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Two kinds of discrete integrable hierarchies of evolution equations and some algebraic-geometric solutions

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Abstract
With the help of a loop algebra we first present a (1 + 1)-dimensional discrete integrable hierarchy with a Hamiltonian structure and generate a (2 + 1)-dimensional discrete integrable hierarchy, respectively. Then we obtain a new differential-difference integrable system with three-potential functions, whose algebraic-geometric solution is derived from the theory of algebraic curves, where we construct the new elliptic coordinates to straighten out the continuous and discrete flows by introducing the Abel maps as well as the Riemann-Jacobi inversion theorem.

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1 Introduction

Integrable nonlinear lattice systems have important applications and rich mathematical structures in mathematical physics, statistical physics, and quantum physics. For example, the Toda lattice equation governs a system of unit masses connected by nonlinear springs whose restoring force is exponential. The equation has been proved to have integrability properties, such as a Lax pair, the Hamiltonian structure, infinite many conservation laws, and so on [1, 2]. The Toda lattice was also solved by using the Casoratian technique systematically on rational or soliton or complex solutions [3, 4]. Therefore, it is interesting how to generate integrable nonlinear lattice systems associated with mathematics and physics by various methods. Suris [5] once derived a new lattice equation related to the relativistic Toda lattice hierarchy via a highly non-trivial Bäcklund transformation. Tu Guizhang [6] applied a compatibility condition of spectral problems and some Lie algebras to propose a powerful method for generating integrable differential-difference hierarchies and the corresponding Hamiltonian structures. Based on the scheme, some related integrable nonlinear lattice hierarchies were obtained; *e.g.* see [7–11]. In the case where lattice equations including the positive and negative lattices by using the semi-direct sums of Lie algebras have been present [12, 13], their mathematical structures such as Hamiltonian structures usually investigated by the variational identity [14]. Ablowitz *et al.* [15] considered some exact linearization of difference equations; Nijhoff and Papageorgiou [16] studied similarity reductions; Levi *et al.* [17] investigated some symmetries of differential and difference equations; Ablowitz and Ladik [18] obtained some differential-difference equations and applied Fourier analysis to review their some integrable properties; Cao Cewen *et al.* [19] applied the nonlinearization method to importantly pave the way for generating differential-difference equations and algebraic-geometric solutions of (1 + 1)-dimensional and (2 + 1)-dimensional difference equations. Next Geng and Dai [20] proposed some new (2 + 1)-dimensional discrete models and obtained some algebraic-geometric solutions by applying the nonlinearization method. Based on this, Geng *et al.* [21–27] further improved the method so as to conveniently investigate algebraic-geometric solutions of differential and difference equations by introducing a new matrix consisting of fundamental solutions of spectral problems which satisfy discrete zero-curvature equations. With the help of the nonlinearization method, some interesting work on algebraic-geometric solutions was performed; *e.g.* see [28, 29].

As for as non-isospectral integrable lattice hierarchies are concerned, as is well known, less work has been done. Gordoa, Pickering and Zhu [30] made great progress in the aspect of constructing new non-isospectral lattice hierarchies in 2 + 1 dimensions. Based on this, Pickering, Zhu [31] constructed two (2 + 1)-dimensional discrete linear spectral problems and generalized some known lattice equations. In the paper, we make use of a loop algebra of the Lie algebra A_1 to deduce a (1 + 1)-dimensional discrete integrable hierarchy and a (2 + 1)-dimensional discrete hierarchy, respectively. Furthermore, we investigate their Hamiltonian structures by the trace identity. The (1 + 1)-dimensional discrete integrable hierarchy obtained in the paper can be reduced to a new (1 + 1)-dimensional integrable nonlinear difference system with three-potential functions, and the (2 + 1)-dimensional discrete integrable hierarchy presented in the paper is obtained by a non-isospectral Lax pair based on the loop algebra and a zero-curvature equation. Finally, we generate the algebraic-geometric solution of the reduced discrete integrable system by introducing Abel coordinates and the Riemann-Jacobi inversion theorem. The latter was once used to investigate binary constrained flows and separation of variables in [32].

2 Two integrable differential-difference hierarchies of evolution equations

We presented a loop algebra of the Lie algebra A_1 as follows in [32]:

$$\tilde{g} = \operatorname{span}\{h_1(n), h_2(n), e(n), f(n)\},$$

where

$$h_1(n) = h_1\lambda^{2n}, \qquad h_2(n) = h_2\lambda^{2n}, \qquad e(n) = e\lambda^{2n-1}, \qquad f(n) = f\lambda^{2n-1},$$
$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which has the commutative operations

$$\begin{aligned} &[h_1, h_2] = 0, \qquad [h_1, e] = e, \qquad [h_1, f] = -f, \qquad [h_2, e] = -e, \\ &[h_2, f] = f, \qquad [e, f] = h \equiv h_1 - h_2, \qquad [h, e] = 2e, \qquad [h, f] = -2f. \end{aligned} \tag{1}$$

In general, we usually apply multiplication operations among elements of the Lie algebra $g = \mathrm{span}\{h_1, h_2, e, f\}$. It is easy to see that

$$\begin{aligned}
&h_1h_1 = h_1, \quad h_2h_2 = h_2, \quad h_1h_2 = h_2h_1 = ee = ff = 0, \quad h_1e = e, \quad eh_1 = 0, \\
&h_1f = 0, \quad fh_1 = f, \quad h_2f = f, \quad fh_2 = 0, \quad h_2e = 0, \quad eh_2 = e, \\
&ef = h_1, \quad fe = h_2.
\end{aligned} \tag{2}$$

In [27], we changed the form of the discrete zero-curvature equation as follows:

$$(\Delta V)U_n = [U_n, V], \tag{3}$$

where $\Delta = E - 1, Ef(n) = f(n+1), U_n$ and V are Lax matrices which appear in the spectral problems

$$\varphi_{n+1} = U_n\varphi_n, \qquad \frac{d\varphi_n}{dt} = V\varphi_n, \qquad \varphi_n = \varphi(n,t). \tag{4}$$

Equation (3) is similar to the stationary zero-curvature equation in continuous spectral problems,

$$V_x = [U, V].$$

The reason why we adopt equation (3) to investigate discrete integrable hierarchies aims at applying the Tu scheme [33] to generate lattice integrable hierarchies, which has been a current way for generating integrable hierarchies of evolution equations. Based on the above version, we had obtained the well-known Toda lattice hierarchy and a differential-difference hierarchy; and further their expanding integrable models were produced, respectively. In the following, we choose U_n and V to be of the form [32]

$$\begin{aligned}
&U_n = h_1(1) + q_nh_2(0) + r_ne(1) + s_nf(1), \\
&V = \sum_{n\geq 0}\big[a_n\big(h_1(-n) - h_2(-n)\big) + b_ne(-n) + c_nf(-n)\big],
\end{aligned}$$

and apply equation (3) and the discrete zero-curvature equation,

$$\frac{dU_n}{dt_m} = (\Delta V_{(m)})U_n - [U_n, V_{(m)}], \tag{5}$$

to obtain the following integrable discrete hierarchy:

$$\begin{cases}
q_{n,t_m} = -r_nc_m^{(1)} + s_nb_m, \\
r_{n,t_m} = b_m, \\
s_{n,t_m} = -c_m^{(1)},
\end{cases} \tag{6}$$

where

$$V_{(m)} = \sum_{n=0}^{m}\big[a_n\big(h_1(m-n) - h_2(m-n)\big) + b_ne(m-n) + c_nf(m-n)\big] - b_me(0) - c_mf(0).$$

