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Almost periodic functions on Hausdorff almost periodic time scales

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Abstract

This paper is devoted to generalizing the notion of almost periodic functions on time scales. We introduce a new class of almost periodic time scales called Hausdorff almost periodic time scales by using the Hausdorff distance and propose a more general notion of almost periodic functions on these new time scales. Then we explore some properties of Hausdorff almost periodic time scales and prove that the family of almost periodic functions on Hausdorff almost periodic time scales is a Banach space. Especially, our analysis also indicates that a function on a Hausdorff almost periodic time scale is almost periodic if and only if its affine extension is Bohr almost periodic on the real numbers $\mathbb{R}$. As an application, we establish the existence of almost periodic solutions for a single species model on Hausdorff almost periodic time scales.

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1 Introduction

It is recognized that the calculus of time scales harmonizes continuous analysis and discrete analysis into a unified framework and plays an important role in theoretical research and practical application [1, 2]. This means that the theory of dynamic equations on time scales not only provides a unifying structure for the study of differential equations and difference equations, but it also explores dynamic equations on general time scales [3, 4]. Almost periodic functions, essentially originating in landmark work of Bohr [5], describe bounded continuous functions with some approximate periodicity and have been extensively studied and discussed for differential equations and difference equations [6, 7]. In view of the two facts above, it is important and of great interest to establish almost periodic functions on time scales and study almost periodic dynamic equations on time scales.

In 2011, Li and Wang [8] defined a class of almost periodic time scales, that is, a time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\},$$

and the corresponding almost periodic functions on this time scale. Guan and Wang [9] explored the structure of time scales under translation and introduced almost periodic functions on two-way translation invariant and positive translation invariant time scales. Their studies point out that almost periodic time scales in [8] and the two-way translation invariant time scales are equal and there exists the smallest positive number τ_0 such that the set $\Pi = \{k\tau_0 : k \in \mathbb{Z}\}$ when $\mathbb{T} \neq \mathbb{R}$. This implies that almost periodic time scales in [8] are too restrictive and cannot well relate general time scales (also see [10]). Recently, Wang and Agarwal [11, 12] and Li and Li [13] introduced a new class of almost periodic time scales by using the distance between two time scales and established the corresponding almost periodic functions on these time scales. Studies suggest that this type of almost periodic time scales is more general, feasible and effective. For more details and applications of almost periodic time scales and almost periodic functions on time scales, we refer the reader to [14–24] and the references therein.

Motivated by the existing nice studies and the above considerations, a main novelty of the present paper is that we develop a new class of almost periodic time scales called Hausdorff almost periodic time scales by using the Hausdorff distance and define almost periodic functions on these new time scales. Hausdorff almost periodic time scales are more general and natural than periodic and positive translation invariant ones, and different from those proposed in [11–13]. And it is of the essence that almost periodic functions still can be well defined on Hausdorff almost periodic time scales. At the same time, there exist two other interesting topics in this paper. The first one is the question whether the family of almost periodic functions on Hausdorff almost periodic time scales is a Banach space. Another important and interesting problem is whether there is a relationship between almost periodic functions on Hausdorff almost periodic time scales and Bohr almost periodic functions on the real numbers.

The organization of this paper is as follows. In Section 2, with the help of the Hausdorff distance of two closed subsets of $\mathbb{R}$, we introduce the concept of Hausdorff almost periodic time scales and explore some of its properties. In Section 3, we define almost periodic functions on Hausdorff almost periodic time scales and show this concept includes Bohr almost periodic functions and sequences when $\mathbb{T} = \mathbb{R}$ and $\mathbb{Z}$. Then we establish a relation between almost periodic functions on Hausdorff almost periodic time scales and Bohr almost periodic functions on the real numbers, and prove that the family of almost periodic functions on Hausdorff almost periodic time scales are a Banach space in Section 4. In the final section, we investigate the existence of almost periodic solutions for a single species model on Hausdorff almost periodic time scales.

2 Hausdorff almost periodic time scales

This section focuses on a new class of almost periodic time scales called Hausdorff almost periodic time scales. To obtain our results, we first introduce the concepts of Hausdorff distance and relatively dense subset, Bohr almost periodic functions on $\mathbb{R}$ and Bohr almost periodic sequences on $\mathbb{Z}$ (see [7]).

Let $\mathcal{S} := \{\mathrm{P} : \mathrm{P} \text{ is a nonempty closed subset of the real numbers } \mathbb{R}\}$. For any $\mathrm{P}_1, \mathrm{P}_2 \in \mathcal{S}$, we define the Hausdorff distance by

$$d_H(\mathrm{P}_1, \mathrm{P}_2) = \max\left\{\sup_{x\in \mathrm{P}_1} d(x, \mathrm{P}_2), \sup_{y\in \mathrm{P}_2} d(y, \mathrm{P}_1)\right\},$$

where $d(z, \mathrm{P}) = \inf_{w\in \mathrm{P}} |z - w|$. If the Hausdorff distance is finite, then it has an equivalent form, which is more visually appealing. Given any $\mathrm{P} \in \mathcal{S}$, we denote the ε-expansion of P by

$$E_\varepsilon(\mathrm{P}) = \bigcup_{x\in\mathrm{P}} (x-\varepsilon, x+\varepsilon).$$

Then the Hausdorff distance $d_H(\mathrm{P}_1, \mathrm{P}_2)$ can be reformulated as

$$d_H(\mathrm{P}_1, \mathrm{P}_2) = \inf\bigl\{\varepsilon > 0 \mid E_\varepsilon(\mathrm{P}_1) \supset \mathrm{P}_2 \text{ and } E_\varepsilon(\mathrm{P}_2) \supset \mathrm{P}_1\bigr\}.$$

A set $A \subset \mathbb{R}$ is said to be relatively dense in $B \subset \mathbb{R}$ if there exists a number $l > 0$ with the property that the intersection of any interval of length l and B contains a $\tau \in A$. The number l is called the inclusion length of A in B.

A continuous function f on $\mathbb{R}$ is said to be Bohr almost periodic if for every $\varepsilon > 0$ the set of ε-translation numbers of f is relatively dense in $\mathbb{R}$, that is, there exists a number $l_\varepsilon > 0$ with the property that any interval of length l_ε contains a $\tau \in \mathbb{R}$ such that $|f(t) - f(t+\tau)| < \varepsilon$ for all $t \in \mathbb{R}$. The number τ is called an ε-translation number of f. Similarly, a sequence $\{a_i\}_{i\in\mathbb{Z}}$ is said to be Bohr almost periodic if for every $\varepsilon > 0$ the set of ε-translation numbers of $\{a_i\}_{i\in\mathbb{Z}}$ is relatively dense in $\mathbb{Z}$.

Now we introduce Hausdorff almost periodic time scales. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of the real numbers $\mathbb{R}$. Let τ a real number and $\{t + \tau : t \in \mathbb{T}\}$ by $\mathbb{T}^\tau$. Then the number τ is called a δ-translation number of $\mathbb{T}$ if $d_H(\mathbb{T}, \mathbb{T}^\tau) < \delta$ for $\delta > 0$.

