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# Solution to the solvability problem for a class of product-type systems of difference equations

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## Abstract

Solution to the solvability problem for a class of product-type systems of difference equations is given by presenting explicit formulas for its solutions. In the main/most complicated cases the problem is solved by using two different methods. This is the last system, out of the three non-equivalent ones, for which an associated polynomial, which essentially determines the structure of the solutions, is of the third order, so that the paper finishes the study of this kind of two-dimensional systems.

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**Keywords:** solvability problem; system of difference equations; product-type system; solutions in closed form

## 1 Introduction

Theory of difference equations is a mathematical area of great interest [1–40]. Concrete systems is one of its subareas which is of some interest nowadays. Papers [12–14] by Paspachinopoulos and Schinas are some of those which have influenced the interest. Later many others have appeared [4, 9–11, 15, 16, 18, 19, 22–28, 30–40]. One of the oldest topics is finding solutions to the equations and systems [1, 5–8]. Since the mid of 2000s there has been increasing interest in the topic [4, 17, 21–31, 33–40]. An interesting fact is that many nonlinear equations and systems are related to linear ones although they are of relatively complex forms, which usually do not suggest it on the first site. A line of the investigation related to a solvable nonlinear difference equation can be followed, for example, in [4, 17, 21, 36] (see also the references therein). A more complicated equation and related methods can be found in [29]. The corresponding and related systems of difference equations were studied, for example, in [23, 33] and [34] (see also the references therein). Another system of interest can be found in [35]. The equations and systems in these papers are solved by employing suitable changes of variables which transform them to some linear equations and systems which are solvable.

Many equations and systems contain as special cases product-type ones [20, 32]. Some methods for the study of the long-term behavior of their solutions have used directly or indirectly the equations and systems which can be solved. In [20] and [32] we studied only positive solutions to the equations and systems appearing therein, so the product-type

ones, which are obtained from them by special choices of their parameters, were solvable. Applying the logarithm is a standard method for solving such equations and systems; it was, among others, also used in solving a system in [22]. In general, a more complex case is studying non-positive solutions to difference equations and systems, which is also the case for the product-type systems, since the previous method is useless because it produces multi-valued solutions. In our papers [26] and [31], we have presented a two- and a three-dimensional solvable product-type system in the complex plane, respectively. Many results on the long-term behavior of solutions to the system in [31] were presented therein, while in [26] we only gave a method for solving the corresponding system. Having published these two papers has motivated us to look for other solvable product-type systems. In [37] we have managed to solve another one. The fact that the systems in [31] and [37] have been special cases of the following system

$$z_n = z_{n-k_1}^a w_{n-k_2}^b, \quad w_n = w_{n-k_3}^c z_{n-k_4}^d, \quad n \in \mathbb{N}_0, \tag{1}$$

has motivated us to start investigating the solvability of other special cases of the system. Some equations in [29] are product-type ones, but they have additional multipliers, which suggested us to study the solvability of the generalizations of system (1) containing some multipliers. The first step in the study was made in [24]. Our further investigations suggested that closed-form formulas for solutions to product-type systems can be presented for all given values of parameters and multipliers, which for two specific systems of the form in (1) was done in [28] and [40] for the first time (in [37] and [39] general formulas were presented in the main cases, but without detailed analysis and without presenting more concrete formulas for each possible case; this type of problems does not appear for the system in [38], where its solutions are presented in all the cases). Product-type systems can be also solved by using the method which we developed in [27], but is somewhat technically complicated with respect to the one developed in [28, 37, 40]. Another class of product-type systems has been quite recently investigated in detail in [30].

This paper is devoted to the study of an extension of system (1), with the following delays  $k_1 = k_2 = 1$  and  $k_3 = k_4 = 2$ , that is, of the system:

$$z_n = \alpha z_{n-1}^a w_{n-1}^b, \quad w_n = \beta w_{n-2}^c z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{2}$$

where  $a, b, c, d \in \mathbb{Z}$ ,  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C}$ . It is also a natural continuation of our study in [24, 27–31, 37–40].

This is the last system, out of the three non-equivalent ones, for which an associated polynomial, which essentially determines the structure of the solutions, is of the third order, so that the paper finishes the study of this kind of systems with two dependent variables. The case when some of initial values or parameters  $\alpha$  and  $\beta$  are equal to zero is excluded from the consideration since such solutions are either not well defined [23] or eventually equal to zero. If a finite sum is of the form  $\sum_{j=1}^{l-1} c_j$ , for some  $l \in \mathbb{Z}$ , we regard that its value is zero.

## 2 Auxiliary results

The following two lemmas are useful in the investigation of product-type systems. Proofs of the first one can be found, for example, in [6] and [37].

**Lemma 1** *Let*

$$Q(t) = \sum_{j=0}^k c_j t^j = c_k \prod_{j=1}^k (t - t_j)$$

such that  $t_i \neq t_j, i \neq j$ , and  $c_k \neq 0$ . Then

$$\sum_{j=1}^k \frac{t_j^s}{Q'(t_j)} = 0$$

for  $0 \leq s \leq k - 2$ , and

$$\sum_{j=1}^k \frac{t_j^{k-1}}{Q'(t_j)} = \frac{1}{c_k}.$$

The second lemma is well known ([6, 8], see also [28] for a more general result).

**Lemma 2** *Let  $i \in \mathbb{N}_0$  and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \dots + n^i z^{n-1}, \quad n \in \mathbb{N},$$

where  $z \in \mathbb{C}$ .

Then

$$s_n^{(0)}(z) = \frac{1 - z^n}{1 - z}, \tag{3}$$

$$s_n^{(1)}(z) = \frac{1 - (n + 1)z^n + nz^{n+1}}{(1 - z)^2}, \tag{4}$$

$$s_n^{(2)}(z) = \frac{1 + z - (n + 1)^2 z^n + (2n^2 + 2n - 1)z^{n+1} - n^2 z^{n+2}}{(1 - z)^3} \tag{5}$$

for every  $z \in \mathbb{C} \setminus \{1\}$  and  $n \in \mathbb{N}$ .

### 3 Main results

Five cases will be considered separately. Before we formulate and prove our main results, note that

$$\begin{aligned} z_0 &= \alpha z_{-1}^a w_{-1}^b, & z_1 &= \alpha^{1+a} \beta^b z_{-2}^{bd} z_{-1}^{a^2} w_{-2}^{bc} w_{-1}^{ab}, \\ w_0 &= \beta w_{-2}^c z_{-2}^d, & w_1 &= \beta w_{-1}^c z_{-1}^d. \end{aligned} \tag{6}$$

**Theorem 1** *Assume that  $a, b, d \in \mathbb{Z}, c = 0, bd = 0, \alpha, \beta, z_{-2}, z_{-1}, w_{-1} \in \mathbb{C} \setminus \{0\}$ . Then*

(a) *if  $a \neq 1$ , the general solution to (2) is given by*

$$z_n = \alpha^{\frac{1-a^{n+1}}{1-a}} \beta^{b \frac{1-a^n}{1-a}} z_{-1}^{a^{n+1}} w_{-1}^{ba^n}, \quad n \in \mathbb{N} \tag{7}$$

and

$$w_n = \beta \alpha^{d \frac{1-a^{n-1}}{1-a}} z_{-1}^{da^{n-1}}, \quad n \geq 2; \tag{8}$$

(b) if  $a = 1$ , the general solution to (2) is given by

$$z_n = \alpha^{n+1} \beta^{bn} z_{-1}^b w_{-1}^b, \quad n \in \mathbb{N}_0 \tag{9}$$

and

$$w_n = \alpha^{d(n-1)} \beta z_{-1}^d, \quad n \geq 2. \tag{10}$$

*Proof* Since  $c = 0$ , we have

$$z_n = \alpha z_{n-1}^a w_{n-1}^b, \quad w_n = \beta z_{n-2}^d, \quad n \in \mathbb{N}_0. \tag{11}$$

From the equations in (11) and the assumption  $bd = 0$ , we get

$$z_n = \alpha \beta^b z_{n-1}^a, \quad n \in \mathbb{N}, \tag{12}$$

and consequently

$$z_n = (\alpha \beta^b)^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N},$$

which along with (6) yields

$$z_n = \alpha^{\sum_{j=0}^n a^j} \beta^{b \sum_{j=0}^{n-1} a^j} z_{-1}^{a^{n+1}} w_{-1}^{ba^n}, \quad n \in \mathbb{N}. \tag{13}$$

From (13) we easily get (7) and (9).