Assume $a_0 = \frac{1}{2}, b_0 = r_n, c_0 = s_{n-1}$, then when $m = 0$, equation (6) can be reduced to

$$q_{n,t_0} = s_n r_n - r_n s_{n-1}, \qquad r_{n,t_0} = r_n, \qquad s_{n,t_0} = -s_n. \tag{7}$$

When $m = 1$, equation (6) gives rise to ($t_1 = t$):

$$\begin{cases} q_{n,t} = s_n q_n r_{n+1} - q_n r_n s_{n-1}, \\ r_{n,t} = q_n r_{n+1} - r_n r_{n+1} s_n - r_n^2 s_{n-1}, \\ s_{n,t} = s_n^2 r_{n+1} + s_n s_{n-1} r_n - q_n s_{n-1}, \end{cases}$$

which can be written as

$$\begin{cases} \partial_t \ln q_n = s_n r_{n+1} - r_n s_{n-1}, \\ \partial_t \ln r_n = -r_{n+1} s_n - r_n s_{n-1} + q_n \frac{r_{n+1}}{r_n}, \\ \partial_t \ln s_n = s_{n-1} r_n + s_n r_{n+1} - q_n \frac{s_{n-1}}{s_n}. \end{cases} \tag{8}$$

In the following, we still make use of the loop algebra $\tilde{g}$ to generate $(2+1)$-dimensional non-isospectral differential-difference hierarchy by adopting the method presented in [34–36].

Consider the non-isospectral Lax problem

$$\begin{cases} \psi_{n+1}(\lambda) = U_n(q_n, r_n, s_n, \lambda)\psi_n(\lambda), \\ \frac{d\psi_n(\lambda)}{dt} = \omega(\lambda)\frac{d\psi_n(\lambda)}{dy} + V_n^{(m)}(q_n, r_n, s_n, \lambda)\psi_n(\lambda), \end{cases} \tag{9}$$

where

$$\lambda = \lambda(t, y), \qquad \frac{d\lambda}{dt} = \lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda).$$

The compatibility condition of (9) yields

$$\frac{\partial U_n}{\partial t} - \omega(\lambda)\frac{\partial U_n}{\partial y} + \beta(\lambda)\frac{\partial U_n}{\partial \lambda} + \left(\Delta V_n^{(m)}\right)U_n - \left[U_n, V_n^{(m)}\right] = 0. \tag{10}$$

Now we take

$$V_n^{(m)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{cases} A = \sum_{j=0}^m a_j(n,t,y)\lambda^{2(m-j)}, \qquad B = \sum_{j=0}^m b_j(n,t,y)\lambda^{2(m-j)+1}, \\ C = \sum_{j=0}^m c_i(n,t,y)\lambda^{2(m-j)+1}, \qquad D = \sum_{j=0}^m d_j(n,t,y)\lambda^{2(m-j)}. \end{cases} \tag{11}$$

Then equation (3) admits

$$\begin{cases} \lambda^2 EA + \lambda s_n EB - \lambda^2 A - \lambda r_n C = 2\lambda\beta(\lambda), \\ \lambda \dot{r_n} - \lambda\omega(\lambda) r_{n,y} + \beta(\lambda) r_n = \lambda r_n EA + q_n EB - \lambda^2 B - \lambda r_n D, \\ \lambda \dot{s_n} - \lambda\omega(\lambda) s_{n,y} + \beta(\lambda) s_n = \lambda^2 EC + \lambda s_n ED - \lambda s_n A - q_n C, \\ \dot{q_n} - \omega(\lambda) q_{n,y} = \lambda r_n EC + q_n ED - \lambda s_n B - q_n D. \end{cases}$$

Set

$$\beta(\lambda) = \sum_{j=0}^{m} \beta_j \lambda^{2(m-j)+1}, \qquad \omega(\lambda) = \lambda^{2m}. \tag{12}$$

Substituting (11) and (12) into equation (5) yields

$$\begin{cases} -r_{n,y} + \beta_0 r_n = r_n E a_0 - b_1 + q_n E b_0 - r_n d_0, \\ r_n \beta_j = r_n E a_j + q_n E b_j - r_n d_j - b_{j+1}, \\ -s_{n,y} + \beta_0 s_n = E c_1 + s_n E d_0 - s_n a_0 - q_n c_0, \\ s_n \beta_j = E c_{j+1} + s_n E d_j - s_n a_j - q_n c_j, \\ -q_{n,y} = r_n E c_1 + q_n E d_0 - s_n b_1 - q_n d_0, \\ r_n E c_{j+1} + q_n E d_j - s_n b_{j+1} - q_n d_j = 0, \quad j = 1, \ldots, m, \end{cases} \tag{13}$$

and

$$\begin{cases} r_{n,t_m} = -r_n \beta_m + r_n E a_m + q_n E b_m - r_n d_m, \\ s_{n,t_m} = s_n E d_m - s_n a_m - q_n c_m, \\ \Delta q_{n,t_m} = q_n \Delta d_m, \quad j = 1, 2, \ldots, m. \end{cases} \tag{14}$$

From equation (13), we find that

$$(q_n - s_n r_n)\Delta d_j = r_n s_n \Delta a_j + q_n s_n E b_j - q_n r_n c_j - 2 s_n r_n \beta_j, \quad j = 1, \ldots, m. \tag{15}$$

For equation (15) to be solvable locally, we let $a_j = -d_j$, then equations (13)-(15) can be simplified, respectively,

$$\begin{cases} -r_{n,y} + \beta_0 r_n = r_n E a_0 - b_1 + q_n E b_0 + r_n a_0, \\ r_n \beta_j = r_n E a_j + q_n E b_j + r_n a_j - b_{j+1}, \\ -s_{n,y} + \beta_0 s_n = E c_1 - s_n E a_0 - s_n a_0 - q_n c_0, \\ s_n \beta_j = E c_{j+1} - s_n E a_j - s_n a_j - q_n c_j, \\ -q_{n,y} = r_n E c_1 - q_n E a_0 - s_n b_1 - q_n d_0, \\ r_n E c_{j+1} - q_n E a_{j+1} - s_n b_{j+1} + q_n a_j = 0, \quad j = 1, \ldots, m, \end{cases} \tag{16}$$

and

$$\begin{cases} r_{n,t_m} = -r_n\beta_m + r_nEa_m + r_na_m + q_nEb_m, \\ s_{n,t_m} = -s_nEa_m - s_na_m - q_nc_m, \\ \Delta q_{n,t_m} = -q_n\Delta a_m, \quad j = 1,2,\ldots,m, \end{cases} \tag{17}$$

$$q_n\Delta a_j = -q_ns_nEb_j + q_nr_nc_j + 2s_nr_n\beta_j, \quad j = 1,2,\ldots,m. \tag{18}$$