Definition 2.1 A time scale $\mathbb{T}$ is said to be Hausdorff almost periodic if, for any $\delta > 0$, the set $\delta_\mathbb{T}$ of all δ-translation numbers of $\mathbb{T}$ is relatively dense in $\mathbb{R}$.

Remark 2.1 A periodic time scale is always a Hausdorff almost periodic time scale. In fact, if $\mathbb{T}$ is a T-periodic time scale and let $\delta_\mathbb{T} = \{zT \mid z \in \mathbb{Z}\}$, then $d_H(\mathbb{T}, \mathbb{T}^\tau) = 0$ for each $\tau \in \delta_\mathbb{T}$.

Here we show that the following two time scales are Hausdorff almost periodic time scales rather than periodic time scales.

Example 2.1 Assume that $\{\alpha_i\}_{i\in\mathbb{Z}}$ and $\{\beta_i\}_{i\in\mathbb{Z}}$ are Bohr almost periodic sequences.

(1) Let $\mathbb{T} := \{\tau_i = iT + \alpha_i, i \in \mathbb{Z}\}$ for some $T > 0$ and $|\alpha_i| < T/2$ for any $i \in \mathbb{Z}$. If $p \in \mathbb{Z}$ is an ε-translation number of the sequence $\{\alpha_i\}_{i\in\mathbb{Z}}$, then

$$\begin{aligned} d_H\bigl(\mathbb{T}, \mathbb{T}^{pT}\bigr) &= \max\Bigl\{\sup_{i\in\mathbb{Z}}\bigl\{d\bigl(\tau_i, \mathbb{T}^{pT}\bigr)\bigr\}, \sup_{i\in\mathbb{Z}}\bigl\{d(\mathbb{T}, \tau_i + pT)\bigr\}\Bigr\} \\ &\le \sup_{i\in\mathbb{Z}} d(\tau_{i+p}, \tau_i + pT) \\ &= \sup_{i\in\mathbb{Z}} |\alpha_{i+p} - \alpha_i| < \varepsilon, \end{aligned}$$

which indicates that $\mathbb{T}$ is a Hausdorff almost periodic time scale. It is noteworthy that this time scale $\mathbb{T}$ is always used as the impulse moments in almost periodic impulsive differential equations (see [25]).

(2) Let $\mathbb{T} := \bigcup_{i\in\mathbb{Z}} [3 + 12i + \alpha_i, 11 + 12i + \beta_i]$, and $\{\alpha_i\}_{i\in\mathbb{Z}}$, $\{\beta_i\}_{i\in\mathbb{Z}}$ be zero mean value with $|\alpha_i|, |\beta_i| < 1$ for any $i \in \mathbb{Z}$. If $\gamma \in \mathbb{Z}$ is a common ε-translation number of $\{\alpha_i\}_{i\in\mathbb{Z}}$ and

$\{\beta_i\}_{i\in\mathbb{Z}}$, then $\mathbb{T} \subset E_\varepsilon(\mathbb{T}^{12\gamma})$ and $E_\varepsilon(\mathbb{T}) \supset \mathbb{T}^{12\gamma}$, that is, $d_H(\mathbb{T}, \mathbb{T}^{12\gamma}) < \varepsilon$. This implies that it is a Hausdorff almost periodic time scale.

Let $\mathbb{T}$ be a time scale, we define the function $f_{\mathbb{T}} : \mathbb{R} \ni t \mapsto d_H(\mathbb{T}, \mathbb{T}^t) \in \mathbb{R}$, which obviously is continuous. The following theorem reveals a relationship of almost periodicity between a time scale $\mathbb{T}$ and its $f_{\mathbb{T}}$.

Theorem 2.1 *A time scale $\mathbb{T}$ is Hausdorff almost periodic if and only if the function $f_{\mathbb{T}}$ is Bohr almost periodic on $\mathbb{R}$.*

Proof *Necessity.* Suppose $\mathbb{T}$ is a Hausdorff almost periodic time scale. For any $\varepsilon > 0$, there is a relatively dense subset P of $\mathbb{R}$ such that

$$d_H(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon, \quad \forall \tau \in \mathrm{P}.$$

For each $t \in \mathbb{R}$ and $\tau \in \mathrm{P}$, we have

$$\begin{aligned} \left|f_{\mathbb{T}}(t) - f_{\mathbb{T}}(t+\tau)\right| &= \left|d_H(\mathbb{T}, \mathbb{T}^t) - d_H(\mathbb{T}, \mathbb{T}^{t+\tau})\right| \\ &\leq d_H(\mathbb{T}^t, \mathbb{T}^{t+\tau}) \\ &= d_H(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon. \end{aligned}$$

Thus,

$$\sup_{t\in\mathbb{R}} \left|f_{\mathbb{T}}(t) - f_{\mathbb{T}}(t+\tau)\right| \leq \varepsilon, \quad \forall \tau \in \mathrm{P},$$

which indicates $f_{\mathbb{T}}$ is Bohr almost periodic on $\mathbb{R}$.

Sufficiency. If $f_{\mathbb{T}}$ is a Bohr almost periodic function on $\mathbb{R}$, and let τ be an ε-translation number of $f_{\mathbb{T}}$, then

$$\begin{aligned} d_H(\mathbb{T}, \mathbb{T}^\tau) &= f_{\mathbb{T}}(\tau) = f_{\mathbb{T}}(\tau) - f_{\mathbb{T}}(0) \\ &\leq \sup_{t\in\mathbb{R}} \left|f_{\mathbb{T}}(t+\tau) - f_{\mathbb{T}}(t)\right| < \varepsilon. \end{aligned}$$

This means that the time scale $\mathbb{T}$ is Hausdorff almost periodic since the set of ε-translation numbers of $f_{\mathbb{T}}$ is relatively dense in $\mathbb{R}$. □

Remark 2.2 From $f_{\mathbb{T}}(0) = 0$ and the proof of the theorem above, we conclude that $\mathbb{T}$ is periodic if and only if there exists a $\tau \neq 0$ such that $f_{\mathbb{T}}(\tau) = 0$. In this case, the function $f_{\mathbb{T}}$ is a continuous periodic function on $\mathbb{R}$.

Corollary 2.1 *If a time scale $\mathbb{T}$ is Hausdorff almost periodic, then* $\inf \mathbb{T} = -\infty$ *and* $\sup \mathbb{T} = +\infty$.

Proof If $\inf \mathbb{T} > -\infty$, then, for each $t < 0$, $\mathbb{T}^t \subsetneq E_{-t/2}(\mathbb{T})$ and $f_{\mathbb{T}}(t) \geq -t/2$. It follows that $\inf_{t\to-\infty} f_{\mathbb{T}}(t) = +\infty$. But Theorem 2.1 implies that $f_{\mathbb{T}}$ is a Bohr almost periodic function which is bounded. This is a contradiction, so $\inf \mathbb{T} = -\infty$. With similar arguments, $\sup \mathbb{T} = +\infty$. □

3 Almost periodic functions on Hausdorff almost periodic time scales

In this section, we first define almost periodic functions on Hausdorff almost periodic time scales. We let a δ-neighborhood of $t \in \mathbb{R}$ be $N_\delta(t)$ and its closure be $\overline{N_\delta(t)}$.

