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Multiple solutions for the fractional differential equation with concave-convex nonlinearities and sign-changing weight functions

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Abstract

In this paper, by using the fibering map and the Nehari manifold, we prove the existence and multiple results of solutions for the following fractional differential equation:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u) = \lambda h(t)|u|^{p-2}u + b(t)|u|^{q-2}u, & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (\frac{1}{2}, 1)$, $0 < p < 2$, $q > 2$, $\lambda > 0$ and $h(t)$, $b(t)$ are sign-changing continuous functions.

Keywords: fractional differential equation; concave-convex nonlinearities; Nehari manifold; fibering map

1 Introduction

The concept of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) is believed to have stemmed from a question raised in 1695 by L'Hôpital to Leibniz, which sought the meaning of Leibniz's derivative notation $\frac{d^n x}{dt^n}$ of order $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$?). There have been many mathematicians who contributed to the study of fractional operator, and we can refer to the monographs of Kilbas [1], Podlubny [2], Samko [3], etc. An important characteristic of a fractional-order differential operator that distinguishes it from an integer-order differential operator is its nonlocal behavior, that is, the future state of a dynamical system or process involving fractional derivatives depends on its current state as well as its past states. During the last three decades or so, due to its demonstrated applications in numerous fields of science and engineering, such as viscoelasticity, neurons, electrochemistry, control (see [4–9]), more attention was paid to the fractional differential equations.

Many important results have been obtained about the existence and multiplicity of solutions for fractional boundary value problems based on the techniques of nonlinear analysis, such as fixed point theory [10–13], topological degree theory [14–16], the method of upper and lower solutions and the monotone iterative method [17].

As is well known, the variational method has turned out to be a very effective tool in studying the existence of solutions for boundary value problems (BVPs for short) of integer order differential equations with variational structure. However, most fractional differential operators do not have a variational structure, for example, the operator $D^\alpha(\alpha \notin \mathbb{N})$, so the variational method cannot be applied. On the other hand, for the operator including both left and right fractional derivatives, the critical point theory can be used. In recent years, many authors have studied the existence of solutions of the fractional boundary value problems (FBVPs for short) by use of the variational method [18–25]. The author of [18, 19] treated fractional order differential equations that contain left and right Riemann-Liouville fractional derivatives. The equations arose as the Euler-Lagrange equation in variational principles with fractional derivatives. They discussed solutions of such equations $({}_tD_T^\alpha({}_0D_t^\alpha y)(x) = \lambda y(x) + g(x))$ or constructed corresponding integral equations and other properties. In the paper [20], for the first time, the authors showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

and obtained the existence of at least one nontrivial solution. What is more, in the paper [23], more precisely they studied the fractional nonlinear Dirichlet problem, and they proved the existence of mountain pass solution for the proposed fractional boundary value problem. Jin Hua [21] discussed the eigenvalue problem for the fractional differential equation containing left and right fractional derivatives with Dirichlet boundary value conditions. For fractional Hamiltonian systems given by

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t,u(t)), \\ u \in H^\alpha(R,R^N), \end{cases}$$

the author of paper [24] proved the existence of solution; and in the paper [26], by the critical point theory, they considered the existence and multiplicity of solutions. Such differential equations mixing both types of derivatives have found interesting applications in fractional variational principles, fractional control theory, fractional Lagrangian and Hamiltonian dynamics as well as in the construction industry (see [27–35]).

However, as far as we know, there are few results about the multiplicity of solutions on the fractional equations involving concave-convex nonlinearities and sign-changing weight functions. In order to improve fractional boundary value problem, we use the fibering map and the Nehari manifold to investigate the existence and multiple results of solutions for the following fractional differential equation when the parameter belongs to a different interval. In this paper, in the fractional Sobolev space $E_0^{\alpha,2}$, we investigate the existence and multiplicity of solutions for the following FBVPs:

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u) = \lambda h(t)|u|^{p-2}u + b(t)|u|^{q-2}u, & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha \leq 1, 0 < p < 2, q > 2$. Note that, when $\alpha = 1$, the fractional differential operator ${}_tD_T^\alpha {}_0D_t^\alpha$ reduces to the standard second differential operator $-D^2$.