Assume $b_0 = \frac{1}{2}s_{n-1}^{-1}, c_0 = -\frac{1}{2}r_n^{-1}$, then one infers from (18) that

$$a_0 = -n + 2\beta_0\Delta^{-1}\frac{s_nr_n}{q_n}.$$

In terms of (16), we have

$$\begin{aligned} b_1 &= -r_n + r_{n,y} - \beta_0r_n + 2\beta_0\frac{s_nr_n^2}{q_n} - \frac{1}{2}\frac{q_n}{s_n}, \\ c_1 &= (\beta_0 - 1)s_{n-1} - s_{n-1,y} + 2\beta_0\frac{r_{n-1}s_{n-1}^2}{q_{n-1}} + \frac{q_{n-1}}{2r_{n-1}}, \\ \Delta a_1 &= -\Delta r_ns_{n-1} + \beta_0(E+1)r_ns_{n-1} - s_nr_{n+1,y} - r_ns_{n-1,y} + \frac{1}{2}q_{n+1}s_ns_{n+1}^{-1} + \frac{1}{2}q_{n-1}r_nr_{n-1}^{-1} \\ &\quad + 2\beta_0r_nr_{n-1}s_{n-1}^2q_{n-1}^{-1} - 2\beta_0s_ns_{n+1}q_{n+1}^{-1}r_{n+1}^2. \end{aligned} \tag{19}$$

Substituting the above results into (17) yields a reduction of the (2 + 1)-dimensional non-isospectral discrete hierarchy (17),

$$\begin{cases} r_{n,t_1} = -\beta_1r_n - q_nr_{n+1} + q_nr_{n+1,y} - \beta_0q_nr_{n+1} + 2\beta_0\frac{q_ns_{n+1}r_{n+1}^2}{q_{n+1}} - \frac{1}{2}\frac{q_nq_{n+1}}{s_{n+1}} + r_n(E+1)a_1, \\ s_{n,t_1} = (1-\beta_0)q_ns_{n-1} + q_ns_{n-1,y} - 2\beta_0\frac{q_nr_{n-1}s_{n-1}^2}{q_{n-1}} - \frac{q_{n-1}a_n}{2r_{n-1}} - s_n(E+1)a_1, \\ \Delta q_{n,t_1} = -q_n\Delta a_1, \end{cases}$$

where a_1 is given by (19).

Remark 1 Via applying the trace identity proposed by Tu [6], we could deduce the Hamiltonian structure of the (1 + 1)-dimensional discrete integrable hierarchy (6). However, how do we search for the Hamiltonian structure of the (2 + 1)-dimensional non-isospectral discrete integrable hierarchy (14)? This is a problem worth of discussing in the future.

3 Algebraic-geometric solution of the (1 + 1)-dimensional nonlinear discrete integrable system (8)

The nonlinear discrete system (8) possesses the following Lax pair:

$$\begin{cases} E\varphi(n) = U_n\varphi(n), \qquad U_n = h_1(1) + q_nh_2(0) + r_ne(1) + s_nf(1), \\ \varphi_t(n) = V_{(1)}\varphi(n), \end{cases} \tag{20}$$

where

$$V_{(1)} = \begin{pmatrix} \frac{1}{2}\lambda^2 - r_ns_{n-1}\lambda & V_{12} \\ V_{21} & -\frac{1}{2}\lambda^2 + r_ns_{n-1}\lambda \end{pmatrix},$$

$$V_{12} = r_n\lambda^2 + \left(\lambda - \frac{1}{\lambda}\right)\left(q_n r_{n+1} - r_n r_{n+1}s_n - r_n^2 s_{n-1}\right),$$

$$V_{21} = s_{n-1}\lambda^2 + \left(\lambda - \frac{1}{\lambda}\right)\left(q_{n-1}s_{n-2} - s_{n-1}^2 r_n - s_{n-1}s_{n-2}r_{n-1}\right).$$

With the help of the approaches presented in [35, 36], we could generate Darboux-Bäcklund transformations and exact soliton solutions of equation (8). Of course, the key problem focuses on how to construct suitable Darboux matrices. The problem will be dealt in another paper.

In the following, we want to seek algebraic-geometric solutions based on theories in [19–23, 37]. We first introduce the Lenard gradient sequence $\bar{S}_j$, $0 \leq j \in \mathbf{Z}$ by the recursion equation

$$K_n\bar{S}_j(n) = J_n\bar{S}_{j+1}, \qquad J_n\bar{S}_0(n) = 0, \quad j \geq 0, \tag{21}$$

with the two operators

$$K_n = \begin{pmatrix} 0 & q_nE & 0 \\ -q_n & 0 & 0 \\ r_nE & -s_n & -q_n\Delta \end{pmatrix}, \qquad J_n = \begin{pmatrix} 0 & -1 & r_nE + r_n \\ E & 0 & -s_nE - s_n \\ r_nE & -s_n & -q_n\Delta \end{pmatrix},$$

$\bar{S}_j(n) = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})^T$.

Equation $J_n\bar{S}_0(n) = 0$ possesses a special solution as follows:

$$\bar{S}_0(n) = \begin{pmatrix} s_{n-1} \\ r_n \\ \frac{1}{2} \end{pmatrix}, \tag{22}$$

and we find that

$$\ker J_n = \left\{c\bar{S}_0(n)\right\},$$

where c is an arbitrary constant. From equation (21), we easily have

$$\bar{S}_1(n) = \begin{pmatrix} -s_{n-1}^2 r_n - s_{n-1}s_{n-2}r_{n-1} + q_{n-1}s_{n-2} \\ -r_n r_{n+1}s_n - r_n^2 s_{n-1} + q_n r_{n+1} \\ -r_n s_{n-1} \end{pmatrix}, \ldots. \tag{23}$$

It is easy to see from (21) that

$$\begin{cases} r_nEs_{j+1}^{(3)} + q_nEs_j^{(2)} - s_{j+1}^{(2)} + r_ns_{j+1}^{(3)} = 0, \\ Es_{j+1}^{(1)} - s_nEs_{j+1}^{(3)} - q_ns_j^{(1)} - s_ns_{j+1}^{(3)} = 0, \\ r_nEs_j^{(1)} - q_nEs_j^{(3)} - s_ns_j^{(2)} + q_ns_j^{(3)} = 0. \end{cases} \tag{24}$$