Definition 3.1 Let $\mathbb{T}$ be a Hausdorff almost periodic time scale, and f be a function from $\mathbb{T}$ to complex field $\mathbb{C}$. If, for any $\varepsilon > 0$, there exist a $\delta = \delta(\varepsilon) > 0$ and a relatively dense subset $\mathrm{R}(\delta)$ of δ-translation number set $\delta_\mathbb{T}$ of $\mathbb{T}$ satisfying the following conditions:

(i) $\mathrm{R}(\delta) \supset (-\delta, \delta)$;

(ii) $\delta(\varepsilon)$ decreases to zero as $\varepsilon \to 0$;

(iii) for each $\tau \in \mathrm{R}(\delta)$, $\sup_{t\in\mathbb{T}} \sup_{s\in\overline{N_\delta(t)}\cap\mathbb{T}^{-\tau}} |f(t) - f(s+\tau)| < \varepsilon$ holds,

then f is called an almost periodic function on the Hausdorff almost periodic time scale $\mathbb{T}$.

We denote the set of all almost periodic functions on Hausdorff almost periodic time scale $\mathbb{T}$ by $\mathrm{AP}(\mathbb{T}, \mathbb{C})$. The set $\overline{N_\delta(t)} \cap \mathbb{T}^{-\tau}$ under condition (iii) is a nonempty closed subset of $\mathbb{T}^{-\tau}$ since $\tau \in \delta_\mathbb{T}$, $d(t, \mathbb{T}^{-\tau}) \leq d_H(\mathbb{T}, \mathbb{T}^{-\tau}) < \delta$ for each $t \in \mathbb{T}$. It is not difficult to show that f is uniformly continuous on $\mathbb{T}$ if $f \in \mathrm{AP}(\mathbb{T}, \mathbb{C})$. Actually, for any $t, t' \in \mathbb{T}$ and $|t - t'| < \delta$, one has $t \in \mathbb{T}^{-(t'-t)}$ and $t' - t \in \mathrm{R}(\delta)$ by condition (i). It follows from condition (iii) that $|f(t) - f(t + (t' - t))| = |f(t) - f(t')| < \varepsilon$.

Now we show that almost periodic functions on Hausdorff almost periodic time scales includes Bohr almost periodic functions on $\mathbb{R}$ and Bohr almost periodic sequences on $\mathbb{Z}$.

Proposition 3.1 *If* $\mathbb{T} = \mathbb{R}$, *then Definition* 3.1 *coincides with Bohr almost periodic functions on* $\mathbb{R}$.

Proof Note that $d_H(\mathbb{R}, \mathbb{R}^\tau) = 0$ for all $\tau \in \mathbb{R}$, then $\delta_\mathbb{T} = \mathbb{R}$ for each $\delta > 0$. Assume that f is almost periodic on $\mathbb{T} = \mathbb{R}$ in Definition 3.1. Then f is continuous on $\mathbb{R}$ and for any $\varepsilon > 0$, there exist a $\delta > 0$ and a relatively dense subset $\mathrm{R}(\delta)$ of $\mathbb{R}$ such that $\sup_{t\in\mathbb{R}} |f(t) - f(t+\tau)| < \varepsilon$ for each $\tau \in \mathrm{R}(\delta)$ since condition (iii) holds in Definition 3.1. This means that f is Bohr almost periodic on $\mathbb{R}$.

On the other hand, let f be a Bohr almost periodic function on $\mathbb{R}$. Then f is uniformly continuous on $\mathbb{R}$ and for any $\varepsilon > 0$, there is an $\eta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ if $|t_1 - t_2| < \eta(\varepsilon)$. Let $\delta(\varepsilon) = \eta(\varepsilon/2)$ and $\mathrm{R}(\delta)$ be the $\varepsilon/2$-translation numbers of f. It is easy to show that these ε, $\delta(\varepsilon)$ and $\mathrm{R}(\delta)$ satisfy the conditions (i)-(iii) in Definition 3.1. This completes the proof. □

Proposition 3.2 *If* $\mathbb{T} = \mathbb{Z}$, *then Definition* 3.1 *coincides with Bohr almost periodic sequences on* $\mathbb{Z}$.

Proof Assume that f is almost periodic on $\mathbb{T} = \mathbb{Z}$ in Definition 3.1. Let $\varepsilon > 0$ be sufficiently small such that $\delta(\varepsilon) < 1/2$. For any $\tau \in \mathrm{R}(\delta)$, we have

$$\sup_{n\in\mathbb{Z}} \sup_{s\in\overline{N_\delta(n)}\cap\mathbb{Z}^{-\tau}} \bigl|f(n) - f(s+\tau)\bigr| < \varepsilon. \tag{3.1}$$

Let $\tilde{\tau}$ be the closest integer of τ. Then $d(\tau, \tilde{\tau}) \leq d_H(\mathbb{Z}^\tau, \mathbb{Z}) < \delta$ and $\{\tilde{\tau} : \tau \in \mathrm{R}(\delta)\}$ is relatively dense in $\mathbb{Z}$. It follows that $s + \tau = (s + \tau - \tilde{\tau}) + \tilde{\tau} \in \mathbb{Z}$ in (3.1), which implies that $s + \tau - \tilde{\tau} \in \mathbb{Z}$. Moreover, since $|(s + \tau - \tilde{\tau}) - n| < |s - n| + |\tau - \tilde{\tau}| < 2\delta < 1$, we get $s + \tau - \tilde{\tau} = n$. So (3.1) is equivalent to $\sup_{n\in\mathbb{Z}}|f(n)-f(n+\tilde{\tau})|<\varepsilon$, which indicates that f is Bohr almost periodic on $\mathbb{Z}$.

Now we let f be a Bohr almost periodic sequence on $\mathbb{Z}$. For any real number $0<\varepsilon<1/2$, let $\delta=\varepsilon$ and $\mathrm{R}(\delta)=E_\delta(T_\varepsilon f)$, where $T_\varepsilon f\subset\mathbb{Z}$ is the set of ε-translation numbers of f. Then $\mathrm{R}(\delta)$ is relatively dense in $\delta_{\mathbb{T}}$ and ε, δ and $\mathrm{R}(\delta)$ satisfy conditions (i) and (ii) in Definition 3.1. By using similar arguments to the previous paragraph, for each $n\in\mathbb{Z}$, $\tau\in\mathrm{R}(\delta)$ and $s\in\overline{N_\delta(n)}\cap\mathbb{Z}^{-\tau}$, we have $s+\tau=n+\tilde{\tau}$ with $\tilde{\tau}\in T_\varepsilon f$. Then, for each $\tau\in\mathrm{R}(\delta)$,

$$\sup_{n\in\mathbb{Z}}\sup_{s\in\overline{N_\delta(n)}\cap\mathbb{Z}^{-\tau}}\bigl|f(n)-f(s+\tau)\bigr|=\sup_{n\in\mathbb{Z}}\bigl|f(n)-f(n+\tilde{\tau})\bigr|<\varepsilon,$$

which completes condition (iii). This completes the proof. □

The following theorem shows that almost periodic functions on Hausdorff almost periodic time scales do exist for other Hausdorff almost periodic time scales besides $\mathbb{R}$ and $\mathbb{Z}$.