In the following, we set $c_{pq} = (\frac{2-p}{q-p})^{\frac{2-p}{q-2}} \frac{q-2}{q-p}$ and $\lambda_1 = c_{pq}(c_b c_q^q)^{\frac{p-2}{q-2}} / c_h c_p^p$, where $c_h = \max\{|h(t)||t \in [0,T]\}, c_b = \max\{|b(t)||t \in [0,T]\}$, c_p, c_q are the Sobolev embedding constants. For the sign-changing weight functions, we suppose the following.

$(f1)$ There exists $u \in E_0^{\alpha,2}$ such that $\int_0^T h(t)|u(t)|^q \, dt > 0$.
$(f2)$ There exists $v \in E_0^{\alpha,2}$ such that $\int_0^T b(t)|v(t)|^p \, dt > 0$.

The main theorems are as follows.

Theorem 1.1 *If* $\lambda \in (0, \lambda_1)$ *and* $h(t)$ *satisfies* $(f1)$, *problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.2 *If* $\lambda \in (0, \frac{p}{2}\lambda_1)$, *and* $h(t), b(t)$ *satisfy* $(f1), (f2)$, *problem* (1.1) *has at least two nontrivial solutions.*

2 Preliminaries

For the convenience of readers, in this section, the definitions of fractional integral and fractional derivative are presented. Since we use the critical point theory to investigate problem (1.1), the appropriate fractional Sobolev space is necessary.

Definition 2.1 ([1]) For $n-1 \leq \alpha < n$, the left (right) Riemann-Liouville fractional integral operator of order α of a function $u : [a,b] \to \mathbb{R}$ is given by

$$
\begin{aligned}
{}_aI_t^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) \, ds, \quad t \in [a,b], \\
{}_tI_b^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) \, ds, \quad t \in [a,b],
\end{aligned}
$$

provided that the right-hand side integral is pointwise defined on $[a,b]$, where $\Gamma(\cdot) > 0$ is the gamma function.

Definition 2.2 ([1]) For $n-1 \leq \alpha < n$, the left (right) Riemann-Liouville fractional derivative operator of order α of a function $u : [a,b] \to \mathbb{R}$ is given by

$$
\begin{aligned}
{}_aD_t^\alpha u(t) &= \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} u(t), \quad t \in [a,b], \\
{}_tD_b^\alpha u(t) &= (-1)^n \frac{d^n}{dt^n} {}_tI_b^{n-\alpha} u(t), \quad t \in [a,b].
\end{aligned}
$$

Next we give the definitions of left and right weak fractional derivatives and the corresponding function spaces. For the details, we refer to [21, 36].

Definition 2.3 Let $n-1 \leq \alpha < n$, $u, v \in L^1[0,T]$, if

$$
\int_0^T v\varphi = \int_0^T u\left({}_tD_T^\alpha \varphi\right), \quad \forall \varphi \in C_0^\infty(0,T),
$$

then v is named the left weak fractional derivative, and we denote it by $v = {}_0\dot{D}_t^\alpha u$.

Definition 2.4 Let $n-1 \leq \alpha < n$, $u, v \in L^1[0,T]$, if

$$\int_0^T v\varphi = \int_0^T u\left({}_0D_t^\alpha \varphi\right), \quad \forall \varphi \in C_0^\infty(0,T),$$

then v is named the right weak fractional derivative denoted by ${}_t\dot{D}_T^\alpha u$.