Using (13) in the second equality in (11), as well as the assumption  $bd = 0$ , we obtain

$$\begin{aligned} w_n &= \alpha^{d \sum_{j=0}^{n-2} a^j} \beta^{1+bd \sum_{j=0}^{n-3} a^j} z_{-1}^{da^{n-1}} w_{-1}^{bda^{n-2}} \\ &= \alpha^{d \sum_{j=0}^{n-2} a^j} \beta z_{-1}^{da^{n-1}}, \quad n \geq 3. \end{aligned} \tag{14}$$

From (14) we easily get (8) and (10) for  $n \geq 3$ . For  $n = 2$ , equalities (8) and (10) are directly verified. □

**Theorem 2** *Assume that  $a, c, d \in \mathbb{Z}$ ,  $c \neq 0$ ,  $b = 0$ ,  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ . Then system (2) is solvable in closed form.*

This theorem was essentially proved in [39], Theorem 2, (see also [39], Corollary 2, for the closed form formulas for the solutions in all the cases), since when  $b = 0$  system (2) is

$$z_n = \alpha z_{n-1}^a, \quad w_n = \beta w_{n-2}^c z_{n-2}^d, \quad n \in \mathbb{N}_0,$$

which is nothing but the corresponding system in Theorem 2 in [39] with indices shifted backward for one. Hence, we omit the proof of the theorem.

**Theorem 3** *Assume that  $a, b, c \in \mathbb{Z}$ ,  $c \neq 0$ ,  $d = 0$ ,  $\alpha, \beta, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ . Then system (2) is solvable in closed form.*

This theorem was essentially proved in [28], since when  $d = 0$  system (2) is

$$z_n = \alpha z_{n-1}^a w_{n-1}^b, \quad w_n = \beta w_{n-2}^c, \quad n \in \mathbb{N}_0,$$

which is nothing but the corresponding system in Theorem 3.7 in [28] whose indices are shifted backward for one (see also [28], Corollary 3.8, for the closed form formulas for the solutions in all the cases). Hence, the proof of the theorem will be also omitted.

**Theorem 4** *Assume that  $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ ,  $ac \neq bd$ ,  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ . Then system (2) is solvable in closed form.*

*Proof* Since  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ , from (2) we get  $z_n \neq 0 \neq w_n$  for  $n \geq -2$ . Hence

$$w_{n-1}^b = \frac{z_n}{\alpha z_{n-1}^a}, \quad n \in \mathbb{N}_0$$

and

$$w_n^b = \beta^b w_{n-2}^{bc} z_{n-2}^{bd}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$z_{n+1} = \alpha^{1-c} \beta^b z_n^a z_{n-1}^c z_{n-2}^{bd-ac}, \quad n \in \mathbb{N}. \tag{15}$$

Let  $\mu = \alpha^{1-c} \beta^b$ ,

$$a_1 = a, \quad b_1 = c, \quad c_1 = bd - ac, \quad y_1 = 1. \tag{16}$$

Then

$$z_{n+1} = \mu^{y_1} z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1}, \quad n \in \mathbb{N}. \tag{17}$$

Equality (17) implies

$$\begin{aligned} z_{n+1} &= \mu^{y_1} \left( \mu z_{n-1}^{a_1} z_{n-2}^{b_1} z_{n-3}^{c_1} \right)^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} \\ &= \mu^{y_1+a_1} z_{n-1}^{a_1 a_1 + b_1} z_{n-2}^{b_1 a_1 + c_1} z_{n-3}^{c_1 a_1} \\ &= \mu^{y_2} z_{n-1}^{a_2} z_{n-2}^{b_2} z_{n-3}^{c_2}, \end{aligned} \tag{18}$$

for  $n \geq 2$ , where

$$a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1 + c_1, \quad c_2 := c_1 a_1, \quad y_2 = y_1 + a_1. \tag{19}$$

Assume

$$z_{n+1} = \mu^{y_k} z_{n+1-k}^{a_k} z_{n-k}^{b_k} z_{n-k-1}^{c_k}, \tag{20}$$

for a  $k \geq 2$  and all  $n \geq k$ , and

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1}, \quad c_k = c_1 a_{k-1}, \tag{21}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{22}$$

If we replace  $n$  by  $n - k$  in (17) and employ it in (20), we get

$$\begin{aligned} z_{n+1} &= \mu^{y_k} \left( \mu z_{n-k}^{a_1} z_{n-k-1}^{b_1} z_{n-k-2}^{c_1} \right)^{a_k} z_{n-k}^{b_k} z_{n-k-1}^{c_k} \\ &= \mu^{y_k+a_k} z_{n-k}^{a_1 a_k + b_k} z_{n-k-1}^{b_1 a_k + c_k} z_{n-k-2}^{c_1 a_k} \\ &= \mu^{y_{k+1}} z_{n-k}^{a_{k+1}} z_{n-k-1}^{b_{k+1}} z_{n-k-2}^{c_{k+1}} \end{aligned} \tag{23}$$

for  $n \geq k + 1$ , where

$$a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k + c_k, \quad c_{k+1} := c_1 a_k, \quad y_{k+1} := y_k + a_k. \tag{24}$$

Relations (18), (19), (23), (24) along with the inductive argument confirm the conjectures in (20)-(22).

Setting  $k = n$  in (20) and using (6), (21) and (22), we have

$$\begin{aligned} z_{n+1} &= \mu^{y_n} z_1^{a_n} z_0^{b_n} z_{-1}^{c_n} \\ &= (\alpha^{1-c} \beta^b)^{y_n} (\alpha^{1+a} \beta^b z_{-2}^{bd} z_{-1}^{a^2} w_{-2}^{bc} w_{-1}^{ab})^{a_n} (\alpha z_{-1}^a w_{-1}^b)^{b_n} z_{-1}^{c_n} \\ &= \alpha^{(1-c)y_n + (1+a)a_n + b_n} \beta^{b(y_n + a_n)} z_{-2}^{bd a_n} z_{-1}^{a^2 a_n + ab_n + c_n} w_{-2}^{bca_n} w_{-1}^{aba_n + bb_n} \\ &= \alpha^{y_{n+2} - cy_n} \beta^{by_{n+1}} z_{-2}^{bd a_n} z_{-1}^{a_{n+2} - ca_n} w_{-2}^{bca_n} w_{-1}^{ba_{n+1}} \end{aligned} \tag{25}$$

for  $n \in \mathbb{N}$ .

The relations in (21) show that

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3}, \quad \text{for } k \geq 4. \tag{26}$$

Since  $c_1 = bd - ac \neq 0$ , (26) yields

$$a_{k-3} = \frac{a_k - a_1 a_{k-1} - b_1 a_{k-2}}{c_1}. \tag{27}$$

This equality enables us to calculate  $a_j$  for  $j \leq 0$ . It is obtained that

$$a_{-1} = a_{-2} = 0, \quad a_0 = 1, \tag{28}$$

(see detailed corresponding calculations in [28] and [40]) from which along with (22), it is obtained

$$y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1 \tag{29}$$

and

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{30}$$

Linear equation (26) can be solved, from which along with (28) a closed form formula for  $a_k$  is obtained. Using (30) and Lemma 2, a closed form formula for  $y_k$  is obtained. Employing such obtained formulas for  $a_k$  and  $y_k$  in (25), we obtain a closed form formula for the solution to (15).

From (2), we also have

$$z_{n-2}^d = \frac{w_n}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0, \tag{31}$$

$$z_n^d = \alpha^d z_{n-1}^{ad} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{32}$$

and consequently

$$w_{n+2} = \alpha^d \beta^{1-a} w_{n+1}^a w_n^c w_{n-1}^{bd-ac}, \quad n \in \mathbb{N}_0. \tag{33}$$

As above, we get

$$w_{n+2} = \eta^{y_k} w_{n+2-k}^{a_k} w_{n+1-k}^{b_k} w_{n-k}^{c_k}, \quad n \geq k - 1, \tag{34}$$

where  $\eta = \alpha^d \beta^{1-a}$ ,  $a_k$ ,  $b_k$  and  $c_k$  are defined by (16) and (21), and  $y_k$  is defined by (22) and (29).