Theorem 3.1 *Let $\mathbb{T}$ be a Hausdorff almost periodic time scale and $\tilde{g}$ be a Bohr almost periodic function on $\mathbb{R}$. Then $g=\tilde{g}|_{\mathbb{T}}$ (the restriction of $\tilde{g}$ on $\mathbb{T}$) is almost periodic.*

Proof From Theorem 2.1, we conclude that $f_{\mathbb{T}}$ is a Bohr almost periodic function on $\mathbb{R}$. Note that each family of finite almost periodic functions are uniformly almost periodic, then $\tilde{g}$ and $f_{\mathbb{T}}$ are equi-uniformly continuous, that is, for any $\nu>0$, there are an $\eta(\cdot)>0$ such that $|\tilde{g}(t)-\tilde{g}(s)|<\nu$ and $|f_{\mathbb{T}}(t)-f_{\mathbb{T}}(s)|<\nu$ when $|t-s|<\eta(\nu)$. For any $\varepsilon>0$, we choose $0<\hat{\varepsilon}<\varepsilon$ such that $\delta(\varepsilon)=\eta(\hat{\varepsilon}/2)<\varepsilon/2$ and let $\mathrm{R}(\delta)=T_\delta f_{\mathbb{T}}\cap T_\delta(2\delta\tilde{g}/\hat{\varepsilon})$, where $T_\delta f_{\mathbb{T}}$ is the set of δ-translation numbers of $f_{\mathbb{T}}$.

It is well known that $\mathrm{R}(\delta)$ is relatively dense in $\mathbb{R}$ (see [7]). And for each $\tau\in\mathrm{R}(\delta)$, we have

$$d_H\bigl(\mathbb{T},\mathbb{T}^\tau\bigr)=\bigl|f_{\mathbb{T}}(0)-f_{\mathbb{T}}(\tau)\bigr|\le\sup_{t\in\mathbb{R}}\bigl|f_{\mathbb{T}}(t)-f_{\mathbb{T}}(t+\tau)\bigr|<\delta,$$

which implies $\mathrm{R}(\delta)\subset\delta_{\mathbb{T}}$. It follows that $\mathrm{R}(\delta)$ is relatively dense in $\delta_{\mathbb{T}}$.

To complete the proof, we only need to show (i)-(iii) of Definition 3.1.

The condition (ii) in Definition 3.1 holds evidently for $\delta(\varepsilon)=\eta(\hat{\varepsilon}/2)<\varepsilon/2$.

For any $\delta'\in(-\delta,\delta)$, we get

$$\begin{aligned}\bigl|f_{\mathbb{T}}\bigl(t+\delta'\bigr)-f_{\mathbb{T}}(t)\bigr|&=\bigl|d_H\bigl(\mathbb{T},\mathbb{T}^{t+\delta'}\bigr)-d_H\bigl(\mathbb{T},\mathbb{T}^t\bigr)\bigr|\\&\le d_H\bigl(\mathbb{T}^{t+\delta'},\mathbb{T}^t\bigr)=d_H\bigl(\mathbb{T}^{\delta'},\mathbb{T}\bigr)\le\delta'<\delta\end{aligned}$$

and

$$\bigl|(2\delta/\hat{\varepsilon})\tilde{g}\bigl(t+\delta'\bigr)-(2\delta/\hat{\varepsilon})\tilde{g}(t)\bigr|<(2\delta\hat{\varepsilon})/(2\hat{\varepsilon})=\delta.$$

Then $\delta'\in\mathrm{R}(\delta)=T_\delta f_{\mathbb{T}}\cap T_\delta(2\delta\tilde{g}/\hat{\varepsilon})$ and $\mathrm{R}(\delta)\supset(-\delta,\delta)$. This implies that condition (i) holds in Definition 3.1.

Now we prove that condition (iii) holds in Definition 3.1. For the above ε, δ and $\mathrm{R}(\delta)$, we have

$$\begin{aligned}\bigl|g(t)-g(s+\tau)\bigr|&\le\bigl|\tilde{g}(t)-\tilde{g}(t+\tau)\bigr|+\bigl|\tilde{g}(t+\tau)-\tilde{g}(s+\tau)\bigr|\\&<\hat{\varepsilon}/2+\hat{\varepsilon}/2<\varepsilon\end{aligned}$$

for every $\tau \in \mathrm{R}(\delta)$ and all $t \in \mathbb{T}$, $s \in \overline{N_\delta(t)} \cap \mathbb{T}^{-\tau}$. Then

$$\sup_{t\in\mathbb{T}} \sup_{s\in\overline{N_\delta(t)}\cap\mathbb{T}^{-\tau}} \left|g(t) - g(s+\tau)\right| < \varepsilon.$$

The proof is completed. □

4 Two properties

In this section, we consider the proofs of two properties of almost periodic functions on Hausdorff almost periodic time scales. One is to establish a relationship between almost periodic functions on Hausdorff almost periodic time scales and Bohr almost periodic functions on the real numbers. Another is to prove that the family of almost periodic functions on Hausdorff almost periodic time scales are a Banach space.

To obtain the first property, we first establish two lemmas.

Lemma 4.1 *Each almost periodic function on Hausdorff almost periodic time scales is bounded.*

Proof Suppose f is almost periodic on a Hausdorff almost periodic time scale $\mathbb{T}$. Let $\varepsilon = 1$ and choose the corresponding δ and $\mathrm{R}(\delta)$ in Definition 3.1, we have

$$\sup_{t\in\mathbb{T}} \sup_{s\in\overline{N_\delta(t)}\cap\mathbb{T}^{-\tau}} \left|f(t) - f(s+\tau)\right| < 1 \tag{4.1}$$

for each $\tau \in \mathrm{R}(\delta)$. Denote the inclusion length of $\mathrm{R}(\delta)$ by l. Since f is continuous, it has a bound M on $[-\delta, l+\delta]_{\mathbb{T}} = [-\delta, l+\delta] \cap \mathbb{T}$. For any $s \in \mathbb{T}$ and each $\tau \in [s-l, s] \cap \mathrm{R}(\delta)$, we have $s - \tau \in [0, l]_{\mathbb{T}^{-\tau}}$. Note that $d_H(\mathbb{T}^{-\tau}, \mathbb{T}) < \delta$, then there is a $t \in [-\delta, l+\delta]_{\mathbb{T}}$ such that $d(s-\tau, t) < \delta$. It follows from (4.1) that

$$\left|f(s)\right| \le \left|f(t) - f\left[(s-\tau)+\tau\right]\right| + \left|f(t)\right| < 1 + M.$$

This shows that f is bounded. □

If $\mathbb{T} \neq \mathbb{R}$ is unbounded above and below, then, for each $t \in \mathbb{R}\backslash\mathbb{T}$, we let $t_* = \sup\{s \in \mathbb{T} : s < t\}$ and $t^* = \inf\{s \in \mathbb{T} : s > t\}$. We define the affine extension $\tilde{f}$ of a function f on a time scale $\mathbb{T}$ by

$$\tilde{f}(t) = \begin{cases} f(t), & t \in \mathbb{T}, \\ \frac{t^*-t}{t^*-t_*}f(t_*) + \frac{t-t_*}{t^*-t_*}f(t^*), & t \in \mathbb{R}\backslash\mathbb{T}, \end{cases} \qquad \mathbb{T} \neq \mathbb{R},$$

and

$$\tilde{f}(t) = f(t), \quad t \in \mathbb{T}, \mathbb{T} = \mathbb{R}.$$

Lemma 4.2 *If f is bounded and uniformly continuous on a time scale $\mathbb{T}$, then its affine extension $\tilde{f}$ also is uniformly continuous on $\mathbb{R}$.*