Definition 2.5 For $1 \leq p \leq \infty$, $0 < \alpha \leq 1$, the space $E^{\alpha,2}$ is defined by

$$E^{\alpha,2} = \left\{u | u \in L^2[0,T], {}_0\dot{D}_t^\alpha u \in L^2[0,T]\right\}$$

with the norm

$$\|u\|^2 = \|u\|^2_{E^{\alpha,2}} = \|u\|^2_{L^2} + \left\|{}_0\dot{D}_t^\alpha u\right\|^2_{L^2},$$

and the product

$$(u,v)_{E^{\alpha,2}} = (u,v)_{L^2} + \left({}_0\dot{D}_t^\alpha u, {}_0\dot{D}_t^\alpha v\right)_{L^2}.$$

Definition 2.6 Fractional Sobolev space $E_0^{\alpha,2}$ is defined by the closure of $C_0^\infty(0,T)$ in $E^{\alpha,2}$ equipped with the norm of $E^{\alpha,2}$. What is more, by studying Remark 3.7 of paper [21], we can obtain that the norm $\|\cdot\|_{E^{\alpha,2}}$ is equivalent to the norm $\|{}_0\dot{D}_t^\alpha u\|_{L^2}$.

Lemma 2.7 ([21]) *If $\alpha \in (\frac{1}{2},1)$, the embedding map from $E_0^{\alpha,2}$ into $C[0,T]$ is compact, and it is also true for the embedding map from $E_0^{\alpha,2}$ into $L^r[0,T]$ $(r \in \mathbb{R}^+)$. So there exists a constant c_r such that $\|u(t)\|_{L^r[0,T]} \leq c_r\|u\|$.*

Remark 2.8 From Theorem 3.11 in [21], we obtain that any $u \in E_0^{\alpha,2}$ $(\alpha \in (1/2,1))$ satisfies $u \in L^2[0,T]$, ${}_0\dot{D}_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$. In the following, we all set $\alpha \in (1/2,1)$.

3 Proof of Theorems 1.1 and 1.2

In this section, we investigate the existence of solutions of equation (1.1) when the parameter λ changes by using the fibering map and the Nehari manifold.

The Euler functional $I_\lambda : E_0^{\alpha,2} \to \mathbb{R}$ associated with problem (1.1) is defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_0^T h(t)\left|u(t)\right|^p\,dt - \frac{1}{q}\int_0^T b(t)\left|u(t)\right|^q\,dt. \tag{3.1}$$

It is easy to see that $I_\lambda(u)$ is C^1 and

$$\left\langle I_\lambda{}'(u), u\right\rangle = \|u\|^2 - \lambda\int_0^T h(t)\left|u(t)\right|^p\,dt - \int_0^T b(t)\left|u(t)\right|^q\,dt, \quad \forall u \in E_0^{\alpha,2}.$$

It is obvious that I_λ is not bounded below on $E_0^{\alpha,2}$, but it is bounded below on an appropriate subset of $E_0^{\alpha,2}$, and a minimizer on this set (if it exists) may give rise to solutions of the corresponding differential equation. A good candidate for the subset is the so-called

Nehari manifold

$$N_\lambda = \{u \in E_0^{\alpha,2} \setminus \{0\} | \langle I_\lambda{}'(u), u\rangle = 0\}.$$

It is clear that all critical points of I_λ must lie on N_λ. On the other hand, if $u \in N_\lambda$, we have

$$\|u\|^2 - \lambda \int_0^T h(t)|u(t)|^p\,dt - \int_0^T b(t)|u(t)|^q\,dt = 0. \tag{3.2}$$

Define the fibering map $\varphi_u(s) = I_\lambda(su)$ by

$$\varphi_u(s) = \frac{s^2}{2}\|u\|^2 - \frac{\lambda s^p}{p}\int_0^T h(t)|u(t)|^p\,dt - \frac{s^q}{q}\int_0^T b(t)|u(t)|^q\,dt.$$

After a simple calculation, we have

$$\begin{aligned}\varphi_u'(s) &= s\|u\|^2 - \lambda s^{p-1}\int_0^T h(t)|u(t)|^p\,dt - s^{q-1}\int_0^T b(t)|u(t)|^q\,dt,\\ \varphi_u''(s) &= \|u\|^2 - \lambda(p-1)s^{p-2}\int_0^T h(t)|u(t)|^p\,dt - (q-1)s^{q-2}\int_0^T b(t)|u(t)|^q\,dt.\end{aligned} \tag{3.3}$$