Setting  $k = n + 1$  in (34) and using (6), we get

$$\begin{aligned} w_{n+2} &= \eta^{y_{n+1}} w_1^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} \\ &= (\alpha^d \beta^{1-a})^{y_{n+1}} (\beta w_{-1}^c z_{-1}^d)^{a_{n+1}} (\beta w_{-2}^c z_{-2}^d)^{b_{n+1}} w_{-1}^{c_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}+b_{n+1}} z_{-2}^{db_{n+1}} z_{-1}^{da_{n+1}} w_{-2}^{cb_{n+1}} w_{-1}^{ca_{n+1}+c_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{y_{n+3}-ay_{n+2}} z_{-2}^{d(a_{n+2}-aa_{n+1})} z_{-1}^{da_{n+1}} w_{-2}^{c(a_{n+2}-aa_{n+1})} w_{-1}^{a_{n+3}-aa_{n+2}} \end{aligned} \tag{35}$$

for  $n \in \mathbb{N}_0$ .

As we have already mentioned, closed form formulas for  $a_k$  and  $y_k$  can be found, from which along with (35) it follows that (33) is solvable. It is not difficult to see that formulas (25) and (35) present a solution to system (2), from which the theorem follows.  $\square$

**Remark 1** Theorem 4 can be also proved by using the method in [27]. We include the proof also for the benefit of the reader and since in the theorem that follows we follow the lines of the method.

First note that

$$\begin{aligned} w_2 &= \beta w_0^c z_0^d = \beta (\beta w_{-2}^c z_{-2}^d)^c (\alpha z_{-1}^a w_{-1}^b)^d = \alpha^d \beta^{1+c} z_{-2}^{cd} z_{-1}^{ad} w_{-2}^{c^2} w_{-1}^{bd}, \\ w_3 &= \beta w_1^c z_1^d = \alpha^{d(1+a)} \beta^{1+c+bd} z_{-2}^{bd^2} z_{-1}^{(a^2+c)d} w_{-2}^{bcd} w_{-1}^{abd+c^2}. \end{aligned} \tag{36}$$

The equalities in (36) show that  $w_2$  and  $w_3$  depend on all the parameters and initial values (we neglect the cases when some of the powers are zero). This fact and (2) show that the same holds for every  $w_n$ ,  $n \geq 2$ , from which along with the first equation in (2) it follows that the same holds for every  $z_n$ ,  $n \geq 3$ . Hence

$$z_n = \alpha^{x_n} \beta^{y_n} z_{-2}^{a_n} z_{-1}^{b_n} w_{-2}^{c_n} w_{-1}^{d_n}, \tag{37}$$

$$w_n = \alpha^{u_n} \beta^{v_n} z_{-2}^{\alpha_n} z_{-1}^{\beta_n} w_{-2}^{\gamma_n} w_{-1}^{\delta_n}, \tag{38}$$

for  $n \geq -2$ , for some sequences  $x_n, y_n, a_n, b_n, c_n, d_n, u_n, v_n, \alpha_n, \beta_n, \gamma_n, \delta_n$ .

Clearly, it must be

$$x_{-2} = 0, \quad y_{-2} = 0, \quad a_{-2} = 1, \quad b_{-2} = 0, \quad c_{-2} = 0, \quad d_{-2} = 0, \tag{39}$$

$$u_{-2} = 0, \quad v_{-2} = 0, \quad \alpha_{-2} = 0, \quad \beta_{-2} = 0, \quad \gamma_{-2} = 1, \quad \delta_{-2} = 0, \tag{40}$$

$$x_{-1} = 0, \quad y_{-1} = 0, \quad a_{-1} = 0, \quad b_{-1} = 1, \quad c_{-1} = 0, \quad d_{-1} = 0, \tag{41}$$

$$u_{-1} = 0, \quad v_{-1} = 0, \quad \alpha_{-1} = 0, \quad \beta_{-1} = 0, \quad \gamma_{-1} = 0, \quad \delta_{-1} = 1, \tag{42}$$

$$x_0 = 1, \quad y_0 = 0, \quad a_0 = 0, \quad b_0 = a, \quad c_0 = 0, \quad d_0 = b, \tag{43}$$

$$u_0 = 0, \quad v_0 = 1, \quad \alpha_0 = d, \quad \beta_0 = 0, \quad \gamma_0 = c, \quad \delta_0 = 0, \tag{44}$$

$$x_1 = 1 + a, \quad y_1 = b, \quad a_1 = bd, \quad b_1 = a^2, \quad c_1 = bc, \quad d_1 = ab, \tag{45}$$

$$u_1 = 0, \quad v_1 = 1, \quad \alpha_1 = 0, \quad \beta_1 = d, \quad \gamma_1 = 0, \quad \delta_1 = c. \tag{46}$$

Applying (37) and (38) in (2), we get

$$\begin{aligned} z_n &= \alpha \left( \alpha^{x_{n-1}} \beta^{y_{n-1}} z_{-2}^{a_{n-1}} z_{-1}^{b_{n-1}} w_{-2}^{c_{n-1}} w_{-1}^{d_{n-1}} \right)^a \left( \alpha^{u_{n-1}} \beta^{v_{n-1}} z_{-2}^{\alpha_{n-1}} z_{-1}^{\beta_{n-1}} w_{-2}^{\gamma_{n-1}} w_{-1}^{\delta_{n-1}} \right)^b \\ &= \alpha^{ax_{n-1} + bu_{n-1} + 1} \beta^{ay_{n-1} + bv_{n-1}} z_{-2}^{aa_{n-1} + b\alpha_{n-1}} z_{-1}^{ab_{n-1} + b\beta_{n-1}} \\ &\quad \times w_{-2}^{ac_{n-1} + b\gamma_{n-1}} w_{-1}^{ad_{n-1} + b\delta_{n-1}} \end{aligned} \tag{47}$$

for  $n \in \mathbb{N}_0$ , and

$$\begin{aligned} w_n &= \beta \left( \alpha^{u_{n-2}} \beta^{v_{n-2}} z_{-2}^{\alpha_{n-2}} z_{-1}^{\beta_{n-2}} w_{-2}^{\gamma_{n-2}} w_{-1}^{\delta_{n-2}} \right)^c \left( \alpha^{x_{n-2}} \beta^{y_{n-2}} z_{-2}^{a_{n-2}} z_{-1}^{b_{n-2}} w_{-2}^{c_{n-2}} w_{-1}^{d_{n-2}} \right)^d \\ &= \alpha^{dx_{n-2} + cu_{n-2}} \beta^{dy_{n-2} + cv_{n-2} + 1} z_{-2}^{da_{n-2} + c\alpha_{n-2}} z_{-1}^{db_{n-2} + c\beta_{n-2}} \\ &\quad \times w_{-2}^{dc_{n-2} + c\gamma_{n-2}} w_{-1}^{dd_{n-2} + c\delta_{n-2}} \end{aligned} \tag{48}$$

for  $n \in \mathbb{N}_0$ .

Motivated by (37)-(48), we define sequences  $x_n, y_n, a_n, b_n, c_n, d_n, u_n, v_n, \alpha_n, \beta_n, \gamma_n, \delta_n$  as follows:

$$x_n := ax_{n-1} + bu_{n-1} + 1, \quad u_n := dx_{n-2} + cu_{n-2}, \tag{49}$$

$$y_n := ay_{n-1} + bv_{n-1}, \quad v_n := dy_{n-2} + cv_{n-2} + 1, \tag{50}$$

$$a_n := aa_{n-1} + b\alpha_{n-1}, \quad \alpha_n := da_{n-2} + c\alpha_{n-2}, \tag{51}$$

$$b_n := ab_{n-1} + b\beta_{n-1}, \quad \beta_n := db_{n-2} + c\beta_{n-2}, \tag{52}$$



$$c_n := ac_{n-1} + b\gamma_{n-1}, \quad \gamma_n := dc_{n-2} + c\gamma_{n-2}, \tag{53}$$

$$d_n := ad_{n-1} + b\delta_{n-1}, \quad \delta_n := dd_{n-2} + c\delta_{n-2}, \tag{54}$$

for  $n \in \mathbb{N}_0$ .