The $(1+1)$-dimensional integrable discrete hierarchy can be viewed as a generation of the following isospectral problems:

$$\begin{cases} \psi(n+1) = U_n\psi(n), & U_n = h_1(1) + q_nh_2(0) + r_ne(1) + s_nf(1), \\ \psi(n)_{t_m} = V_n^{(m)}\psi(n), & V_n^{(m)} = A_n^{(m)}h_1(1) + B_n^{(m)}e(1) + C_n^{(m)}f(1) - A_n^{(m)}h_2(1), \end{cases} \tag{25}$$

where

$$A_n^{(m)} = \sum_{j=0}^{m} s_j^{(3)}(n)\lambda^{2(m-j)}, \qquad B_n^{(m)} = \sum_{j=0}^{m} s_j^{(2)}(n)\lambda^{2(m-j)}, \qquad C_n^{(m)} = \sum_{j=0}^{m} s_j^{(1)}(n)\lambda^{2(m-j)}.$$

The compatibility condition of (25) admits equation (6), which can be expressed as

$$\begin{pmatrix} q_n \\ r_n \\ s_n \end{pmatrix}_{t_m} = X_m(n) = \begin{pmatrix} -r_nc_m^{(1)} + s_nb_m \\ b_m \\ -c_m^{(1)} \end{pmatrix}.$$

3.1 Decomposition of the differential-difference equations

In the subsection, we shall decompose the $(1 + 1)$-dimensional lattice system (8) into solvable ordinary differential equations. Suppose (25) has two basic solutions $\psi(n) = (\psi^{(1)}(n), \psi^{(2)}(n))^T$ and $\varphi(n) = (\varphi^{(1)}(n), \varphi^{(2)}(n))^T$. We define a Lax matrix W_n in terms of $\psi(n)$ and $\varphi(n)$, which has some generalizations in [38], by

$$W_n = \begin{pmatrix} f(n) & g(n) \\ h(n) & -f(n) \end{pmatrix} = \frac{1}{2}\big(\varphi(n)\psi(n)^T + \psi(n)\varphi(n)^T\big)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

From equation (25) we can verify that

$$W_{n+1}U_n - U_nW_n = 0, \qquad W_{n,t_m} = \big[V_n^{(m)}, W_n\big], \tag{27}$$

which means that the function $\det W_n$ is a constant independent of n and t_m. It is easy to see that equation (27) can be written as

$$\begin{cases} \lambda^2\Delta f(n) + \lambda s_nEg(n) - \lambda r_nh(n) = 0, \\ \lambda r_nEf(n) + q_nEg(n) - \lambda^2g(n) + \lambda r_nf(n) = 0, \\ \lambda^2Eh(n) - \lambda s_nEf(n) - q_nh(n) - \lambda s_nf(n) = 0, \\ \lambda r_nEh(n) - q_nEf(n) - \lambda s_ng(n) + q_nf(n) = 0, \end{cases} \tag{28}$$

and

$$\begin{cases} f(n)_{t_m} = B_n^{(m)}h(n) - C_n^{(m)}g(n), \\ g(n)_{t_m} = 2g(n)A_n^{(m)} - 2B_n^{(m)}f(n), \\ h(n)_{t_m} = 2C_n^{(m)}f(n) - 2A_n^{(m)}h(n), \end{cases} \tag{29}$$

where

$$\begin{aligned} f(n) &= \sum_{j=0}^{N} f_j(n)\lambda^{2(N-j)+2}, \qquad g(n) = \sum_{j=0}^{N} g_j(n)\lambda^{2(N-j)+1}, \\ h(n) &= \sum_{j=0}^{N} h_j(n)\lambda^{2(N-j)+1}. \end{aligned} \tag{30}$$

Substituting (30) into (28) and comparing the coefficients of the same powers of λ give rise to

$$K_n G_j(n) = J_n G_{j+1}(n), \qquad J_n G_0(n) = 0, \qquad K_n G_N(n) = 0, \tag{31}$$

where $G_j(n) = (h_j(n), g_j(n), f_j(n))^T$. It is easy to see that equation $J_n G_0(n) = 0$ has the general solution

$$G_0(n) = \alpha_0 \bar{S}_0(n), \tag{32}$$

here α_0 is a constant. Acting with $(J_n^{-1}K_n)^k$ on equation (32), we obtain

$$G_k(n) = \alpha_0 \bar{S}_k(n) + \alpha_1 \bar{S}_{k-1}(n) + \cdots + \alpha_k \bar{S}_0(n), \tag{33}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are constants. Inserting (33) into equation $K_n G_N(n) = 0$ gives a discrete stationary equation

$$\alpha_0 X_N(n) + \alpha_1 X_{N-1}(n) + \cdots + \alpha_N X_0(n) = 0, \tag{34}$$

which implies (q_n, r_n, s_n) is the finite-band solution. Assume $\alpha_0 = 1$, we can obtain from (32) and (33) that

$$\begin{cases} f_0(n) = \frac{1}{2}, \qquad g_0(n) = r_n, \qquad h_0(n) = s_{n-1}, \\ f_1(n) = -r_n s_{n-1}, \\ g_1(n) = -r_n r_{n+1} s_n - r_n^2 s_{n-1} + q_n r_{n+1}, \\ h_1(n) = -s_{n-1}^2 r_n - s_{n-1} s_{n-2} r_{n-1} + q_{n-1} s_{n-2}. \end{cases} \tag{35}$$

We apply $g(n)$ and $h(n)$ as polynomials of λ to define the elliptic coordinates $\{\mu_j(n)\}$ and $\{\nu_j(n)\}$:

$$\begin{cases} g(n) = r_n \pi_{j=1}^{N} (\lambda^2 - \mu_j(n)^2) \equiv r_n \pi_{j=1}^{N} (\tilde{\lambda} - \tilde{\mu}_j(n)), \\ h(n) = s_{n-1} (\pi_{j=1}^{N} (\lambda^2 - \nu_j(n)^2)) \equiv s_{n-1} \pi_{j=1}^{N} (\tilde{\lambda} - \tilde{\nu}_j(n)), \end{cases} \tag{36}$$

where we denote $\lambda^2, \mu_j(n)^2, \nu_j(n)^2$ by $\tilde{\lambda}, \tilde{\mu}_j(n)$ and $\tilde{\nu}_j(n)$, respectively. By comparing coefficients of the same power for λ, we have

$$\begin{cases} g_1(n) = -r_n \sum_{j=1}^{N} \tilde{\mu}_j(n), \qquad h_1(n) = -s_{n-1} \sum_{j=1}^{N} \tilde{\nu}_j(n), \\ g_2(n) = r_n \sum_{i<j} \tilde{\mu}_i(n) \tilde{\mu}_j(n), \qquad h_2(n) = s_{n-1} \sum_{i<j} \tilde{\nu}_i(n) \tilde{n} u_j(n). \end{cases} \tag{37}$$

Combined with (33), equation (37) can be written as

$$\begin{cases} r_{n+1}s_n + r_n s_{n-1} - \frac{r_{n+1}}{r_n} q_n = \sum_{j=1}^{N} \tilde{\mu}_j(n) + \alpha_1, \\ s_{n-1}r_n + s_{n-2}r_{n-1} - \frac{s_{n-2}}{s_{n-1}} q_{n-1} = \sum_{j=1}^{N} \tilde{\nu}_j(n) + \alpha_1. \end{cases} \tag{38}$$