It is easy to see that $su \in N_\lambda$ if and only if $\varphi_u'(s) = 0$ and $u \in N_\lambda$ if and only if $\varphi_u'(1) = 0$. That is to say, if u is the minimizer point of I_λ, $\varphi_u(s)$ has the local minimum or maximum at $s = 1$. Thus it is natural to split N_λ into three subsets N_λ^+, N_λ^-, N_λ^0 corresponding to local minima, local maxima and points of inflexion of a fibering map. Hence we define

$$\begin{aligned}N_\lambda^+ &= \{u \in N_\lambda | \varphi_u''(1) > 0\},\\ N_\lambda^- &= \{u \in N_\lambda | \varphi_u''(1) < 0\},\\ N_\lambda^0 &= \{u \in N_\lambda | \varphi_u''(1) = 0\}.\end{aligned}$$

Lemma 3.1 (see [37]) *Suppose that $u \in N_\lambda$ is a local minimizer of I_λ on N_λ and $u \notin N_\lambda^0$, then u is a critical point of I_λ.*

Lemma 3.2 *$I_\lambda(u)$ is coercive and bounded from below on N_λ.*

Proof Let $u \in N_\lambda$, then we have

$$\|u\|^2 - \lambda \int_0^T h(t)|u(t)|^p\,dt = \int_0^T b(t)|u(t)|^q\,dt.$$

From Lemma 2.7, we obtain

$$\left|\int_0^T h(t)|u(t)|^p\,dt\right| \le c_h \int_0^T |u(t)|^p\,dt \le c_h c_p^p \|u\|^p,$$

where $c_h = \max\{|h(t)| | t \in [0,T]\}$ and c_p is the Sobolev embedding constant.

For $u \in N_\lambda$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\int_0^T h(t)\left|u(t)\right|^p dt - \frac{1}{q}\left(\|u\|^2 - \lambda\int_0^T h(t)\left|u(t)\right|^p dt\right) \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^T h(t)\left|u(t)\right|^p dt \\ &\geq \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \lambda c_h c_p^p\left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^p. \end{aligned}$$

Since $0 < p < 2, q > 2$, it is easy to see that the functional $I_\lambda(u)$ is coercive and bounded from below on N_λ. The proof is finished. □

Before studying the behavior of a Nehari manifold by using a fibering map, we consider the function $\psi_u(s) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_u(s) = s^{2-p}\|u\|^2 - s^{q-p}\int_0^T b(t)\left|u(t)\right|^q dt.$$

It is obvious that $\psi_u(0) = 0$ and

$$\psi_u'(s) = (2-p)s^{1-p}\|u\|^2 - (q-p)s^{q-p-1}\int_0^T b(t)\left|u(t)\right|^q dt. \tag{3.4}$$

It follows from (3.3) that $\varphi_u'(s) = s^{p-1}(\psi_u(s) - \lambda\int_0^T h(t)|u(t)|^p dt)$. Therefore, $su \in N_\lambda$ if and only if $\psi_u(s) = \lambda\int_0^T h(t)|u(t)|^p dt$. Thus, if $\psi_u(s)$ and $\lambda\int_0^T h(t)|u(t)|^p dt$ have the same sign, $\varphi_u(s)$ has stationary points, and if $\psi_u(s)$ and $\lambda\int_0^T h(t)|u(t)|^p dt$ have the opposite signs, $\varphi_u(s)$ has no stationary points.

For the convenience of investigating the fibering map according to the sign of $\int_0^T h(t)|u(t)|^p dt$ and $\int_0^T b(t)|u(t)|^q dt$, we introduce some notations.

$$\begin{aligned} B^\pm &= \left\{u \in E_0^{\alpha,2}\backslash\{0\} : \int_0^T b(t)\left|u(t)\right|^q dt \gtrless 0\right\}, \\ B^0 &= \left\{u \in E_0^{\alpha,2}\backslash\{0\} : \int_0^T b(t)\left|u(t)\right|^q dt = 0\right\}, \\ H^\pm &= \left\{u \in E_0^{\alpha,2}\backslash\{0\} : \int_0^T h(t)\left|u(t)\right|^p dt \gtrless 0\right\}, \\ H^0 &= \left\{u \in E_0^{\alpha,2}\backslash\{0\} : \int_0^T h(t)\left|u(t)\right|^p dt = 0\right\}. \end{aligned}$$

Case 1: If $u \in B^{\underline{0}} \cap H^{\underline{0}}$, $\psi_u(s) \geq 0$ and is strictly increasing for all $s > 0$. As $\lambda\int_0^T h(t)|u(t)|^p dt \leq 0$, so $\varphi_u(s)$ has no stationary points.