Relations (39)-(44), (49)-(54) define  $x_n, y_n, a_n, b_n, c_n, d_n, u_n, v_n, \alpha_n, \beta_n, \gamma_n, \delta_n$  uniquely, while  $z_n$  and  $w_n$  in (2) are defined uniquely by initial values  $z_{-2}, z_{-1}, w_{-2}, w_{-1}$ . Since  $z_n$  and  $w_n$  defined by (47) and (48) are solutions to (2) such that their values for  $n = 0, 1$  are the same, it follows that they are closed form formulas for solutions to system (2). Hence, we need only to prove that systems (49)-(54) are solvable.

The condition  $d \neq 0$  implies

$$x_{n-2} = \frac{u_n - cu_{n-2}}{d}, \quad n \in \mathbb{N}_0, \tag{55}$$

which along with the first equality in (49) yields

$$u_{n+2} = au_{n+1} + cu_n + (bd - ac)u_{n-1} + d, \quad n \in \mathbb{N}_0, \tag{56}$$

and also

$$y_{n-2} = \frac{v_n - cv_{n-2} - 1}{d}, \quad n \in \mathbb{N}_0, \tag{57}$$

which along with the first equality in (50) yields

$$v_{n+2} = av_{n+1} + cv_n + (bd - ac)v_{n-1} + 1 - a, \quad n \in \mathbb{N}_0. \tag{58}$$

We also have

$$a_{n-2} = \frac{\alpha_n - c\alpha_{n-2}}{d}, \tag{59}$$

$$b_{n-2} = \frac{\beta_n - c\beta_{n-2}}{d}, \tag{60}$$

$$c_{n-2} = \frac{\gamma_n - c\gamma_{n-2}}{d}, \tag{61}$$

$$d_{n-2} = \frac{\delta_n - c\delta_{n-2}}{d}, \tag{62}$$

for  $n \in \mathbb{N}_0$ , and consequently

$$\alpha_{n+2} = a\alpha_{n+1} + c\alpha_n + (bd - ac)\alpha_{n-1}, \tag{63}$$

$$\beta_{n+2} = a\beta_{n+1} + c\beta_n + (bd - ac)\beta_{n-1}, \tag{64}$$

$$\gamma_{n+2} = a\gamma_{n+1} + c\gamma_n + (bd - ac)\gamma_{n-1}, \tag{65}$$

$$\delta_{n+2} = a\delta_{n+1} + c\delta_n + (bd - ac)\delta_{n-1}, \tag{66}$$

for  $n \in \mathbb{N}_0$ .

It is important to point out that all the transformations that we have employed transform systems (49)-(54) into equivalent ones. Therefore, there is a bijection between the sets of solutions to the original and transformed systems.

Let

$$A(t_n) = t_{n+2} - at_{n+1} - ct_n + (ac - bd)t_{n-1}, \quad n \geq -1, \tag{67}$$

where  $(t_n)_{n \geq -2}$  is a sequence. Then  $A$  is a linear operator, (56) can be written as  $A(u_n) = d$ ,  $n \in \mathbb{N}_0$ , (58) as  $A(v_n) = 1 - a$ ,  $n \in \mathbb{N}_0$ , and (63)-(66) as

$$A(\alpha_n) = A(\beta_n) = A(\gamma_n) = A(\delta_n) = 0, \quad n \in \mathbb{N}_0.$$

Since the equation

$$A(t_n) = 0 \tag{68}$$

is linear of third order, it can be solved, and specially closed form formulas for  $\alpha_n, \beta_n, \gamma_n, \delta_n$  can be obtained for the corresponding initial conditions (39)-(44). Employing the formulas for  $\alpha_n, \beta_n, \gamma_n, \delta_n$  in (59)-(62), formulas for  $a_n, b_n, c_n, d_n$  are obtained.

When  $h \in \mathbb{R}$ , the equation

$$A(t_n) = h \tag{69}$$

can be also solved (especially for  $h = d$  and  $h = 1 - a$ ), since a particular solution to the equation can be found in the following form:

$$t_n = g_0 + g_1n + g_2n^2 + g_3n^3$$

for some constants  $g_j, j = \overline{0,3}$  (see, e.g., [1, 8]). From this and using (39)-(44), formulas for  $u_n$  and  $v_n$  can be found, from which along with (55) and (57) formulas for  $x_n$  and  $y_n$  are obtained. Using the obtained formulas for these twelve sequences in (47) and (48), we obtain closed form formulas for solutions to system (2).

### 3.1 Forms of solutions to (2) for the case $ac \neq bd$

The characteristic polynomial associated to (26) is

$$P_3(\lambda) = \lambda^3 - a\lambda^2 - c\lambda + ac - bd, \tag{70}$$

which is of the third order due to the assumption  $ac \neq bd$ . By using the change of variables  $\lambda = t + \frac{a}{3}$ , the equation  $P_3(\lambda) = 0$  becomes

$$t^3 + pt + q = 0,$$

where  $p = -(a^2 + 3c)/3$  and  $q = (18ac - 27bd - 2a^3)/27$ , which is solved by looking for solutions in the form  $t = u + v$  and posing the condition  $3uv = -p$ . Then  $u^3$  and  $v^3$  are solutions of the equation  $z^2 + qz - \frac{p^3}{27} = 0$ , from which it follows that

$$\lambda_j = \frac{a}{3} + \varepsilon^{j-1} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \varepsilon^{j-1} \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad j = \overline{1,3}, \tag{71}$$

where  $\varepsilon^3 = 1, \varepsilon \neq 1$ .

If the discriminant

$$\Delta := 4p^3 + 27q^2 \tag{72}$$

is such that  $\Delta \neq 0$ , then all three zeros are different (if  $\Delta > 0$  two zeros are complex conjugate, if  $\Delta < 0$  all three zeros are real), whereas if  $\Delta = 0$ , at least two zeros are equal [41].

Case  $\Delta \neq 0$ . Since in this case the zeros of  $P_3$  are different, the general solution to (26) is

$$u_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n, \quad n \in \mathbb{N}, \tag{73}$$

where  $\alpha_i, i = \overline{1,3}$ , are constants (in fact, (73) holds for every  $n \in \mathbb{Z}$  due to the assumption  $ac \neq bd$ ; see the relation in (27)).

Lemma 1 employed to  $P_3$  shows that

$$\sum_{j=1}^3 \frac{\lambda_j^l}{P'_3(\lambda_j)} = 0, \quad \text{for } l = 0, 1, \quad \text{and} \quad \sum_{j=1}^3 \frac{\lambda_j^2}{P'_3(\lambda_j)} = 1. \tag{74}$$

From (28), (73) and (74) it follows that

$$\begin{aligned} a_n &= \sum_{j=1}^3 \frac{\lambda_j^{n+2}}{P'_3(\lambda_j)} = \frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\ &\quad + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{aligned} \tag{75}$$

for  $n \geq -2$ .

Equalities (30) and (75) yield

$$y_n = \sum_{i=0}^{n-1} \sum_{j=1}^3 \frac{\lambda_j^{i+2}}{P'_3(\lambda_j)}, \quad n \in \mathbb{N}. \tag{76}$$

If  $P_3(1) \neq 0$ , that is,  $(a - 1)(c - 1) \neq bd$ , then (76) yields

$$\begin{aligned} y_n &= \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - 1)} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - 1)} \\ &\quad + \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - 1)}. \end{aligned} \tag{77}$$

Using (74), it is easily seen that (77) holds for  $n \geq -2$ .

If  $P_3(1) = 0$ , that is,  $(a - 1)(c - 1) = bd$ , and  $P'(1) = 3 - 2a - c \neq 0$ , then

$$P_3(\lambda) = \lambda^3 - a\lambda^2 - c\lambda + a + c - 1 = (\lambda - 1)(\lambda^2 + (1 - a)\lambda + 1 - a - c), \tag{78}$$

so that  $\lambda_1 = 1$ ,

$$\lambda_{2,3} = \frac{a - 1 \pm \sqrt{a^2 + 2a + 4c - 3}}{2} \neq 1 \tag{79}$$

and

$$y_n = \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_3)(\lambda_2 - 1)^2} + \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3 - \lambda_2)(\lambda_3 - 1)^2} + \frac{n}{(\lambda_2 - 1)(\lambda_3 - 1)}. \tag{80}$$

Using (74) it is easily checked that (80) holds for  $n \geq -2$  (for a detailed explanation, see [40]).