Thus, equation (8) can be written as

$$\begin{cases} \partial_t \ln q_n = s_n r_{n+1} - r_n s_{n-1}, \\ \partial_t \ln r_n = -\sum_{j=1}^{N} \tilde{\mu}_j(n) - \alpha_1, \\ \partial_t \ln s_n = E\sum_{j=1}^{N} \tilde{\nu}_j(n) + \alpha_1. \end{cases} \tag{39}$$

Consider the function det W_n, which is a $(4N + 4)$th-order polynomial in λ:

$$-\det W_n = f^2(n) + g(n)h(n) = \frac{1}{4}\lambda^2 \mathop{\pi}_{j=1}^{2N+1}\left(\lambda^2 - \lambda_j^2\right) = \frac{1}{4}\tilde{\lambda} \mathop{\pi}_{j=1}^{2N+1}(\tilde{\lambda} - \tilde{\lambda}_j) = \frac{1}{4}R(\tilde{\lambda}). \tag{40}$$

Substituting (30) into (40) yields

$$\alpha_1 = -\frac{1}{2}\sum_{j=1}^{2N+1} \tilde{\lambda}_j.$$

One infers that

$$f(n)|_{\tilde{\lambda}=\tilde{\mu}_k(n)} = \frac{1}{2}\sqrt{R\left(\tilde{\mu}_k(n)\right)}, \qquad f(n)|_{\tilde{\lambda}=\tilde{\nu}_j(n)} = \frac{1}{2}\sqrt{R\left(\tilde{\nu}_j(n)\right)}, \tag{41}$$

and

$$\begin{cases} g(n)_{t_0}|_{\tilde{\lambda}=\tilde{\mu}_k(n)} = (2s_0^{(3)}(n)g(n) - 2f(n)s_0^{(2)}(n))|_{\tilde{\lambda}=\tilde{\nu}_j(n)} \\ \qquad\qquad = g(n)_{t_0}|_{\tilde{\lambda}=\tilde{\mu}_k(n)} = r_n(\partial_{t_0}\tilde{\mu}_k(n)) \mathop{\pi}_{i\neq j, i=1}^{N}(\tilde{\mu}_k(n) - \tilde{\mu}_i(n)), \\ h(n)_{t_0}|_{\tilde{\lambda}=\tilde{\nu}_k(n)} = (2f(n)s_0^{(1)} - 2h(n)s_0^{(3)})|_{\tilde{\lambda}=\tilde{\nu}_k(n)} = s_{n-1}(\partial_{t_0}\tilde{\nu}_k(n)) \mathop{\pi}_{i\neq j, i=1}^{N}(\tilde{\nu}_k(n) - \tilde{\nu}_i(n)), \end{cases}$$

from which we have

$$\begin{cases} \frac{\partial_{t_0}\tilde{\mu}_k(n)}{\sqrt{R(\tilde{\mu}_k(n))}} = -\frac{1}{\mathop{\pi}_{i\neq k, i=1}^{N}(\tilde{\mu}_k(n) - \tilde{\mu}_i(n))}, \\ \frac{\partial_{t_0}\tilde{\nu}_k(n)}{\sqrt{R(\tilde{\nu}_k(n))}} = \frac{1}{\mathop{\pi}_{i\neq k, i=1}^{N}(\tilde{\nu}_k(n) - \tilde{\nu}_i(n))}. \end{cases} \tag{42}$$

Taking $t = t_1$, in terms of (29), we get

$$\begin{aligned} g(n)_t|_{\tilde{\lambda}=\tilde{\mu}_k(n)} &= 2g(n)\left[\frac{1}{2}\tilde{\lambda}^2 - r_n s_{n-1}\tilde{\lambda}\right] \\ &\quad - 2f(n)\left[r_n\tilde{\lambda}\sqrt{\tilde{\lambda}}\left(-r_n r_{n+1}s_n - r_n^2 s_{n-1} + q_n r_{n+1}\right)|_{\tilde{\lambda}=\tilde{\mu}_k(n)}\right], \end{aligned} \tag{43}$$

$$\begin{aligned} h(n)_t|_{\tilde{\lambda}=\tilde{\nu}_k(n)} &= 2f\left(\tilde{\nu}_k(n)\right)\left[s_{n-1}\tilde{\nu}_k(n)\sqrt{\tilde{\nu}_k(n)}\right] \\ &\quad + \left(-s_{n-1}^2 r_n - s_{n-1}s_{n-2}r_{n-1} + q_{n-1}s_{n-2}\right)\sqrt{\tilde{\nu}_k(n)}. \end{aligned} \tag{44}$$

Again from (36) and (43), (44), we have the following ODEs:

$$\begin{cases} \frac{\partial_t \tilde{\mu}_k(n)}{\sqrt{R(\tilde{\mu}_k(n))}} = -\frac{\tilde{\mu}_k(n)-\sum_{j=1}^N \tilde{\mu}_j(n)-\alpha_1}{\underset{i\neq j,i=1}{\overset{N}{\pi}}(\tilde{\mu}_k(n)-\tilde{\mu}_i(n))}, \\ \frac{\partial_t \tilde{\nu}_k(n)}{\sqrt{R(\tilde{\nu}_k(n))}} = \frac{\tilde{\nu}_k(n)-\sum_{j=1}^N \tilde{\nu}_j(n)-\alpha_1}{\underset{i\neq j,i=1}{\overset{N}{\pi}}(\tilde{\nu}_k(n)-\tilde{\nu}_i(n))}. \end{cases} \tag{45}$$

Therefore, if $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{2N+1}$ are $2N+1$ distinct parameters, and $\tilde{\mu}_k(n), \tilde{\nu}_k(n)$ are compatible solutions of (42) and (45), then q_n, r_n, s_n determined by (38), (39) solve the (1 + 1)-dimensional lattice system (8).

3.2 Straightening out of the continuous flow

We introduce the Riemann surface Γ of the hyper-elliptic curve with genus N:

$$\xi^2 = R(\tilde{\lambda}), \qquad R(\tilde{\lambda}) = \tilde{\lambda} \overset{2N+1}{\underset{j=1}{\pi}} (\tilde{\lambda} - \tilde{\lambda}_j),$$

which has two infinite points ∞_1 and ∞_2, not branch points of Γ. We fix a set of regular cycle paths: $a_1,\ldots,a_N; b_1,\ldots,b_N$ which are independent and have the intersection numbers

$$a_k \circ a_j = b_k \circ b_j = 0, \qquad a_k \circ b_j = \delta_{kj}, \quad 1 \le k,j \le N.$$

On Γ, we choose the holomorphic differentials:

$$\tilde{\omega}_l = \frac{\tilde{\lambda}^{l-1} d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \quad l = 1,\ldots,N,$$

and we denote

$$A_{kj} = \int_{a_j} \tilde{\omega}_k, \qquad B_{kj} = \int_{b_j} \tilde{\omega}_k.$$