Case 2: If $u \in B^{\underline{0}} \cap H^+$, $\psi_u(s) \geq 0$ and is strictly increasing for all $s > 0$. Since $\lambda\int_0^T h(t)|u(t)|^p dt \geq 0$, there exists a unique point s_1^* such that $s_1^* u \in N_\lambda$. We also get that for $0 < s < s_1^*$, $\varphi_u'(s) < 0$ and for $s > s_1^*$, $\varphi_u'(s) > 0$. So $\varphi_u(s)$ attains its minimum at s_1^*, which means that $s_1^* u \in N_\lambda^+$.

Case 3: If $u \in B^+ \cap H^{\underline{0}}$, $\psi_u(s) \geq 0$ for s small enough and $\psi_u(s) \to -\infty$ as $s \to +\infty$. From (3.4), $\psi_u(s)$ has a unique maximum stationary point at $s^* = (\frac{(2-p)\|u\|^2}{(q-p)\int_0^T b(t)|u|^q dt})^{\frac{1}{q-2}}$ such that

$\psi_u'(s^*) = 0$. For $s > 0$, we infer that

$$\max\{\psi_u(s)\} = \psi_u(s^*) = \left(\frac{2-p}{q-p}\right)^{\frac{2-p}{q-2}} \frac{q-2}{q-p} \frac{\|u\|^{\frac{2(q-p)}{q-2}}}{(\int_0^T b(t)|u|^q)^{\frac{2-p}{q-2}}}$$
$$= c_{pq} \frac{\|u\|^{\frac{2(q-p)}{q-2}}}{(\int_0^T b(t)|u|^q)^{\frac{2-p}{q-2}}}.$$

Since $u \in H^0_-$, there exists a unique s_2^* and $\psi_u(s_2^*) = \lambda \int_0^T h(t)|u(t)|^p\,dt$ satisfying $\varphi_u'(s_2^*) = 0$, which means $s_2^* u \in N_\lambda$. Moreover, we obtain that for $0 < s < s_2^*$, $\varphi_u'(s) > 0$ and for $s > s_2^*$, $\varphi_u'(s) < 0$. So $\varphi_u(s)$ gets its maximum at s_2^*, that is to say, $s_2^* u \in N_\lambda^-$.

Case 4: If $u \in B^+ \cap H^+$, similarly to Case 3, $\psi_u(s)$ has a unique maximum stationary point at s^* such that $\psi_u'(s^*) = 0$. Hence, if

$$0 < \lambda \int_0^T h(t)\left|u(t)\right|^p dt < \psi_u\left(s^*\right), \tag{3.5}$$

there exist two points s_1, s_2 such that $\lambda \int_0^T h(t)|u(t)|^p\,dt = \psi_u(s_1) = \psi_u(s_2)$. From $\varphi_u'(s_1) = 0$ and $\psi_u'(s_1) > 0$, we have $s_1 u \in N_\lambda^+$. By $\varphi_u'(s_2) = 0$ and $\psi_u'(s_2) < 0$, we get $s_2 u \in N_\lambda^-$.

Lemma 3.3 *If* $\lambda \in (0, \lambda_1)$, *then* $N_\lambda^0 = \emptyset$.