**Corollary 1** *Assume that  $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ ,  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ ,  $ac \neq bd$  and  $\Delta \neq 0$ . Then the following statements hold.*

- (a) *If  $(a - 1)(c - 1) \neq bd$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (75),  $(y_n)_{n \geq -2}$  is given by (77), and  $\lambda_j, j = \overline{1, 3}$ , are given by (71).*
- (b) *If  $(a - 1)(c - 1) = bd$  and  $2a + c \neq 3$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (75) with  $\lambda_1 = 1$ ,  $(y_n)_{n \geq -2}$  is given by (80),  $\lambda_1 = 1$ , while  $\lambda_{2,3}$  are given by (79).*

**Remark 2** As we have already mentioned,  $P_3$  has a zero equal to 1 if and only if  $(a - 1)(c - 1) = bd$ , so that (78) holds. The condition  $\Delta \neq 0$  will be satisfied if and only if

$$(18ac - 27(a - 1)(c - 1) - 2a^3)^2 \neq 4(a^2 + 3c)^3.$$

For example, if  $a = c \in \mathbb{Z} \setminus \{0, 1\}$  and  $bd = (a - 1)^2 \neq 0$ , then  $\Delta \neq 0$ ,

$$P_3(\lambda) = \lambda^3 - a\lambda^2 - a\lambda + 2a - 1 = (\lambda - 1)(\lambda^2 - (a - 1)\lambda + 1 - 2a),$$

and conditions of Corollary 1(b) are satisfied. Note that  $\lambda_1 = 1$  and

$$\lambda_{2,3} = \frac{a - 1 \pm \sqrt{a^2 + 6a - 3}}{2}.$$

Thus  $P_3$  can have three different zeros, one of which is equal to 1.

*Case  $\Delta = 0$ .* In this case,  $P_3$  has at least two equal zeros. We may assume that  $\lambda_2 = \lambda_3$ . If  $p \neq 0$ , then  $\lambda_2$  is not a triple zero of  $P_3$ , so the general solution to (26) is

$$a_n = \hat{g}_1 \lambda_1^n + (\hat{g}_2 + \hat{g}_3 n) \lambda_2^n, \quad n \in \mathbb{N}, \tag{81}$$

where  $\hat{g}_j, j = \overline{1, 3}$ , are constants. Since the solution satisfies (28), we can find it by letting  $\lambda_3 \rightarrow \lambda_2$  in (75), and obtain

$$a_n = \frac{\lambda_1^{n+2} - (n + 2)\lambda_1 \lambda_2^{n+1} + (n + 1)\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)^2}, \quad n \geq -2, \tag{82}$$

from which along with (30), we get

$$y_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + (\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1))\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \quad n \in \mathbb{N}. \tag{83}$$

If we assume  $\lambda_1 \neq 1 \neq \lambda_2 = \lambda_3$ , then (83) and Lemma 2 imply

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(\lambda_2 - 2\lambda_1)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2}, \quad n \in \mathbb{N}. \tag{84}$$

Moreover, using (74) it is easily checked that (84) holds also for  $n \geq -2$ .

If we assume that  $\lambda_1 \neq 1$  and  $\lambda_2 = \lambda_3 = 1$ , then from (83) it follows that

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - 1)^3} + \frac{(1 - 2\lambda_1)n}{(\lambda_1 - 1)^2} + \frac{(n-1)n}{2(1 - \lambda_1)}. \tag{85}$$

Moreover, (85) holds also for  $n \geq -2$ . This case is obtained when  $(a - 1)(c - 1) = bd$  and  $2a + c = 3$ , which implies

$$P_3(\lambda) = \lambda^3 - a\lambda^2 + (2a - 3)\lambda + 2 - a = (\lambda - 1)^2(\lambda + 2 - a),$$

so that  $\lambda_1 = a - 2$ .

If we assume that  $\lambda_1 = 1$  and  $\lambda_2 = \lambda_3 \neq 1$ , then from (83) it follows that

$$y_n = \frac{n}{(\lambda_2 - 1)^2} + \frac{(\lambda_2 - 2)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - 1)^3} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - 1)^3}. \tag{86}$$

Moreover, (86) holds also for  $n \geq -2$ .

If  $\Delta = p = 0$ , then  $\lambda_1 = \lambda_2 = \lambda_3$ , so the general solution to (26) has the following form:

$$a_n = (\hat{\beta}_1 + \hat{\beta}_2 n + \hat{\beta}_3 n^2)\lambda_1^n, \quad n \in \mathbb{N}, \tag{87}$$

where  $\hat{\beta}_1, \hat{\beta}_2$  and  $\hat{\beta}_3$  are constants.

To find the solution to equation (26) satisfying (28) in this case, we can let  $\lambda_2 \rightarrow \lambda_1$  in (82) and obtain

$$a_n = \frac{(n+1)(n+2)}{2}\lambda_1^n, \tag{88}$$

for  $n \geq -2$ , from which along with (30) we obtain

$$y_n = \sum_{j=0}^{n-1} \frac{(j+1)(j+2)}{2}\lambda_1^j \tag{89}$$

for every  $n \in \mathbb{N}$ . Moreover, (89) holds also for  $n \geq -2$ .

If we assume that  $\lambda_1 = \lambda_2 = \lambda_3 \neq 1$ , then from (87) and Lemma 2 it follows that

$$y_n = \frac{2s_n^{(0)} + 3\lambda_1 s_{n-1}^{(1)} + \lambda_1 s_{n-1}^{(2)}}{2} = \frac{2 - (n+1)(n+2)\lambda_1^n + 2n(n+2)\lambda_1^{n+1} - n(n+1)\lambda_1^{n+2}}{2(1 - \lambda_1)^3} \tag{90}$$

for every  $n \in \mathbb{N}$ . In fact, (90) holds also for  $n \geq -2$ .

If we assume  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ , then from (87) and some calculation, we obtain

$$y_n = \sum_{j=0}^{n-1} \frac{(j+1)(j+2)}{2} = \frac{n(n+1)(n+2)}{6} \tag{91}$$

for every  $n \in \mathbb{N}$ , which obviously holds for  $j = -2, -1, 0$  (for more details, see [28, 40]).

**Corollary 2** *Assume that  $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ ,  $\alpha, \beta, z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ ,  $ac \neq bd$  and  $\Delta = 0$ . Then the following statements are true.*

- (a) *If  $(a-1)(c-1) \neq bd$  and  $a^2 + 3c \neq 0$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (82),  $(y_n)_{n \geq -2}$  is given by (84), while  $\lambda_j, j = \overline{1, 3}$  are given by (71).*
- (b) *If  $(a-1)(c-1) = bd$ ,  $2a + c = 3$  and  $a \neq 3$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (82) with  $\lambda_2 = 1$ ,  $(y_n)_{n \geq -2}$  is given by (85), and  $\lambda_1 = a - 2$ .*
- (c) *If  $(a-1)(c-1) = bd$ ,  $2a + c \neq 3$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (82) with  $\lambda_1 = 1$ ,  $(y_n)_{n \geq -2}$  is given by (86),  $\lambda_1 = 1$ , while  $\lambda_{2,3}$  are given by (79).*
- (d) *If  $(a-1)(c-1) \neq bd$  and  $a^2 + 3c = 0$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (88),  $(y_n)_{n \geq -2}$  is given by (90), while  $\lambda_j = a/3, j = \overline{1, 3}$ .*
- (e) *If  $(a-1)(c-1) = bd$  and  $2a + c = 3$  and  $a = 3$ , then the general solution to system (2) is given by (25) and (35), where  $(a_n)_{n \geq -2}$  is given by (88) with  $\lambda_1 = 1$ , while  $(y_n)_{n \geq -2}$  is given by (91).*

**Remark 3** Recall that equation (70) will have a zero equal to one if  $(a-1)(c-1) = bd$ . If  $\lambda = m \in \mathbb{Z} \setminus \{0\}$  is a double zero of  $P_3$ , then it must be

$$P_3(\lambda) = \lambda^3 - a\lambda^2 - c\lambda + cm + am^2 - m^3, \tag{92}$$

$$P'_3(m) = 3m^2 - 2am - c = 0. \tag{93}$$

From (93) we have that  $c = 3m^2 - 2am$ . Using this in (92), we get

$$P_3(\lambda) = \lambda^3 - a\lambda^2 + (2am - 3m^2)\lambda + 2m^3 - am^2 = (\lambda - m)^2(\lambda - a + 2m). \tag{94}$$

From this, and since  $ac \neq bd$ , it follows that  $a \neq 2m$ . Since  $P'_3(m) \neq 0$ , we also have that it must be  $a \neq 3m$ . Hence, for every  $m \in \mathbb{Z} \setminus \{0\}$  and  $a \in \mathbb{Z}$  such that  $2m \neq a \neq 3m$ , the polynomials in (94) are of type (70), and they have exactly two equal zeros  $\lambda_1 = \lambda_2 = m \neq \lambda_3 = a - 2m$ , and if additionally  $m \neq 1$  and  $a \neq 2m + 1$  hold, we see that the conditions of Corollary 2(a) are satisfied. For  $m = 1$  and  $a \neq 3$ , it is obtained that  $P_3$  can have 1 as a double zero, which means that such polynomials satisfy the conditions of Corollary 2(b). For  $a = 2m + 1$ , we get a polynomial which has a double zero different from 1, which is its third zero. This means that such polynomials satisfy the conditions of Corollary 2(c).