Proof It is clear that the conclusion holds when $\mathbb{T} = \mathbb{R}$. We only need to consider $\mathbb{T} \neq \mathbb{R}$. Since f is uniformly continuous, for any $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that, for any $\tilde{x}, \tilde{y} \in \mathbb{T}$, $|f(\tilde{x}) - f(\tilde{y})| < \varepsilon/6$ when $|\tilde{x} - \tilde{y}| < \eta$. Let $0 < \delta < \eta$ satisfy $\delta\|f\|_\infty/(\eta - \delta) < \varepsilon/12$, where $\|f\|_\infty$ is the supremum norm of f on $\mathbb{T}$.

To prove that $\tilde{f}$ is uniformly continuous on $\mathbb{R}$, for the above ε, δ and any $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we consider the following three cases:

Case 1: $x \in \mathbb{T}$, $y \in \mathbb{T}$. Then when $|x - y| < \delta$, we have

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| = \left|f(x) - f(y)\right| < \varepsilon.$$

Case 2: $x \in \mathbb{T}$, $y \notin \mathbb{T}$ or $x \notin \mathbb{T}$, $y \in \mathbb{T}$. Without loss of generality, we only consider $x \in \mathbb{T}$, $y \notin \mathbb{T}$ and $x < y$, since the rest of the arguments are similar. In this case, we have $x \leq y_* < y < x + \delta < x + \eta$ and $|f(x) - f(y_*)| < \varepsilon/6$. If $y^* - x < \eta$, then $|f(x) - f(y^*)| < \varepsilon/6$. It follows that

$$\begin{aligned}\left|\tilde{f}(x) - \tilde{f}(y)\right| &= \left|f(x) - \frac{y^* - y}{y^* - y_*}f(y_*) - \frac{y - y_*}{y^* - y_*}f\left(y^*\right)\right| \\ &\leq \frac{y^* - y}{y^* - y_*}\left|f(x) - f(y_*)\right| + \frac{y - y_*}{y^* - y_*}\left|f(x) - f\left(y^*\right)\right| \\ &< \varepsilon/6.\end{aligned}$$

Else if $y^* - x \geq \eta$, then $y^* - y_* \geq \eta - \delta$. It follows that

$$\begin{aligned}\left|\tilde{f}(y) - f(y_*)\right| &= \left|\frac{y^* - y}{y^* - y_*}f(y_*) + \frac{y - y_*}{y^* - y_*}f\left(y^*\right) - f(y_*)\right| \\ &\leq \frac{y - y_*}{y^* - y_*}\left|f(y_*) - f\left(y^*\right)\right| \\ &\leq 2\delta\|f\|_\infty/(\eta - \delta) < \varepsilon/6.\end{aligned}$$

Then

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| \leq \left|f(x) - f(y_*)\right| + \left|\tilde{f}(y) - f(y_*)\right| < \varepsilon/3.$$

In conclusion, when $|x - y| < \delta$, we have

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| < \varepsilon/3 < \varepsilon.$$

Case 3: $x, y \notin \mathbb{T}$. We only consider $x < y$ since $x > y$ is similar. If $x < x^* \leq y_* < y$, then $|f(x^*) - f(y_*)| < \varepsilon/6$. It follows from Case 2 that

$$\left|f(y_*) - \tilde{f}(y)\right| < \varepsilon/3 \quad \text{and} \quad \left|f\left(x^*\right) - \tilde{f}(x)\right| < \varepsilon/3,$$

since x^* and y^* are in $\mathbb{T}$ with $|x - x^*| < \delta$ and $|y - y^*| < \delta$. Then

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| \leq \left|\tilde{f}(x) - f\left(x^*\right)\right| + \left|f\left(x^*\right) - f(y_*)\right| + \left|f(y_*) - \tilde{f}(y)\right| < \varepsilon.$$

If $x_* = y_* < x < y < y^* = x^*$, then

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| \leq \left|f(y_*) - f\left(y^*\right)\right| < \varepsilon \quad \text{when } y^* - y_* < \eta$$

and

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| \leq \frac{y - x}{y^* - y_*}\left|f\left(y^*\right) - f(y_*)\right| \leq 2\delta\|f\|_\infty/\eta < \varepsilon \quad \text{when } y^* - y_* \geq \eta.$$

In conclusion, $|\tilde{f}(x) - \tilde{f}(y)| < \varepsilon$ when $|x - y| < \delta$.

The proof is completed. □

Theorem 4.1 *If f is a function on a Hausdorff almost periodic time scale $\mathbb{T}$, then f is almost periodic if and only if its affine extension $\tilde{f}$ is Bohr almost periodic.*

Proof It follows from Proposition 3.1 that the conclusion holds when $\mathbb{T} = \mathbb{R}$. If $\mathbb{T} \neq \mathbb{R}$, the sufficiency is true since Theorem 3.1 holds. Then we only need to prove the necessity for $\mathbb{T} \neq \mathbb{R}$.

Assume that f is almost periodic on $\mathbb{T}$. Then f is bounded and uniformly continuous on $\mathbb{T}$. It follows from Lemma 4.2 that $\tilde{f}$ is uniformly continuous. This means that, for any $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that, for any $t, s \in \mathbb{R}$, we have

$$\left|\tilde{f}(t) - \tilde{f}(s)\right| < \varepsilon/6$$

if $|t - s| < \eta$. We choose a $\varepsilon' > 0$ with $\varepsilon' < \varepsilon/10$ and the corresponding $\delta = \delta(\varepsilon')$, $\mathrm{R}(\delta)$ in Definition 3.1 with $4\delta(\varepsilon') < \eta(\varepsilon)$ and $M\delta/(\eta - 2\delta) \leq \varepsilon/18$, where M is a bound of f in Lemma 4.1.

To complete the proof, we will show that, for any $t \in \mathbb{R}$ and each $\tau \in \mathrm{R}(\delta)$, $|\tilde{f}(t) - \tilde{f}(t+\tau)| < \varepsilon$ holds. For different cases t and $t + \tau$, this proof is achieved in three steps.