Proof Suppose not, that is, $N_\lambda^0 \neq \emptyset$. Letting $u \in N_\lambda^0$, we have

$$\|u\|^2 = \lambda \int_0^T h(t)\left|u(t)\right|^p dt + \int_0^T b(t)\left|u(t)\right|^q dt = \lambda \frac{q-p}{q-2} \int_0^T h(t)\left|u(t)\right|^p dt, \tag{3.6}$$

$$\begin{aligned}\|u\|^2 &= \lambda(p-1) \int_0^T h(t)\left|u(t)\right|^p dt + (q-1) \int_0^T b(t)\left|u(t)\right|^q dt \\ &= \frac{q-p}{2-p} \int_0^T b(t)\left|u(t)\right|^q dt.\end{aligned} \tag{3.7}$$

Since $0 < p < 2, q > 2$, from (3.6) and (3.7) we obtain

$$\|u\| \le \left(\lambda \frac{q-p}{q-2} c_h c_p^p\right)^{\frac{1}{2-p}}, \tag{3.8}$$

$$\|u\| \ge \left(\frac{2-p}{(q-p) c_b c_q^q}\right)^{\frac{1}{q-2}}. \tag{3.9}$$

It is easy to verify that, if $\lambda \in (0, \lambda_1)$, (3.8) and (3.9) are contradictory. The proof is finished. □

By Lemmas 3.2 and 3.3, for any $\lambda \in (0, \lambda_1)$, we know that $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and $I_\lambda(u)$ is coercive and bounded from below on N_λ^+ and N_λ^-.

Now, we complete the proof of Theorem 1.1.

Proof Since $\lambda \in (0, \lambda_1)$, $N_\lambda^0 = \emptyset$. In the following, we show that there exists u_1 belonging to N_λ^+ and satisfying $I_\lambda(u_1) = \inf_{u \in N_\lambda^+} I_\lambda(u) < 0$. From Lemma 3.1, u_1 is the critical point of I_λ.

By $(f1)$, there exists $u \in E_0^{\alpha,2}$ satisfying $\int_0^T h(t)|u(t)|^p\,dt > 0$, that is, $u \in H^+$. If $u \in B^{\underline{0}}$, from the argument in Case 2, there exists a unique point s_1^* such that $s_1^* u \in N_\lambda^+$. If $u \in B^+$, from the argument in Case 4, we get that

$$\begin{aligned}
\frac{\psi_u(s^*)}{\int_0^T h(t)|u(t)|^p\,dt} &= c_{pq}\frac{\|u\|^{\frac{2(q-p)}{q-2}}}{(\int_0^T b(t)|u|^q)^{\frac{2-p}{q-2}}\int_0^T h(t)|u(t)|^p\,dt} \\
&\geq c_{pq}\frac{\|u\|^{\frac{2(q-p)}{q-2}}}{c_h c_p^p\|u\|^p(c_b c_q^q)^{\frac{2-p}{q-2}}\|u\|^{\frac{q(2-p)}{q-2}}} \\
&= \frac{c_{pq}}{c_h c_p^p(c_b c_q^q)^{\frac{2-p}{q-2}}} = \lambda_1.
\end{aligned}$$

Since $\lambda \in (0, \lambda_1)$, from (3.5), we can deduce that there also exists a point, still represented by s_1^*, such that $s_1^* u \in N_\lambda^+$. So N_λ^+ is nonempty.

For $u \in N_\lambda^+$, from (3.2) and

$$\|u\|^2 - \lambda(p-1)\int_0^T h(t)\big|u(t)\big|^p\,dt - (q-1)\int_0^T b(t)\big|u(t)\big|^q\,dt > 0,$$

we obtain

$$\|u\|^2 < \frac{\lambda(q-p)}{q-2}\int_0^T h(t)\big|u(t)\big|^p\,dt.$$

Consequently,

$$\begin{aligned}
I_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^T h(t)\big|u(t)\big|^p\,dt \\
&\leq \frac{q-2}{2q}\frac{\lambda(q-p)}{q-2}\int_0^T h(t)\big|u(t)\big|^p\,dt - \frac{\lambda(q-p)}{pq}\int_0^T h(t)\big|u(t)\big|^p\,dt \\
&= \frac{\lambda(p-2)(q-p)}{2pq}\int_0^T h(t)\big|u(t)\big|^p\,dt \\
&< 0.
\end{aligned}$$

Thus we have $\inf_{u\in N_\lambda^+} I_\lambda(u) < 0$.