The polynomial has three equal zeros only if  $q = 0$ , which along with  $\Delta = 0$  implies  $p = 0$ . Hence, we have  $c = -a^2/3$  and  $8ac = 9bd$ , which implies that  $c_1 = a^3/27$ , so that

$$P_3(\lambda) = \lambda^3 - a\lambda^2 + \frac{a^2}{3}\lambda - \frac{a^3}{27} = \left(\lambda - \frac{a}{3}\right)^3.$$

Since  $c$  must be an integer, it follows that  $a = 3\hat{a}$  for some  $\hat{a} \in \mathbb{Z}$ , so that  $c = -3\hat{a}^2$  and  $c_1 = \hat{a}^3$ . Hence, for each  $\hat{a} \in \mathbb{Z} \setminus \{0\}$ , these relations give polynomials of the form in (70) which have three equal zeros. This means that such polynomials satisfy the conditions of Corollary 2(d). When  $a = 3$ , polynomial  $P_3$  has 1 as a triple zero, which means that there is a polynomial satisfying the conditions of Corollary 2(e).

Before we formulate our next result, we want to say that the solution to the difference equation

$$z_{n+2} + az_{n+1} + bz_n = 0, \quad n \in \mathbb{N}_0, \tag{95}$$

with initial values  $z_0, z_1 \in \mathbb{C}$ , where  $a \in \mathbb{C}$  and  $b \in \mathbb{C} \setminus \{0\}$ , is given by

$$z_n = \frac{\lambda_2 z_0 - z_1}{\lambda_2 - \lambda_1} \lambda_1^n + \frac{z_1 - \lambda_1 z_0}{\lambda_2 - \lambda_1} \lambda_2^n \tag{96}$$

if  $\lambda_1 \neq \lambda_2$ , and

$$z_n = (z_1 n + \lambda_1 z_0 (1 - n)) \lambda_1^{n-1} \tag{97}$$

if  $\lambda_1 = \lambda_2$ , where  $\lambda_1$  and  $\lambda_2$  are the zeros of the characteristic polynomial  $P_2(\lambda) = \lambda^2 + a\lambda + b$ . This simple result is well known and can be essentially found in any book on difference equations and many papers [6, 8, 25].

**Theorem 5** *Assume that  $a, b, c, d \in \mathbb{Z}$ ,  $ac = bd$ ,  $cd \neq 0$ ,  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  and  $z_{-2}, z_{-1}, w_{-2}, w_{-1} \in \mathbb{C} \setminus \{0\}$ . Then the following statements hold.*

- (a) *If  $a^2 \neq -4c$  and  $a + c \neq 1$ , then the general solution to system (2) is given by the following formulas:*

$$\begin{aligned} z_n = & \alpha^{\frac{1}{1-a-c}} \left( a \frac{(\lambda_2-1)\lambda_1^{n+1} - (\lambda_1-1)\lambda_2^{n+1}}{\lambda_1-\lambda_2} + 1 - c \right) \\ & \times \beta^{\frac{ac}{d(a+c-1)}} \left( \frac{(\lambda_2-1)\lambda_1^{n+1} - (\lambda_1-1)\lambda_2^{n+1}}{\lambda_2-\lambda_1} - 1 \right) z_{-2}^{\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}} \\ & \times z_{-1}^{\frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}} w_{-2}^{\frac{ac^2 \lambda_1^n - \lambda_2^n}{d(\lambda_1 - \lambda_2)}} w_{-1}^{\frac{ac \lambda_1^{n+1} - \lambda_2^{n+1}}{d(\lambda_1 - \lambda_2)}} \end{aligned} \tag{98}$$

and

$$\begin{aligned} w_n = & \alpha^{\frac{d}{1-a-c}} \left( \frac{(\lambda_2-1)\lambda_1^n - (\lambda_1-1)\lambda_2^n}{\lambda_1-\lambda_2} + 1 \right) \beta^{\frac{1}{a+c-1}} \left( c \frac{(\lambda_2-1)\lambda_1^n - (\lambda_1-1)\lambda_2^n}{\lambda_2-\lambda_1} + a - 1 \right) \\ & \times z_{-2}^{\frac{cd \lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2}} z_{-1}^{\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}} w_{-2}^{\frac{c^2 \lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2}} w_{-1}^{\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}}, \end{aligned} \tag{99}$$

where

$$\lambda_{1,2} = \frac{a + \sqrt{a^2 + 4c}}{2}. \tag{100}$$

(b) If  $a^2 = -4c$  and  $a + c \neq 1$ , then the general solution to system (2) is given by the following formulas:

$$\begin{aligned}
 z_n &= \alpha \frac{1}{a+c-1} ((-n+1)\lambda_1^3 + (n+2)\lambda_1^2 + c(n-1)\lambda_1 - cn)\lambda_1^{n-1+c-1} \\
 &\quad \times \beta \frac{c}{d(a+c-1)} ((-n+1)\lambda_1^3 + (n+2)\lambda_1^2 + c(n-1)\lambda_1 - cn)\lambda_1^{n-1-a} z_{-2}^{(c(n-1)-(n+1)\lambda_1^2)\lambda_1^n} \\
 &\quad \times z_{-1}^{((n+2)\lambda_1^2 - cn)\lambda_1^{n-1}} w_{-2}^{\frac{c}{d}(c(n-1)-(n+1)\lambda_1^2)\lambda_1^n} w_{-1}^{\frac{c}{d}((n+2)\lambda_1^2 - cn)\lambda_1^{n-1}}
 \end{aligned} \tag{101}$$

and

$$\begin{aligned}
 w_n &= \alpha \frac{d}{a+c-1} ((\lambda_1 + n(1-\lambda_1))\lambda_1^{n-1} - 1) \beta \frac{1}{a+c-1} (c(\lambda_1 + (1-\lambda_1)n)\lambda_1^{n-1+a-1}) z_{-2}^{d(1-n)\lambda_1^n} \\
 &\quad \times z_{-1}^{dn\lambda_1^{n-1}} w_{-2}^{c(1-n)\lambda_1^n} w_{-1}^{cn\lambda_1^{n-1}},
 \end{aligned} \tag{102}$$

where

$$\lambda_1 = a/2. \tag{103}$$

(c) If  $a^2 \neq -4c$  and  $a + c = 1$ , then the general solution to system (2) is given by the following formulas:

$$\begin{aligned}
 z_n &= \alpha \frac{a(a-1)^{n+1} - n(a-1)^2 - 3(a-1) + n + 1}{(a-2)^2} \\
 &\quad \times \beta \frac{-a(a-1)^{n+2} + na(1-a)(2-a) + a(a-1)^2}{d(2-a)^2} z_{-2}^{a(a-1)\frac{(a-1)^n - 1}{2-a}} \\
 &\quad \times z_{-1}^{a\frac{(a-1)^{n+1} - 1}{a-2}} w_{-2}^{\frac{a(a-1)^2((a-1)^n - 1)}{d(a-2)}} w_{-1}^{\frac{a(a-1)((a-1)^{n+1} - 1)}{d(2-a)}}
 \end{aligned} \tag{104}$$

and

$$\begin{aligned}
 w_n &= \alpha \frac{d((a-1)^n + n(1-a) + n - 1)}{(a-2)^2} \beta \frac{-(a-1)^{n+1} + (2-a)^2 - 1 + a + n(1-a)(2-a)}{(2-a)^2} z_{-2}^{d(a-1)\frac{(a-1)^{n-1} - 1}{2-a}} \\
 &\quad \times z_{-1}^{d\frac{(a-1)^n - 1}{a-2}} w_{-2}^{\frac{(a-1)^2((a-1)^{n-1} - 1)}{a-2}} w_{-1}^{\frac{(a-1)((a-1)^n - 1)}{2-a}}.
 \end{aligned} \tag{105}$$