The first step is $t, t + \tau \in \mathbb{T}$. In this case, it is clear that

$$\left|\tilde{f}(t) - \tilde{f}(t + \tau)\right| = \left|f(t) - f(t + \tau)\right| < \varepsilon.$$

The second step is $t \in \mathbb{T}$, $t + \tau \notin \mathbb{T}$ or $t \notin \mathbb{T}$, $t + \tau \in \mathbb{T}$. The arguments for $t \in \mathbb{T}$, $t + \tau \notin \mathbb{T}$ and $t \notin \mathbb{T}$, $t + \tau \in \mathbb{T}$ are similar, so we only consider the former. It follows from $d_H(\mathbb{T}, \mathbb{T}^\tau) < \delta$ that $|(t + \tau)_* - (t + \tau)| < \delta$ or $|(t + \tau)^* - (t + \tau)| < \delta$. We only consider the case $|(t + \tau)_* - (t + \tau)| < \delta$, since the other case, $|(t + \tau)^* - (t + \tau)| < \delta$, is similar. It follows that

$$\left|\tilde{f}(t) - f\left((t + \tau)_*\right)\right| \leq \sup_{s \in \overline{N_\delta(t)} \cap \mathbb{T}^{-\tau}} \left|f(t) - f(s + \tau)\right| < \varepsilon'.$$

Since $t + \tau \notin \mathbb{T}$, $(t + \tau)_* \in \mathbb{T}$ and $|(t + \tau)_* - (t + \tau)| < \delta$, from case 2 in the proof of Lemma 4.2, we have $|\tilde{f}(t + \tau) - f((t + \tau)_*)| < \varepsilon/3$. It follows from the two inequalities above that $|\tilde{f}(t) - \tilde{f}(t + \tau)| < \varepsilon$.

The third step is $t \notin \mathbb{T}$, $t + \tau \notin \mathbb{T}$. We divide this step into two different cases.

Case (a): $t^* - t_* < \eta - 2\delta$. It follows that $|f(t^*) - f(t_*)| < \varepsilon/6$. We consider the following three cases: (a_1) $(t + \tau)_* < t_* + \tau - \delta$, ($a_2$) $(t + \tau)^* > t^* + \tau + \delta$, ($a_3$) $t_* + \tau - \delta \leq (t + \tau)_* < (t + \tau)^* \leq t^* + \tau + \delta$.

(a_1) It is clear that $(t+\tau)_* + \delta < t_* + \tau < t + \tau < (t+\tau)^*$. By $d(t_* + \tau, \mathbb{T}) < \delta$, we get $(t+\tau)^* - (t_* + \tau) < \delta$. It follows from $\tau \in \mathrm{R}(\delta)$ and the uniform continuity of $\tilde{f}$ that $|f(t_*) - f((t+\tau)^*)| < \varepsilon'$ and $|\tilde{f}(t+\tau) - f((t+\tau)^*)| < \varepsilon/6$. Then

$$\begin{aligned} \left|\tilde{f}(t) - \tilde{f}(t+\tau)\right| &= \left|\frac{t^* - t}{t^* - t_*} f(t_*) + \frac{t - t_*}{t^* - t_*} f\left(t^*\right) - \tilde{f}(t+\tau)\right| \\ &\leq \frac{t - t_*}{t^* - t_*}\left|f\left(t^*\right) - f(t_*)\right| + \left|f(t_*) - f\left((t+\tau)^*\right)\right| \\ &\quad + \left|f\left((t+\tau)^*\right) - \tilde{f}(t+\tau)\right| < \varepsilon. \end{aligned}$$

(a_2) By similar arguments to case (a_1), we get $|\tilde{f}(t) - \tilde{f}(t+\tau)| < \varepsilon$.

(a_3) It follows from $d(t_* + \tau, \mathbb{T}) < \delta$ that there exists an $s \in \mathbb{T}$ such that $|s - (t_* + \tau)| < \delta$ and $|f(s) - f(t_*)| < \varepsilon'$. Note that $s, (t+\tau)_*, (t+\tau)^* \in (t_* + \tau - \delta, t^* + \tau + \delta)$ and this interval length is less than η, we have

$$\left|f(s) - f\left((t+\tau)_*\right)\right| < \varepsilon/6 \quad \text{and} \quad \left|f(s) - f\left((t+\tau)^*\right)\right| < \varepsilon/6,$$

since $\tilde{f}$ is the uniform continuous. Then

$$\left|f(t_*) - f\left((t+\tau)_*\right)\right| < 4\varepsilon/15, \qquad \left|f(t_*) - f\left((t+\tau)^*\right)\right| < 4\varepsilon/15$$

and

$$\begin{aligned} \left|\tilde{f}(t) - \tilde{f}(t+\tau)\right| &= \left|\frac{t^* - t}{t^* - t_*} f(t_*) + \frac{t - t_*}{t^* - t_*} f\left(t^*\right) - \frac{(t+\tau)^* - (t+\tau)}{(t+\tau)^* - (t+\tau)_*} f\left((t+\tau)_*\right)\right. \\ &\quad \left. - \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*} f\left((t+\tau)^*\right)\right| \\ &\leq \frac{t - t_*}{t^* - t_*}\left|f\left(t^*\right) - f(t_*)\right| + \frac{(t+\tau)^* - (t+\tau)}{(t+\tau)^* - (t+\tau)_*}\left|f(t_*) - f\left((t+\tau)_*\right)\right| \\ &\quad + \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*}\left|f(t_*) - f\left((t+\tau)^*\right)\right| \\ &< \varepsilon. \end{aligned}$$

In summary of (a_1)-(a_3), we have $|\tilde{f}(t) - \tilde{f}(t+\tau)| < \varepsilon$.

Case (b): $t^* - t_* \geq \eta - 2\delta$. It follows from $4\delta < \eta$ that there are the following three cases: (b_1) $t_* + \delta < t < t^* - \delta$, ($b_2$) $t \leq t_* + \delta$ and (b_3) $t^* - \delta \leq t$.

(b_1) It follows that $(t_* + \tau) + \delta < t + \tau < (t^* + \tau) - \delta$. Since $d(t_* + \tau, \mathbb{T}) < \delta$ and $d(t^* + \tau, \mathbb{T}) < \delta$, there exist two numbers κ and ν in $\mathbb{T}$ such that $t_* + \tau - \delta < \kappa < t + \tau$ and $t + \tau < \nu < t^* + \tau + \delta$. Thus,

$$(t_* + \tau) - \delta < (t+\tau)_* < (t+\tau)^* < \left(t^* + \tau\right) + \delta. \tag{4.2}$$

Next we show that $(t_* + \tau) + \delta > (t+\tau)_*$. If it is not true, that is, $(t_* + \tau) + \delta \leq (t+\tau)_*$, then $t_* + \delta \leq (t+\tau)_* - \tau$ and there exists an $s \in \mathbb{T}$ such that $d(s, (t+\tau)_* - \tau) < \delta$ since $d(\mathbb{T}^{-\tau}, \mathbb{T}) < \delta$. This implies that $t_* < s$ and $t^* \leq s$. Thus $(t+\tau)_* < t + \tau < t^* + \tau \leq s + \tau$ and $d(s+\tau, (t+\tau)_*) = d(s, (t+\tau)_* - \tau) < \delta$, which means that $t^* - t < \delta$. This contradicts $t < t^* - \delta$

in (b_1). With similar arguments, we have $(t^* + \tau) - \delta < (t + \tau)^*$. Together with (4.2) gives $(t_* + \tau) - \delta < (t + \tau)_* < (t_* + \tau) + \delta$ and $(t^* + \tau) - \delta < (t + \tau)^* < (t^* + \tau) + \delta$. From $\tau \in \mathrm{R}(\delta)$, we have $|f(t_*) - f((t + \tau)_*)| < \varepsilon'$ and $|f(t^*) - f((t + \tau)^*)| < \varepsilon'$. It follows that