Since $I_\lambda(u)$ is coercive and bounded from below on N_λ^+, there exist a minimizing sequence $\{u_k\} \subset N_\lambda^+$ and $u_1 \in E_0^{\alpha,2}$ such that $I_\lambda(u_k) \to \inf_{u\in N_\lambda^+} I_\lambda(u)$ and $u_k \rightharpoonup u_1$ (up to a subsequence). From Lemma 2.7, $u_k \to u_1$ in $L^r[0,T]$ $(r = p, q)$. Hence

$$\begin{aligned}
&\int_0^T h(t)\big|u_k(t)\big|^p\,dt \to \int_0^T h(t)\big|u_1(t)\big|^p\,dt, \\
&\int_0^T b(t)\big|u_k(t)\big|^q\,dt \to \int_0^T b(t)\big|u_1(t)\big|^q\,dt.
\end{aligned}$$

It follows from (3.1) and (3.2) that

$$\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^T h(t)\big|u_k(t)\big|^p\,dt = \left(\frac{1}{2}-\frac{1}{q}\right)\|u_k\|^2 - I_\lambda(u_k).$$

Letting $k \to \infty$, we get $\int_0^T h(t)|u_1(t)|^p \, dt > 0$. By the same argument as above, there exists s_1^* such that $s_1^* u_1 \in N_\lambda^+$. That is to say, $\varphi_{u_1}(s)$ attains its local (or global) minimum at s_1^*. Therefore, there exists $t_1 > s_1^*$ such that

$$\varphi_{u_1}'(s) \begin{cases} < 0, & 0 < s < s_1^*, \\ > 0, & s_1^* < s < t_1. \end{cases} \tag{3.10}$$

Next we claim that $u_k \to u_1$ in $E_0^{\alpha,2}$. Otherwise, $\|u_1\| < \liminf_{k\to\infty} \|u_k\|$. Since $u_k \in N_\lambda^+$, $\varphi_{u_k}(s)$ attains its local (or global) minimum at $s = 1$. So, there exists $t_2 > 1$ such that

$$\varphi_{u_k}'(s) \begin{cases} < 0, & 0 < s < 1, \\ > 0, & 1 < s < t_2. \end{cases} \tag{3.11}$$

What is more,

$$\begin{aligned} \lim_{k\to\infty} \varphi_{u_k}'\bigl(s_1^*\bigr) &= \lim_{k\to\infty} s_1^* \|u_k\|^2 - \lambda s_1^{*(p-1)} \int_0^T h(t)\bigl|u_k(t)\bigr|^p \, dt \\ &\quad - s_1^{*(q-1)} \int_0^T b(t)\bigl|u_k(t)\bigr|^q \, dt \\ &> s_1^* \|u_1\|^2 - \lambda s_1^{*(p-1)} \int_0^T h(t)\bigl|u_1(t)\bigr|^p \, dt - s_1^{*(q-1)} \int_0^T b(t)\bigl|u_1(t)\bigr|^q \, dt \\ &= \varphi_{u_1}'\bigl(s_1^*\bigr) = 0. \end{aligned}$$

Hence, for k large enough, $\varphi_{u_k}'(s_1^*) > 0$. Together with (3.11), we have $s_1^* > 1$. Together with (3.10), we get

$$I_\lambda\bigl(s_1^* u_1\bigr) \le I_\lambda(u_1) < \liminf_{k\to\infty} I_\lambda(u_k) = \inf_{u \in N_\lambda^+} I_\lambda(u),$$

which is a contradiction. Hence, $u_k \to u_1$ strongly in $E_0^{\alpha,2}$. This implies

$$I_\lambda(u_k) \to I_\lambda(u_1) = \inf_{u \in N_\lambda^+} I_\lambda(u).$$

Namely, u_1 is a minimizer of I_λ on N_λ^+. From Lemmas 3.1 and 3.3, u_1 is a critical point of $I_\lambda(u)$. The proof is finished. □

Completion of the proof of Theorem 1.2.