(d) If  $a^2 = -4c$  and  $a + c = 1$ , then the general solution to system (2) is given by the following formulas:

$$z_n = \alpha^{n^2+n+1} \beta^{b\frac{n(n+1)}{2}} z_{-2}^{-2n} z_{-1}^{2n+2} w_{-2}^{-bn} w_{-1}^{b(n+1)} \tag{106}$$

and

$$w_n = \alpha^{\frac{d(n-1)n}{2}} \beta^{\frac{2+n-n^2}{2}} z_{-2}^{d(1-n)} z_{-1}^{dn} w_{-2}^{n-1} w_{-1}^{-n}. \tag{107}$$

*Proof* First note that in this case equalities (49)-(66) also hold. Since  $ac = bd$  and  $c \neq 0$ , we have

$$u_{n+2} = au_{n+1} + cu_n + d, \tag{108}$$



$$v_{n+2} = av_{n+1} + cv_n + 1 - a, \tag{109}$$

$$\alpha_{n+2} = a\alpha_{n+1} + c\alpha_n, \tag{110}$$

$$\beta_{n+2} = a\beta_{n+1} + c\beta_n, \tag{111}$$

$$\gamma_{n+2} = a\gamma_{n+1} + c\gamma_n, \tag{112}$$

$$\delta_{n+2} = a\delta_{n+1} + c\delta_n \tag{113}$$

for  $n \in \mathbb{N}_0$ , while operator  $A$  defined in (67) becomes

$$A(t_n) = t_{n+2} - at_{n+1} - ct_n.$$

Now, equation (68) is of the second order, so solvable in closed form. Consequently, (110)-(113) are solvable, so that closed form formulas for  $\alpha_n, \beta_n, \gamma_n, \delta_n$  can be found by employing the corresponding conditions in (39)-(44). Such obtained formulas along with (59)-(62) yield formulas for  $a_n, b_n, c_n, d_n$ . Further, (69) is also solvable, since a particular solution to the equation has the form

$$t_n = \hat{d}_0 + \hat{d}_1n + \hat{d}_2n^2$$

for some constants  $\hat{d}_j, j = \overline{0,2}$ . Specially, it is solved for  $f = d$  and  $f = 1 - a$ , which gives formulas for the sequences  $u_n$  and  $v_n$ , and consequently closed form formulas for  $x_n$  and  $y_n$ . Using such obtained formulas in (47) and (48), we get formulas for solutions to system (2) in this case.

(a) *Case  $a^2 + 4c \neq 0, a + c \neq 1$ .* In this case, a particular solution to equation (108) is

$$u_n^p = \frac{d}{1 - a - c}, \tag{114}$$

so that

$$u_n = c_1\lambda_1^n + c_2\lambda_2^n + \frac{d}{1 - a - c}, \quad n \in \mathbb{N}_0,$$

where

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4c}}{2}, \tag{115}$$

from which along with  $u_0 = u_1 = 0$  it follows that

$$c_1 = \frac{d(\lambda_2 - 1)}{(a + c - 1)(\lambda_2 - \lambda_1)} \quad \text{and} \quad c_2 = \frac{d(1 - \lambda_1)}{(a + c - 1)(\lambda_2 - \lambda_1)},$$

and consequently

$$u_n = \frac{d}{1 - a - c} \left( \frac{(\lambda_2 - 1)\lambda_1^n - (\lambda_1 - 1)\lambda_2^n}{\lambda_1 - \lambda_2} + 1 \right). \tag{116}$$

Using (116) in (55), we get

$$x_n = \frac{1}{1-a-c} \left( \frac{(\lambda_2 - 1)(\lambda_1^2 - c)\lambda_1^n - (\lambda_1 - 1)(\lambda_2^2 - c)\lambda_2^n}{\lambda_1 - \lambda_2} + 1 - c \right). \tag{117}$$

Particular solution to equation (109) in this case is

$$v_n^p = \frac{1-a}{1-a-c}, \tag{118}$$

so that

$$v_n = d_1 \lambda_1^n + d_2 \lambda_2^n + \frac{1-a}{1-a-c}, \quad n \in \mathbb{N}_0,$$

where  $\lambda_{1,2}$  are given by (115), from which along with  $v_0 = v_1 = 1$  it follows that

$$d_1 = \frac{c(\lambda_2 - 1)}{(a+c-1)(\lambda_2 - \lambda_1)} \quad \text{and} \quad d_2 = \frac{c(1 - \lambda_1)}{(a+c-1)(\lambda_2 - \lambda_1)},$$

and consequently

$$v_n = \frac{1}{a+c-1} \left( c \frac{(\lambda_2 - 1)\lambda_1^n - (\lambda_1 - 1)\lambda_2^n}{\lambda_2 - \lambda_1} + a - 1 \right). \tag{119}$$

Using (119) in (57), we get

$$y_n = \frac{c}{d(a+c-1)} \left( \frac{(\lambda_2 - 1)(\lambda_1^2 - c)\lambda_1^n - (\lambda_1 - 1)(\lambda_2^2 - c)\lambda_2^n}{\lambda_2 - \lambda_1} - a \right). \tag{120}$$

By using the corresponding conditions in (43)-(46) in formula (96), we get

$$\alpha_n = d \lambda_1 \lambda_2 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_2 - \lambda_1}, \tag{121}$$

$$\beta_n = d \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \tag{122}$$

$$\gamma_n = c \lambda_1 \lambda_2 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_2 - \lambda_1}, \tag{123}$$

$$\delta_n = c \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \tag{124}$$

By using (121)-(124) in (59)-(62), we get

$$a_n = \lambda_1 \lambda_2 \frac{(\lambda_1^2 - c)\lambda_1^{n-1} - (\lambda_2^2 - c)\lambda_2^{n-1}}{\lambda_2 - \lambda_1}, \tag{125}$$

$$b_n = \frac{(\lambda_1^2 - c)\lambda_1^n - (\lambda_2^2 - c)\lambda_2^n}{\lambda_1 - \lambda_2}, \tag{126}$$

$$c_n = \frac{c \lambda_1 \lambda_2}{d} \frac{(\lambda_1^2 - c)\lambda_1^{n-1} - (\lambda_2^2 - c)\lambda_2^{n-1}}{\lambda_2 - \lambda_1}, \tag{127}$$

$$d_n = \frac{c}{d} \frac{(\lambda_1^2 - c)\lambda_1^n - (\lambda_2^2 - c)\lambda_2^n}{\lambda_1 - \lambda_2}. \tag{128}$$

Using (116), (117), (119)-(128) in (47) and (48), as well as the fact that  $\lambda_{1,2}$  are roots of the polynomial  $\lambda^2 - a\lambda - c$ , and after some standard calculations, formulas (98) and (99) are obtained.