$$\begin{aligned} \bigl|\tilde{f}(t) - \tilde{f}(t+\tau)\bigr| &= \biggl| f(t_*) + \frac{t - t_*}{t^* - t_*}\bigl[f\bigl(t^*\bigr) - f(t_*)\bigr] - f\bigl((t+\tau)_*\bigr) \\ &\quad - \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*}\bigl[f\bigl((t+\tau)^*\bigr) - f\bigl((t+\tau)_*\bigr)\bigr]\biggr| \\ &\le \bigl|f(t_*) - f\bigl((t+\tau)_*\bigr)\bigr| + I_1 + I_2, \end{aligned}$$

where

$$I_1 = \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*}\bigl|\bigl[f\bigl(t^*\bigr) - f(t_*)\bigr] - \bigl[f\bigl((t+\tau)^*\bigr) - f\bigl((t+\tau)_*\bigr)\bigr]\bigr| \le 2\varepsilon'$$

and

$$\begin{aligned} I_2 &= \biggl(\frac{t - t_*}{t^* - t_*} - \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*}\biggr)\bigl|f\bigl(t^*\bigr) - f(t_*)\bigr| \\ &\le 2M\biggl(\frac{|(t - t_*) - [(t+\tau) - (t+\tau)_*]|}{t^* - t_*} \\ &\quad + \frac{|(t^* - t_*) - [(t+\tau)^* - (t+\tau)_*]|}{t^* - t_*} \cdot \frac{(t+\tau) - (t+\tau)_*}{(t+\tau)^* - (t+\tau)_*}\biggr) \\ &\le 2M\biggl(\frac{\delta}{\eta - 2\delta} + \frac{2\delta}{\eta - 2\delta}\biggr). \end{aligned}$$

The last inequality follows from the inequalities showed above that $(t_* + \tau) - \delta < (t + \tau)_* < (t_* + \tau) + \delta$ and $(t^* + \tau) - \delta < (t + \tau)^* < (t^* + \tau) + \delta$. Then $|\tilde{f}(t) - \tilde{f}(t + \tau)| < 2\varepsilon/3$.

(b_2) and (b_3) In each of these two cases, there is a $\hat{t} \in (t_* + \delta, t^* - \delta)$ and $|t - \hat{t}| < \delta$. It follows from the proof of case (b_1) that $|\tilde{f}(\hat{t}) - \tilde{f}(\hat{t} + \tau)| < 2\varepsilon/3$. By using the uniform continuity of $\tilde{f}$, we have $|\tilde{f}(\hat{t}) - \tilde{f}(t)| < \varepsilon'$ and $|\tilde{f}(\hat{t} + \tau) - \tilde{f}(t + \tau)| < \varepsilon'$. Then

$$\bigl|\tilde{f}(t) - \tilde{f}(t+\tau)\bigr| < \bigl|\tilde{f}(t) - \tilde{f}(\hat{t})\bigr| + \bigl|\tilde{f}(\hat{t}) - \tilde{f}(\hat{t}+\tau)\bigr| + \bigl|\tilde{f}(\hat{t}+\tau) - \tilde{f}(t+\tau)\bigr| < \varepsilon.$$

This means that we have $|\tilde{f}(t) - \tilde{f}(t + \tau)| < \varepsilon$ for case (b).

Combining (a) and (b) gives the conclusion of the theorem. This proof is complete. □

Now we establish the second property.

Theorem 4.2 *Let* $\mathbb{T}$ *be a Hausdorff almost periodic time scale. Then the family* $\mathrm{AP}(\mathbb{T}, \mathbb{C})$ *of almost periodic functions on* $\mathbb{T}$ *is a Banach space.*

Proof We first prove that $\mathrm{AP}(\mathbb{T}, \mathbb{C})$ is a linear space. Let $f, g \in \mathrm{AP}(\mathbb{T}, \mathbb{C})$ and $\lambda \in \mathbb{C}$ a constant. For the affine extensions, it is easy to see that $\widetilde{f + g} = \tilde{f} + \tilde{g}$ and $\widetilde{\lambda f} = \lambda\tilde{f}$. From Theorem 4.1, we conclude that $\tilde{f}$ and $\tilde{g}$ both are almost periodic on $\mathbb{R}$. Note that the family $\mathrm{AP}(\mathbb{R}, \mathbb{C})$ of almost periodic functions on $\mathbb{R}$ is a Banach space, then $\tilde{f} + \tilde{g}$ and $\lambda\tilde{f}$ are almost periodic on $\mathbb{R}$. It follows from Theorem 4.1 that $f + g, \lambda f \in \mathrm{AP}(\mathbb{T}, \mathbb{C})$.

Now we show that $\mathrm{AP}(\mathbb{T},\mathbb{C})$ is completed. Let $\{f_n\}$ be a Cauchy sequence in $\widetilde{\mathrm{AP}(\mathbb{T},\mathbb{C})}$. Let $\|\varphi\|_\infty$ denote the supremum norm of φ on its domain. By $\|\tilde{f}_n - \tilde{f}_m\|_\infty = \|\widetilde{f_n - f_m}\|_\infty = \|f_n - f_m\|_\infty$, $\{\tilde{f}_n\}$ also is a Cauchy sequence in $\mathrm{AP}(\mathbb{R},\mathbb{C})$ with a limit function denoted by f_0. For the restriction of $\tilde{f}_n$ and f_0 on $\mathbb{T}$, we have $\tilde{f}_n|_{\mathbb{T}} = f_n$ and $f_n \to f_0|_{\mathbb{T}}$ as $n \to +\infty$. This implies that $f_0|_{\mathbb{T}} \in \mathrm{AP}(\mathbb{T},\mathbb{C})$ since Theorem 4.1 holds. The proof is completed. □

5 An application

As an application, we consider a single species with hibernation in an almost periodic environment. We assume that the active stage of this species is a time scale $\mathbb{T}$ given by (2) in Example 2.1. We consider the following model:

$$x^{\Delta}(t) = \begin{cases} rx(1-x), & t \in [3+12k+\alpha_k, 11+12k+\beta_k), \\ [(s_k-1)/(4+\alpha_{k+1}-\beta_k)]x(t), & t = 11+12k+\beta_k, \end{cases} \tag{5.1}$$

where r is the inherent growth rate, s_k is the survival rate during the hibernation and an almost periodic sequence. Let

$$\lambda_k = 8 + \beta_{k+1} - \alpha_{k+1}, \qquad \underline{\lambda} = \inf\{\lambda_k\},$$
$$\underline{s} = \inf\{s_k\} > 0, \qquad l = \bigl(1 - \mathrm{e}^{-r\underline{\lambda}}/\underline{s}\bigr)/\bigl(1 - \mathrm{e}^{-r\underline{\lambda}}\bigr).$$

Theorem 5.1 *If* $\mathrm{e}^{-r\underline{\lambda}} < \underline{s}l^2$, *then equation* (5.1) *has a unique almost periodic solution on* $\mathbb{T}$ *in the region* $D = \{x \in \mathrm{AP}(\mathbb{T},\mathbb{R}) : l \le x(t) \le 1, \textit{for all } t \in \mathbb{T}\}$.