Proof From Theorem 1.1, we know that $u_1 \in N_\lambda^+$ is a critical point of $I_\lambda(u)$ when $\lambda \in (0, \frac{p}{2}\lambda_1)$ since $p < 2$. Next, we show that if $\lambda \in (0, \frac{p}{2}\lambda_1)$, there exists another critical point u_2 of I_λ which belongs to N_λ^- and satisfies $I_\lambda(u_2) = \inf_{u \in N_\lambda^-} I_\lambda(u) > 0$. Let $u \in E_0^{\alpha,2}$ satisfying $\int_0^T b(t)|u(t)|^q \, dt > 0$, namely, $u \in B^+$. From the argument in Cases 3 and 4, we obtain that N_λ^- is not empty. Since $I_\lambda(u)$ is coercive and bounded from below on N_λ^-, there exist a minimizing sequence $\{u_k\}$ and $u_2 \in E_0^{\alpha,2}$ such that $I_\lambda(u_k) \to \inf_{u \in N_\lambda^-} I_\lambda(u)$ and $u_k \rightharpoonup u_2$(up to a subsequence). Next we claim that $I_\lambda(u_k) \to I_\lambda(u_2)$ and $\inf_{u\in N_\lambda^-} I_\lambda(u) > 0$. For $u \in N_\lambda^-$, from (3.2) and

$$\|u\|^2 - \lambda(p-1)\int_0^T h(t)\left|u(t)\right|^p\,dt - (q-1)\int_0^T b(t)\left|u(t)\right|^q\,dt < 0,$$

we have

$$\|u\|^2 < \frac{q-p}{2-p}\int_0^T b(t)\left|u(t)\right|^q\,dt.$$

For $q > 2$ and from Lemma 2.7, we have

$$\|u\| > \left(\frac{2-p}{(q-p)c_b c_q^q}\right)^{\frac{1}{q-2}} = \delta_1 > 0.$$

Thus

$$\begin{aligned}
I_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^T h(t)\left|u(t)\right|^p\,dt \\
&> \frac{q-2}{2q}\|u\|^2 - \frac{\lambda(q-p)}{pq}c_h c_p^p\|u\|^p \\
&= \|u\|^p\left(\frac{q-2}{2q}\|u\|^{2-p} - \frac{\lambda(q-p)}{pq}c_h c_p^p\right) \\
&> \delta_1^p\left(\frac{q-2}{2q}\delta_1^{2-p} - \frac{\lambda(q-p)}{pq}c_h c_p^p\right) \\
&= \delta.
\end{aligned}$$

Since $\lambda < \frac{p}{2}\lambda_1$, then $I_\lambda(u) > \delta > 0$ and $\inf_{u\in N_\lambda^-} I_\lambda(u) > 0$. It follows from (3.1) and (3.2) that

$$\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^T b(t)\left|u_k(t)\right|^q\,dt = I_\lambda(u_k) + \left(\frac{1}{p}-\frac{1}{2}\right)\|u_k\|^2.$$

Letting $k \to \infty$, from $\inf_{u\in N_\lambda^-} I_\lambda(u) > 0$, we get

$$\int_0^T b(t)\left|u_2(t)\right|^q\,dt > 0.$$

Since $\lambda \in (0, \frac{p}{2}\lambda_1)$, from the argument in Cases 3 and 4, we infer that there exists $s_2 u_2 \in N_\lambda^-$. Next we claim that $u_k \to u_2$ strongly in $E_0^{\alpha,2}$. If not, $\|u_2\| < \liminf_{k\to\infty}\|u_k\|$. Since $u_k \in N_\lambda^-$, then $I_\lambda(su_k)$ attains its global maximum at $s = 1$. Hence,

$$I_\lambda(s_2u_2) < \liminf_{k\to\infty} I_\lambda(s_2u_k) \le \liminf_{k\to\infty} I_\lambda(u_k) = \inf_{u\in N_\lambda^-} I_\lambda(u),$$

which is a contradiction. So, $u_k \to u_2$ strongly in $E_0^{\alpha,2}$ and $u_2 \in N_\lambda^-$. Namely, u_2 is a minimizer of I_λ on N_λ^-. From Lemmas 3.1 and 3.3, u_2 is a critical point of $I_\lambda(u)$. The proof is finished. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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