(b) *Case  $a^2 + 4c = 0, a + c \neq 1$ .* In this case, we have  $\lambda_1 = \lambda_2 \neq 1$  and a particular solution to equation (108) is given by (114), so that

$$u_n = (c_1 + c_2n)\lambda_1^n + \frac{d}{1 - a - c}, \quad n \in \mathbb{N}_0,$$

where  $\lambda_1$  is given by (103), from which along with  $u_0 = u_1 = 0$  it follows that

$$c_1 = \frac{d}{a + c - 1} \quad \text{and} \quad c_2 = \frac{d(1 - \lambda_1)}{(a + c - 1)\lambda_1},$$

and consequently

$$u_n = \frac{d}{a + c - 1} ((\lambda_1 + n(1 - \lambda_1))\lambda_1^{n-1} - 1). \tag{129}$$

Using (129) in (55), we get

$$x_n = \frac{1}{a + c - 1} ((- (n + 1)\lambda_1^3 + (n + 2)\lambda_1^2 + c(n - 1)\lambda_1 - cn)\lambda_1^{n-1} + c - 1). \tag{130}$$

Particular solution to equation (109) in this case is given by (118), so that

$$v_n = (d_1 + d_2n)\lambda_1^n + \frac{1 - a}{1 - a - c}, \quad n \in \mathbb{N}_0,$$

where  $\lambda_1$  is given by (103), from which along with  $v_0 = v_1 = 1$  it follows that

$$d_1 = \frac{c}{a + c - 1} \quad \text{and} \quad d_2 = \frac{c(1 - \lambda_1)}{(a + c - 1)\lambda_1},$$

and consequently

$$v_n = \frac{1}{a + c - 1} (c(\lambda_1 + (1 - \lambda_1)n)\lambda_1^{n-1} + a - 1). \tag{131}$$

Using (131) in (57), we get

$$y_n = \frac{c}{d(a + c - 1)} ((- (n + 1)\lambda_1^3 + (n + 2)\lambda_1^2 + c(n - 1)\lambda_1 - cn)\lambda_1^{n-1} - a). \tag{132}$$

By using the corresponding conditions in (43)-(46) in formula (97), we get

$$\alpha_n = d(1 - n)\lambda_1^n, \tag{133}$$

$$\beta_n = dn\lambda_1^{n-1}, \tag{134}$$

$$\gamma_n = c(1 - n)\lambda_1^n, \tag{135}$$

$$\delta_n = cn\lambda_1^{n-1}. \tag{136}$$

By using (133)-(136) in (59)-(62), we get

$$a_n = (c(n - 1) - (n + 1)\lambda_1^2)\lambda_1^n, \tag{137}$$

$$b_n = ((n + 2)\lambda_1^2 - cn)\lambda_1^{n-1}, \tag{138}$$

$$c_n = \frac{c}{d}(c(n - 1) - (n + 1)\lambda_1^2)\lambda_1^n, \tag{139}$$

$$d_n = \frac{c}{d}((n + 2)\lambda_1^2 - cn)\lambda_1^{n-1}. \tag{140}$$

Using the equalities in (129)-(140), in (47) and (48), formulas (101) and (102) are obtained.

(c) *Case*  $a^2 + 4c \neq 0, a + c = 1$ . In this case, we have that  $\lambda_1 \neq 1 = \lambda_2$  and that a particular solution to equation (108) is given by

$$u_n^p = \frac{dn}{2 - a}, \tag{141}$$

so that

$$u_n = c_1\lambda_1^n + c_2 + \frac{dn}{2 - a}, \quad n \in \mathbb{N}_0,$$

where  $\lambda_1 = -c = a - 1$ , from which along with  $u_0 = u_1 = 0$  it follows that

$$c_1 = -c_2 = \frac{d}{(a - 2)(\lambda_1 - 1)} = \frac{d}{(a - 2)^2},$$

and consequently

$$\begin{aligned} u_n &= \frac{d}{a - 2} \left( \frac{\lambda_1^n - 1}{\lambda_1 - 1} - n \right) = \frac{d(\lambda_1^n - n\lambda_1 + n - 1)}{(a - 2)^2} \\ &= \frac{d((a - 1)^n + n(1 - a) + n - 1)}{(a - 2)^2}. \end{aligned} \tag{142}$$

Using (142) in (55), we get

$$x_n = \frac{a(a - 1)^{n+1} - n(a - 1)^2 - 3(a - 1) + n + 1}{(a - 2)^2}. \tag{143}$$

Particular solution to equation (109) in this case is given by

$$v_n^p = \frac{n(1 - a)}{2 - a}, \tag{144}$$

so that

$$v_n = d_1\lambda_1^n + d_2 + \frac{n(1 - a)}{2 - a}, \quad n \in \mathbb{N}_0,$$

from which along with  $v_0 = v_1 = 1$  it follows that

$$d_1 = \frac{1 - a}{(2 - a)^2} \quad \text{and} \quad d_2 = 1 - \frac{1 - a}{(2 - a)^2},$$

and consequently

$$v_n = \frac{-(a-1)^{n+1} + (2-a)^2 - 1 + a + n(1-a)(2-a)}{(2-a)^2}. \tag{145}$$

Using (145) in (57), we get

$$y_n = \frac{-a(a-1)^{n+2} + na(1-a)(2-a) + a(a-1)^2}{d(2-a)^2}. \tag{146}$$

By using the corresponding conditions in (43)-(46) in formula (96), we get

$$\alpha_n = d\lambda_1 \frac{\lambda_1^{n-1} - 1}{1 - \lambda_1} = d(a-1) \frac{(a-1)^{n-1} - 1}{2-a}, \tag{147}$$

$$\beta_n = d \frac{\lambda_1^n - 1}{\lambda_1 - 1} = d \frac{(a-1)^n - 1}{a-2}, \tag{148}$$

$$\gamma_n = c\lambda_1 \frac{\lambda_1^{n-1} - 1}{1 - \lambda_1} = (a-1)^2 \frac{(a-1)^{n-1} - 1}{a-2}, \tag{149}$$

$$\delta_n = c \frac{\lambda_1^n - 1}{\lambda_1 - 1} = (a-1) \frac{(a-1)^n - 1}{2-a}. \tag{150}$$

By using (147)-(150) in (59)-(62), we get

$$a_n = a(a-1) \frac{(a-1)^n - 1}{2-a}, \tag{151}$$

$$b_n = a \frac{(a-1)^{n+1} - 1}{a-2}, \tag{152}$$

$$c_n = \frac{a(a-1)^2((a-1)^n - 1)}{d(a-2)}, \tag{153}$$

$$d_n = \frac{a(a-1)((a-1)^{n+1} - 1)}{d(2-a)}. \tag{154}$$

Using (142), (143), (145)-(154) in (47) and (48), formulas (104) and (105) are obtained.

(d) *Case*  $a^2 + 4c = 0, a + c = 1$ . In this case, we have that  $a = 2$  and  $c = -1$ . We also have that  $\lambda_1 = \lambda_2 = 1$ , and that a particular solution to equation (108) is given by

$$u_n^p = \frac{dn^2}{2}, \tag{155}$$

so that

$$u_n = c_1 + c_2n + \frac{dn^2}{2}, \quad n \in \mathbb{N}_0,$$

from which along with  $u_0 = u_1 = 0$  it follows that  $c_1 = 0$  and  $c_2 = -\frac{d}{2}$ , and consequently

$$u_n = \frac{d(n-1)n}{2}. \tag{156}$$

Using (156) in (55), we get

$$x_n = n^2 + n + 1. \tag{157}$$

Particular solution to equation (109) in this case is given by

$$v_n^p = -\frac{n^2}{2}, \tag{158}$$

so that

$$v_n = d_1 + d_2n - \frac{n^2}{2}, \quad n \in \mathbb{N}_0,$$

from which along with  $v_0 = v_1 = 1$  it follows that  $d_1 = 1$  and  $d_2 = 1/2$ , and consequently

$$v_n = \frac{2 + n - n^2}{2}. \tag{159}$$

Using (159) in (57), we get

$$y_n = -\frac{n(n+1)}{d} = \frac{bn(n+1)}{2}. \tag{160}$$

By using the corresponding conditions in (43)-(46) in formula (97), we get

$$\alpha_n = d(1 - n), \tag{161}$$

$$\beta_n = dn, \tag{162}$$

$$\gamma_n = n - 1, \tag{163}$$

$$\delta_n = -n. \tag{164}$$

By using (161)-(164) in (59)-(62), we get

$$a_n = -2n, \tag{165}$$

$$b_n = 2n + 2, \tag{166}$$

$$c_n = \frac{2n}{d} = -bn, \tag{167}$$

$$d_n = -\frac{2(n+1)}{d} = b(n+1). \tag{168}$$

Using (156), (157), (159)-(168) in (47) and (48), formulas (106) and (107) are obtained.

By some calculation it can be proved that formulas (99)-(107) really satisfy system (2), which was checked by the author. □

**Competing interests**

The author declares that he has no competing interests.

**Author's contributions**

The author contributed solely to the writing of this paper. He read and approved the manuscript.

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