Proof We easily show that $x(t)$ is a solution of equation (5.1) if and only if

$$x(t) = \frac{1}{1 + [1/(s_k x(11+12k+\beta_k)) - 1]\mathrm{e}^{-r[t-(3+12(k+1)+\alpha_{k+1})]}} \tag{5.2}$$

for every $t \in [3+12(k+1)+\alpha_{k+1}, 11+12(k+1)+\beta_{k+1}]$, $k \in \mathbb{Z}$. Let $t = 11+12(k+1)+\beta_{k+1}$ and $x_k = x(11+12k+\beta_k)$ in (5.2). A direct calculation shows that $\{x_k\}_{k\in\mathbb{Z}}$ is a solution of the difference equation

$$x_{k+1} = \frac{1}{1 + (1/(s_k x_k) - 1)\mathrm{e}^{-r\lambda_k}} \tag{5.3}$$

if $x(t)$ is a solution of (5.1). Conversely, if $\{x_k\}_{k\in\mathbb{Z}}$ is a solution of (5.3), then

$$x(t) = \frac{1}{1 + [1/(s_k x_k) - 1]\mathrm{e}^{-r[t-(3+12(k+1)+\alpha_{k+1})]}},$$

for $t \in [3+12(k+1)+\alpha_{k+1}, 11+12(k+1)+\beta_{k+1}]$ is a solution of (5.1).

We define an operator J on $\widetilde{D} = \{\{x_k\}_{k\in\mathbb{Z}} \in \mathrm{AP}(\mathbb{Z},\mathbb{R}) : l \le x_k \le 1, k \in \mathbb{Z}\}$ by

$$J(x_k) = \frac{1}{1 + (1/(s_{k-1}x_{k-1}) - 1)\mathrm{e}^{-r\lambda_{k-1}}}, \quad k \in \mathbb{Z},$$

for any sequence $\{x_k\}_{k\in\mathbb{Z}} \in \widetilde{D}$. It follows from Theorem 1.18 of [7] that $J : \widetilde{D} \to \mathrm{AP}(\mathbb{Z},\mathbb{R})$. If $\{x_k\}_{k\in\mathbb{Z}} \in \widetilde{D}$, then we have $J(x_k) \le 1$ and

$$J(x_k) \ge \frac{1}{1 + (1/(\underline{s}l) - 1)\mathrm{e}^{-r\underline{\lambda}}} = l$$

for any $k \in \mathbb{Z}$, which implies that $J : \widetilde{D} \to \widetilde{D}$. For any $\{w_k\}_{k\in\mathbb{Z}}, \{v_k\}_{k\in\mathbb{Z}} \in \widetilde{D}$, then

$$\begin{aligned}\sup_{k\in\mathbb{Z}}\left|J(w_k) - J(v_k)\right| &\le \sup_{k\in\mathbb{Z}} \frac{[1/(s_k w_k v_k)]e^{-r\lambda_k}|w_k - v_k|}{[1 + (1/(s_k w_k) - 1)e^{-r\lambda_k}][1 + (1/(s_k v_k) - 1)e^{-r\lambda_k}]} \\ &\le \left[e^{-r\underline{\lambda}}/\left(\underline{s}l^2\right)\right] \sup_{k\in\mathbb{Z}} |w_k - v_k|.\end{aligned}$$

Thus J is a contraction mapping on $\widetilde{D}$. By the contraction mapping principle, J has a unique fixed point $\{x_k^*\}_{k\in\mathbb{Z}}$ in $\widetilde{D}$, that is, $\{x_k^*\}_{k\in\mathbb{Z}}$ is a unique solution of (5.3) in $\widetilde{D}$. Then (5.1) has a unique solution,

$$x(t) = \frac{1}{1 + [1/(s_k x_k^*) - 1]e^{-r[t-(3+12(k+1)+\alpha_{k+1})]}} \tag{5.4}$$

for $t \in [3 + 12(k + 1) + \alpha_{k+1}, 11 + 12(k + 1) + \beta_{k+1}]$ and $k \in \mathbb{Z}$. It is easy to see that $l \le x(t) \le 1$ for any $t \in \mathbb{T}$.

Now we show that $x \in \mathrm{AP}(\mathbb{T}, \mathbb{C})$ in (5.4). It is clear that $x(\cdot)$ is uniformly continuous and $x(t)$ continuously depends on the initial value $1/(s_{k-1}x_{k-1}^*)$ for $t \in [3 + 12k + \alpha_k, 11 + 12k + \beta_k]$ and $k \in \mathbb{Z}$. Then for any $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon)$ such that $|x(t_1) - x(t_2)| < \varepsilon/2$ when $t_i \in [3 + 12k_i + \alpha_{k_i}, 11 + 12k_i + \beta_{k_i}]$ with

$$\left|1/\left(s_{k_1-1}x_{k_1-1}^*\right) - 1/\left(s_{k_2-1}x_{k_2-1}^*\right)\right| < \eta \tag{5.5}$$

and

$$\left|\left[t_1 - (3 + 12k_1 + \alpha_{k_1})\right] - \left[t_2 - (3 + 12k_2 + \alpha_{k_2})\right]\right| < \eta. \tag{5.6}$$

Let $\delta = \min\{2\eta/5, 2/5\}$, $\mathrm{R}(\delta) = E_{\delta/2}(Q_\delta) \cup (-\delta, \delta)$, where

$$Q_\delta = \left\{12p : p \in \mathbb{Z} \cap T_{5\delta/2}\left(\left\{1/\left(s_k x_k^*\right)\right\}\right) \cap T_{\delta/2}\left(\{\alpha_k\}\right) \cap T_{\delta/2}\left(\{\beta_k\}\right)\right\}.$$

It is not difficult to show that $\mathrm{R}(\delta) \subset \delta_{\mathbb{T}}$ and conditions (i) and (ii) in Definition 3.1 hold.

Next we prove that condition (iii) in Definition 3.1 also holds. For each $\tau \in \mathrm{R}(\delta)$, it can be uniquely decomposed as $\tau = 12p_\tau + \tau'$ with $p_\tau \in Q_\delta$ and $|\tau'| < \delta$. It follows from $|\alpha_k| < 1$ and $|\beta_k| < 1$ that, for any $t \in [3 + 12k + \alpha_k, 11 + 12k + \beta_k]$, $s \in \overline{N_\delta(t)} \cap \mathbb{T}^{-\tau}$, we have $s + \tau \in [3 + 12(k + p_\tau) + \alpha_{k+p_\tau}, 11 + 12(k + p_\tau) + \beta_{k+p_\tau}]$. Then (5.5) and (5.6) hold, if we replace k_1, k_2, t_1 and t_2 with k, $k + p_\tau$, t and $s + \tau$, respectively. Thus $|x(t) - x(s + \tau)| < \varepsilon/2$. This means that, for each $\tau \in \mathrm{R}(\delta)$, we have

$$\sup_{t\in\mathbb{T}} \sup_{s\in\overline{N_\delta(t)}\cap\mathbb{T}^{-\tau}} \left|x(t) - x(s + \tau)\right| < \varepsilon.$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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