It can be verified that the matrices $A = (A_{kj})$ and $B = (B_{ij})$ are all $N \times N$ invertible. If we denote matrices C and τ by $C = (c_{kj}) = A^{-1}, \tau = (\tau_{kj}) = CB$, then the matrix τ can be proved to be symmetric and have positive defined imaginary part. Now we normalize $\tilde{\omega}_j$ into the new basis ω_j:

$$\omega_j = \sum_{i=1}^N c_{jl}\tilde{\omega}_l, \quad l = 1,\ldots,N,$$

so that they satisfy

$$\int_{a_k} \omega_j = \sum_{l=1}^N c_{jl} \int_{a_k} \tilde{\omega}_l = \sum_{l=1}^N c_{jl} A_{lk} = \delta_{jk}, \qquad \int_{b_k} \omega_j = \tau_{jk}. \tag{46}$$

We again introduce the Abel map $\mathcal{A}(P)$:

$$\mathcal{A}(P) = \int_{P_0}^P \omega,$$

which can be extended to the whole divisor group of $\Gamma : \mathcal{A} : \mathrm{Div}(\Gamma) \to \bar{\mathcal{J}}(\Gamma) = C^N/\mathcal{J}$, where the lattice $\mathcal{J}$ is spanned by the periodic vectors $\{\delta_k, \tau_k\}$ given by (46). The Abel-Jacobi coordinates are defined as

$$\begin{cases} \rho^{(1)}(n) = \mathcal{A}(\sum_{k=1}^N P(\tilde{\mu}_k(n))) = \sum_{k=1}^N \int_{P_0}^{P(\tilde{\mu}_k(n))} \omega, \\ \rho^{(2)}(n) = \mathcal{A}(\sum_{k=1}^N P(\tilde{\nu}_k(n))) = \sum_{k=1}^N P(\tilde{\mu}_k(n)) = \sum_{k=1}^N \int_{P_0}^{P(\tilde{\nu}_k(n))} \omega, \end{cases}$$

explicitly,

$$\begin{cases} \rho^{(1)}(n) = \sum_{k=1}^N \int_{\tilde{\lambda}(P_0)}^{\tilde{\mu}_k(n)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N c_{jl} \int_{\tilde{\lambda}(P_0)}^{\tilde{\mu}_k(n)} \frac{\tilde{\lambda}^{l-1}\, d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \\ \rho^{(2)}(n) = \sum_{k=1}^N \int_{\tilde{\lambda}(P_0)}^{\tilde{\nu}_k(n)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N c_{jl} \int_{\tilde{\lambda}(P_0)}^{\tilde{\nu}_k(n)} \frac{\tilde{\lambda}^{l-1}\, d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \end{cases} \tag{47}$$

where $\tilde{\lambda}(P_0)$ is the local coordinate of $P_0, P(\tilde{\mu}_k(n)) = (\tilde{\lambda} = \tilde{\mu}_k(n), \xi = \sqrt{R(\tilde{\mu}_k(n))}), P(\tilde{\nu}_k(n)) = (\tilde{\lambda} = \tilde{\nu}_k(n), \xi = \sqrt{R(\tilde{\nu}_k(n))}) \in \Gamma$. We obtain

$$\begin{aligned} \partial_{t_0} \rho^{(1)}(n) &= \sum_{l=1}^N \sum_{k=1}^N c_{jl} \frac{\tilde{\mu}_k(n)^{l-1} \partial_{t_0} \tilde{\mu}_k(n)}{\sqrt{R(\tilde{\mu}_k(n))}} \\ &= -\sum_{l=1}^N \sum_{k=1}^N c_{jl} \frac{\tilde{\mu}_k(n)^{l-1}}{\underset{i\neq k, i=1}{\overset{N}{\pi}} (\tilde{\mu}_k(n) - \tilde{\mu}_i(n))} = -c_{jN} \equiv \Omega_j^{(1)}, \end{aligned} \tag{48}$$

$$\begin{aligned} \partial_t \rho^{(1)}(n) &= \partial_t \sum_{l=1}^N \sum_{k=1}^N c_{jl} \int_{\tilde{\lambda}(P_0)}^{\tilde{\mu}_k(n)} \frac{\tilde{\lambda}^{l-1}\, d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}} \\ &= -\sum_{k=1}^N c_{jl} \frac{\tilde{\mu}_k(n)^{l-1} (\tilde{\mu}_k(n) - \sum_{j=1}^N \tilde{\mu}_j(n) - \alpha_1)}{\underset{i\neq k, i=1}{\overset{N}{\pi}} (\tilde{\mu}_k(n) - \tilde{\mu}_i(n))} \equiv \Omega_j^{(2)}, \quad 1 \leq j \leq N. \end{aligned} \tag{49}$$

Similarly, we can obtain

$$\partial_{t_0} \rho^{(2)}(n) = -\Omega_j^{(1)}, \qquad \partial_t \rho^{(2)}(n) = -\Omega_j^{(2)}, \quad j = 1, 2, \ldots, N.$$

Remark 2 Equation (49) is a finite sum, but we do not know how to express it by some linear combinations of the elements c_{ij}.

3.3 Straightening out of the discrete flow

Suppose the fundamental solution matrix of the first equation in (25) is given by [24]

$$Q_n = \big(\phi(n), \tilde{\phi}(n)\big) = \begin{pmatrix} \phi^{(1)} & \hat{\phi}^{(1)}(n) \\ \phi^{(2)} & \hat{\phi}^{(2)}(n) \end{pmatrix}, \qquad Q_0 = I,$$

which satisfies

$$Q_{n+1} = U_n U_{n-1} \ldots U_0. \tag{50}$$

We can compute that

$$\phi^{(1)}(1) = \lambda^2, \qquad \phi^{(2)}(1) = \lambda s_0, \qquad \hat{\phi}^{(1)}(1) = \lambda r_0, \qquad \hat{\phi}^{(2)}(1) = q_0,$$
$$\phi^{(1)}(2) = \lambda^4 + \lambda^2 r_1 s_0, \qquad \phi^{(2)}(2) = \lambda^3 s_1 + \lambda q_1 s_0,$$
$$\hat{\phi}^{(1)}(2) = \lambda^3 r_0 + \lambda r_1 q_0, \qquad \hat{\phi}^{(2)} = \lambda^2 s_1 r_0 + q_1 q_0, \ldots.$$

Assume δ is the eigenvalue of the Lax matrix W_n in the solution space of equation $\psi(n+1) = U_n\psi(n)$, which is invariant under the action of W_n due to $(EW_n)U_n = U_nW_n$. The corresponding eigenfunction is $\psi(n)$ called the Baker function which obeys

$$\psi(n+1) = U_n\psi(n), \qquad W_n\psi(n) = \delta\psi(n). \tag{51}$$

It is easy to see that

$$\det|\delta - W_n| = \delta^2 - f^2(n) - g(n)h(n) = 0$$

has two eigenvalues $\delta^{\pm} = \pm\delta$, where

$$\delta = \sqrt{f^2(n) + g(n)h(n)} = \frac{1}{2}\sqrt{R(\tilde{\lambda})}. \tag{52}$$

