D_{0^+}^{\beta_i} u \leq {}^C D_{0^+}^{\beta_i} w, \quad i = 1, 2, \ldots, n-1. \tag{16}$$

Applying to T to (16), we have

$$T^n v \leq T^n u \leq T^n w, \qquad {}^C D_{0^+}^{\beta_i}\bigl(T^n v\bigr) \leq {}^C D_{0^+}^{\beta_i}\bigl(T^n u\bigr) \leq {}^C D_{0^+}^{\beta_i}\bigl(T^n w\bigr), \quad i = 1, 2, \ldots, n-1.$$

Further, we have

$$v_n \le u \le w_n, \qquad {}^C D_{0^+}^{\beta_i} v_n \le^C D_{0^+}^{\beta_i} u \le^C D_{0^+}^{\beta_i} w_n.$$

Letting $n \to \infty$, we obtain

$$\underline{u} \le u \le \overline{u}, \qquad {}^C D_{0^+}^{\beta_i} \underline{u} \le^C D_{0^+}^{\beta_i} u \le^C D_{0^+}^{\beta_i} \overline{u}, \quad i = 1,2,\ldots,n-1.$$

Thus, we see that $\underline{u}$, $\overline{u}$ are minimum and maximum solutions between v and w, respectively. The proof is complete. □

By the proof procedure of Theorem 7, we have the following result.

Corollary 8 *Let v, w be lower and upper solutions of problem* (1) *such that $v(t) \le w(t)$ for $t \in I$. Assume that the nonlinear term $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies assumption* (H). *Then using the linear iterative equation starting from $u_0 = v$ and $u_0 = w$, respectively,*

$$\begin{cases} {}^C D_{0^+}^{\alpha} u_n(t) = f(t, u_{n-1}(t), {}^C D_{0^+}^{\beta_1} u_{n-1}(t), \ldots, {}^C D_{0^+}^{\beta_{n-1}} u(t)), \quad 0 < t < 1, \\ u_n^{(j)}(0) = 0, \qquad u_n^{(n-1)}(0) = \rho I_{0^+}^{\gamma} u_n(1), \quad j = 0,1,\ldots,n-2, \end{cases}$$

we define iterative sequences $\{v_n\}$, $\{w_n\}$. By this procedure we can obtain

$$\lim_{n\to\infty} v_n(t) = \underline{u}, \qquad \lim_{n\to\infty} w_n(t) = \overline{u},$$
$$\lim_{n\to\infty} \bigl({}^C D_{0^+}^{\beta_i} v_n(t)\bigr) =^C D_{0^+}^{\beta_i} \underline{u}, \qquad \lim_{n\to\infty} \bigl({}^C D_{0^+}^{\beta_i} w_n(t)\bigr) =^C D_{0^+}^{\beta_i} \overline{u},$$

uniformly for every $t \in I$, where $\underline{u}$, $\overline{u}$ are minimum and maximum solutions between v and w, $i = 1,2,\ldots,n-1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally and approved the final version of the manuscript.

Acknowledgements
The authors are very grateful to the anonymous referees for their valuable suggestions.
Research supported by NNSFs of China (11501455, 11661071), Key Project of Gansu Provincial National Science Foundation (1606RJZA015) and Project of NWNU-LKQN-14-6.

Received: 23 September 2016 Accepted: 21 November 2016 Published online: 01 December 2016

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