The corresponding Baker function can be taken as

$$\phi^{\pm}(n) = \phi(n) + b^{\pm}\hat{\phi}(n), \qquad b^{\pm} = \frac{\pm\delta - f(0)}{g(0)},$$

or

$$\phi^{\pm}(n) = \phi(n) + \bar{b}^{\pm}\hat{\phi}(n), \qquad \bar{b}^{\pm} = \frac{h(0)}{\pm\delta + f(0)};$$
$$\hat{\phi}^{\pm}(n) = c^{\pm}\phi(n) + \hat{\phi}(n), \qquad c^{\pm} = \frac{\pm\delta + f(0)}{h(0)},$$

or

$$\hat{\phi}^{\pm}(n) = \bar{c}^{\pm}\phi(n) + \hat{\phi}(n), \qquad \bar{c}^{\pm} = \frac{g(0)}{\pm\delta - f(0)}.$$

By following [19], we can prove the following formula of Dubrovin-Novikov type:

$$\begin{cases} p^+(n)p^-(n) = \frac{r_n}{r_0}\prod_{j=1}^{N}\frac{\tilde{\lambda}-\tilde{\mu}_j(n)}{\tilde{\lambda}-\tilde{\mu}_j(0)}, \\ q^+(n)q^-(n) = \frac{s_{n-1}}{s_{-1}}\prod_{j=1}^{N}\frac{\tilde{\lambda}-\tilde{\nu}_j(n)}{\tilde{\lambda}-\tilde{\nu}_j(0)}, \end{cases} \tag{53}$$

where

$$\begin{cases} p^+(n) = \phi^{(1)}(n) + b^+\hat{\phi}^{(2)}(n), \qquad p^-(n) = \phi^{(1)}(n) + b^-\hat{\phi}^{(2)}(n), \\ q^+(n) = c^+\phi^{(1)}(n) + \hat{\phi}^{(2)}(n), \qquad q^-(n) = c^-\phi^{(1)}(n) + \hat{\phi}^{(2)}(n). \end{cases} \tag{54}$$

Now we consider the approximation of $b^{\pm}$ and $c^{\pm}$, then we discuss the approximations of the functions (54) so that we have some properties of the Baker function as follows. A direct calculation gives rise to

$$b^{+} = \frac{h(0)}{\delta + f(0)} = 2s_{-1}\tilde{\lambda}\big(1 + O(\tilde{\lambda}^{-1})\big), \tag{55}$$

$$b^{-} = \frac{-\delta - f(0)}{g(0)} = -\frac{1}{r_0}\tilde{\lambda}\big\{1 + O(\tilde{\lambda}^{-1})\big\}, \tag{56}$$

$$c^{+} = \frac{\delta + f(0)}{h(0)} = \frac{\tilde{\lambda}}{s_{-1}}\big\{1 + O(\tilde{\lambda}^{-1})\big\}, \tag{57}$$

$$c^{-} = -\frac{g(0)}{\delta + f(0)} = -r_0\tilde{\lambda}^{-1}\big\{1 + O(\tilde{\lambda}^{-1})\big\}. \tag{58}$$

From (55)-(58) and (53), one infers that

$$p^{+}(n)p^{-}(n) = \frac{r_n}{r_0}\big\{1 + O(\tilde{\lambda}^{-1})\big\},$$
$$q^{+}(n)q^{-}(n) = \frac{s_{n-1}}{s_{-1}}\big\{1 + O(\tilde{\lambda}^{-1})\big\}.$$

The functions $p^{+}(n), p^{-}(n)$ and $q^{+}(n), q^{-}(n)$ can be regarded as values of the singly valued functions $p(n,P)$ and $q(n,P)$ on the upper and lower sheets of Γ, respectively. Hence, we have the following assertion:

$$\begin{cases} p^{+}(n) = (1 + 2s_{-1}s_n r_0)\tilde{\lambda}^{n} + O(\tilde{\lambda}^{n-1}), \\ p^{-}(n) = (1 - s_n)\tilde{\lambda}^{n} + O(\tilde{\lambda}^{n-1}), \\ q^{+}(n) = \frac{1}{s_{-1}}\tilde{\lambda}^{n+1} + (s_{-1}^{-1}r_n s_{n-1} + s_n r_0)\tilde{\lambda}^{n} + O(\tilde{\lambda}^{n-1}), \\ q^{-}(n) = (1 - s_n)r_0\tilde{\lambda}^{n-1} + O(\tilde{\lambda}^{n-2}). \end{cases} \tag{59}$$

As stated by Cao and Geng [19, 20], we can prove the following assertions based on (53)-(59).

Proposition 1 *The Baker function $p(n,P)$ has*

(i) *N simple zeros at $\tilde{\mu}_1(n),\dots,\tilde{\mu}_N(n)$ and a pole of the nth order at $\infty_2 = (z = 0, 1), z = \tilde{\lambda}^{-1}$ on the upper sheet of Γ;*
(ii) *N simple zeros at $\tilde{\nu}_1(n),\dots,\tilde{\nu}_N(n)$ and a zero of the nth order at $\infty_1 = (z = 0, -1)$ on the lower sheet of Γ.*

Proposition 2 *The Baker function $q(n,P)$ has*

(i) *N simple poles at $\tilde{\nu}_1(0),\dots,\tilde{\nu}_N(n)$ and a pole of nth order at ∞_2 on the upper sheet of Γ;*
(ii) *N simple zeros at $\tilde{\nu}_1(n),\dots,\tilde{\nu}_k(n)$ and a zero of the nth order at ∞_1 on the lower sheet of Γ.*

Theorem (Straightening out the discrete flow)

$$\Delta\rho^{(1)} = \rho^{(1)}(n+1) - \rho^{(1)}(n) = \Omega^{(0)}(\mathrm{mod}\,\mathcal{J});$$

$$\Delta\rho^{(2)} = \rho^{(2)}(n+1) - \rho^{(2)}(n) = \Omega^{(0)}(\operatorname{mod}\mathcal{J}),$$

where $\Omega^{(0)} = \int_{\infty_1}^{\infty_2}\omega$.

3.4 Algebraic-geometric solution of equation (8)

The well-known Riemann theta function of Γ is defined by

$$\theta(\xi|\tau) = \sum_{z\in\mathbf{Z}^N}\exp\bigl(2\pi i\langle\tau z,z\rangle + 2\pi i\langle\xi,z\rangle\bigr),\quad \xi\in\mathbf{C}^N,$$

in which $\xi = (\xi_1,\dots,\xi_N)^T$, $\langle\xi,z\rangle = \sum_{j=1}^N \xi_j z_j$.

According to the Riemann theorem, there exists a constant $M^{(i)}\in\mathbf{C}^N$ so that

(i) $F_1 = \theta(\mathcal{A}(P) - \rho^{(1)}(n) - M^{(1)})$ has exactly N zeros at $\tilde\lambda = \tilde\mu_1(n),\dots,\tilde\mu_N(n)$;

(ii) $F_2 = \theta(\mathcal{A}(P) - \rho^{(2)}(n) - M^{(2)})$ has exactly zeros at $\tilde\lambda = \tilde\nu_1(n),\dots,\tilde\nu_N(n)$.

The surface Γ is cut along all a_k, b_k to become a simple connected region so that the function defined on Γ is simple valued. Denote the boundary of Γ by γ, then the integrals

$$\frac{1}{2\pi i}\int_\gamma \tilde\lambda\, d\ln F_m = I_k(\Gamma),\quad m=1,2; k=1,2,$$

are constants which are independent of $\rho^{(1)}(n)$ and $\rho^{(2)}(n)$ with $I_k(\Gamma) = \sum_{j=1}^N\int_{a_j}\tilde\lambda^k\omega_j$.

According to the inversion theorem, we have

$$\begin{cases}\sum_{j=1}^N \tilde\mu_j(n)^k = I_k(\Gamma) - \sum_{s=1}^2 \operatorname{Res}_{\tilde\lambda=\infty_s}\tilde\lambda^k\, d\ln F_1(\tilde\lambda),\\ \sum_{j=1}^N \tilde\nu_j(n)^k = I_k(\Gamma) - \sum_{s=1}^2 \operatorname{Res}_{\tilde\lambda=\infty_s}\tilde\lambda^k\, d\ln F_2(\tilde\lambda).\end{cases}\tag{60}$$

In the following, we calculate the residues in (60). We introduce local coordinate $z = \tilde\lambda^{-1}$ at ∞_s. Then the hyper-elliptic curve $\xi^2 = R(\tilde\lambda)$ in the neighborhood of infinity is given by $\tilde\xi^2 = \tilde R(z)$ along with $\tilde\xi = z^{2N+2}, \tilde R(z) = z^{2N}\pi_{j=1}^{2N+1}(1-\tilde\lambda_j z)$, and $\infty_s = (z=0, (-1)^{s-1}\sqrt{\tilde R(\tilde\lambda)}|_{z=0}) = (0,(-1)^{s-1}), s=1,2$. We can infer that

$$\begin{aligned}\mathcal{A}\bigl(P\bigl(z^{-1}\bigr)\bigr) &= \biggl(-\int_{\infty_s}^{P_0} + \int_{\infty_s}^{P}\biggr)\omega\\ &= -\eta_s - (-1)^{s-1}\sum_{l=1}^N c_{jl}\int_0^z \frac{z^{N-1}\,dz}{\sqrt{\tilde R(z)}}\\ &= -\eta_s - (-1)^{s-1}\bigl[c_{jN}z + O\bigl(z^2\bigr)\bigr],\quad \eta_s = \int_{\infty_s}^{P_0}\omega.\end{aligned}$$

Since the theta function is an even function, $F_m(\tilde\lambda)$ can be written as

$$\begin{aligned}F_m\bigl(z^{-1}\bigr) &= \theta\bigl(\dots,\rho_j^{(m)} + M_j^{(m)} + \eta_s^{(j)} + (-1)^{s-1}c_{jN}z + O\bigl(z^2\bigr),\dots\bigr)\\ &= \theta_s^{(m)} + z(-1)^{s-1}\sum_{j=1}^N c_{jN}D_j\theta_s^{(m)} + O\bigl(z^2\bigr),\end{aligned}$$

where $\theta_s^{(m)} = \theta(\rho^{(m)}(n) + M^{(m)} + \eta_s^{(m)}), \eta_s^{(m)} = \int_{\infty_m}^{P_0} \omega, m = 1, 2$. D_j stands for the derivative with respect to the jth argument of $\theta_s^{(m)}$. It is easy to compute that

$$\frac{\partial}{\partial_{t_0}} \theta_s^{(m)} = \sum_{j=1}^{N} c_{jN} D_j \theta_s^{(m)}.$$

Thus, we have

$$\begin{aligned} &F_m(z^{-1}) = \theta_s^{(m)} - z(-1)^{s-1} \partial_{t_0} \partial_t \theta_s^{(m)} + O(z^2). \\ &\operatorname{Res}_{\tilde{\lambda}=\infty_s} \tilde{\lambda}\, d \ln F_m(\tilde{\lambda}) = -(-1)^{s-1} \partial_{t_0} \ln \theta_s^{(m)} + O(z), \quad 1 \le s, m \le 2, \end{aligned} \tag{61}$$

where

$$\begin{aligned} \theta_s^{(1)} &= \theta(\Omega^{(0)} n + t_0 \Omega^{(1)} + t\Omega^{(2)} + \rho_0^{(1)}), \\ \theta_s^{(2)} &= \theta(\Omega^{(0)} n - t_0 \Omega^{(1)} - t\Omega^{(2)} + \rho^{(2)}). \end{aligned}$$

Hence, equations (60) and (61) lead to

$$\begin{cases} \sum_{j=1}^{N} \tilde{\mu}_j(n) = I_1(\Gamma) - \partial_{t_0} \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}}, \\ \sum_{j=1}^{N} \tilde{\nu}_j(n) = I_1(\Gamma) - \partial_{t_0} \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}}. \end{cases} \tag{62}$$

Substituting (62) into (39) yields

$$\begin{aligned} r_n &= \exp\left[-\partial_{t^{-1}} \partial_{t_0} \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2} t \sum_{j=1}^{2N+1} \tilde{\lambda}_j\right], \\ s_n &= \exp\left[-\partial_{t^{-1}} \partial_{t_0} \ln \frac{E\theta_2^{(1)}}{E\theta_1^{(1)}} - \frac{1}{2} t \sum_{j=1}^{2N+1} \tilde{\lambda}_j\right], \\ q_n &= \exp\left\{\partial_{t^{-1}} \Delta\left[\exp\left(-\partial_t^{-1} \partial_{t_0} \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{2} t \sum_{j=1}^{2N+1}\right) \exp\left(-\partial_{t^{-1}} \partial_{t_0} \ln \frac{E\theta_2^{(1)}}{E\theta_1^{(1)}} - \frac{1}{2} t \sum_{j=1}^{2N+1} \tilde{\lambda}_j\right)\right]\right\}, \end{aligned}$$

which is the algebro-geometric solution to equation (8).

Remark 3 We have obtained the algebraic-geometric solutions of the $(1 + 1)$-dimensional nonlinear discrete system (8). It is also an interesting and challenging work to address how to directly generate algebraic-geometric solutions of some $(2 + 1)$-dimensional reduced discrete integrable systems of the $(2 + 1)$-dimensional differential-difference hierarchy (17) just like the model for generating algebraic-geometric solutions in $1 + 1$ dimensions. In addition, it is important for investigating numerical solutions of the discrete integrable system (8) like the way presented in [39]. It is also interesting to discuss some properties presented in [40–43]. These problems will be discussed in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The idea to deduce two discrete hierarchies of the evolution equations and solve the algebraic-geometric solutions of the given discrete equations in the paper belongs to YZ. The Hamiltonian structures were proposed by XZ and YZ together. The two authors read and approved the final manuscript.

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