# Multiple-soliton solutions and a generalized double Wronskian determinant to the $(2+1)$-dimensional nonlinear Schrödinger equations 

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#### Abstract

A $(2+1)$-dimensional nonlinear Schrödinger equation is mainly discussed. Based on the Hirota direct method and the Wronskian technique, multiple-soliton solutions and a generalized double Wronskian determinant are obtained, respectively.


Keywords: the $(2+1)$-dimensional nonlinear Schrödinger equations; multiple-soliton solutions; double Wronskian determinant

## 1 Introduction

As one of the most important integrable nonlinear equations, the nonlinear Schrödinger (NLS) equation

$$
\begin{equation*}
i u_{t}+u_{x x}+\varepsilon|u|^{2} u=0, \tag{1}
\end{equation*}
$$

is a well-known mathematical model for describing the evolution of pulses in nonlinear optical fibers and of surface gravity waves in fluid dynamics [1]. To investigate different complex nonlinear phenomena of our realistic world, some generalizations of the widely used ( $1+1$ )-dimensional Eq. (1) into a $(2+1)$-dimensional one are obtained [2-4]. Theoretical and experimental research of these higher-dimensional integrable generalizations has been carried out due to their attraction and application in many fields such as plasma physics, nonlinear optics, fluid dynamics, and Bose-Einstein condensates [5-8]. The NLS equation admits the following $(2+1)$-dimensional extension:

$$
\begin{align*}
& i u_{t}+u_{x x}-u Q_{x}=0, \\
& i u_{t}^{*}-u_{x x}^{*}+u^{*} Q_{x}=0,  \tag{2}\\
& u u^{*}=Q_{y},
\end{align*}
$$

where $u=u(x, y, t)$ is a complex function, $Q=Q(x, y, t)$ is a real function, and $x, y, t$ are real. For system (2), Ruan and Chen [9] have discussed its Painlevé properties and presented
infinitely many truncated symmetries in terms of arbitrary functions of time $t$ and variable $y$. Zhang [10] has revealed the rich dromion structures of system (2) based on the Hirota bilinear form.

One of vital aspects of soliton theory is searching exact solutions to soliton equations. Generally speaking, the Hirota bilinear method and the Wronskian technique are efficient and direct methods to construct exact solutions [11-14]. Recently, Chen et al. [15] extended the traditional condition equation to the arbitrary matrix equation and established the rational solutions and complexitons in terms of double Wronskian forms for the AKNS system. At present, general rational soliton solutions to various soliton equations are also discussed within the Casoratian structure and the Grammian or Pfaffian formulation [16-18].

In this paper, using the Hirota bilinear method and Chen's method, we discuss multiplesoliton solutions and a generalized double Wronskian determinant solution to system (2), respectively. The paper is organized as follows. In Section 2, by a dependent variable transformation, system (2) is transformed into a bilinear equation. Utilizing the perturbation method, we derive soliton solutions of system (2) based on the Hirota bilinear form. In Section 3, we present a double Wronskian form of system (2) whose entries satisfy a general matrix equation. A conclusion and remarks are given in Section 4.

## 2 Multi-soliton solutions

Via the dependent variable transformations [10]

$$
\begin{equation*}
u=\frac{g}{f}, \quad u^{*}=\frac{g^{*}}{f}, \quad Q=-2(\ln f)_{x} \tag{3}
\end{equation*}
$$

where $g=g(x, y, t)$ is a differentiable complex function, and $f=f(x, y, t)$ is a differentiable real function, the bilinear form for system (2) is as follows:

$$
\begin{align*}
& \left(D_{x}^{2}+i D_{t}\right) g \cdot f=0,  \tag{4a}\\
& \left(D_{x}^{2}-i D_{t}\right) g^{*} \cdot f=0,  \tag{4b}\\
& D_{x} D_{x} f \cdot f=-g g^{*} \tag{4c}
\end{align*}
$$

where $D_{x}, D_{y}$, and $D_{t}$ are the Hirota bilinear differential operators [19].
We expand $f$ and $g$ in the form of a power series as

$$
\begin{align*}
& f(x, y, t)=1+f^{(2)} \varepsilon^{2}+f^{(4)} \varepsilon^{4}+\cdots+f^{(2 j)} \varepsilon^{2 j}+\cdots  \tag{5a}\\
& g(x, y, t)=g^{(1)} \varepsilon+g^{(3)} \varepsilon^{3}+\cdots+g^{(2 j+1)} \varepsilon^{2 j+1}+\cdots \tag{5b}
\end{align*}
$$

Substituting (5a)-(5b) into Eqs. (4a)-(4c) and collecting terms of each order of $\varepsilon$, we get

$$
\begin{align*}
& i g_{t}^{(1)}+g_{x x}^{(1)}=0,  \tag{6a}\\
& i g_{t}^{(3)}+g_{x x}^{(3)}=-\left(D_{x}^{2}+i D_{t}\right) g^{(1)} \cdot f^{(2)}, \tag{6b}
\end{align*}
$$

$$
\begin{equation*}
-i g_{t}^{(1) *}+g_{x x}^{(1) *}=0, \tag{6c}
\end{equation*}
$$



Figure 1 The plots of expression (8) with special parameters: $k_{1}=1, p_{1}=2, t=1$.

$$
\begin{equation*}
-i g_{t}^{(3) *}+g_{x x}^{(3) *}=-\left(D_{x}^{2}-i D_{t}\right) g^{(1) *} \cdot f^{(2)} \tag{6d}
\end{equation*}
$$

$2 f_{x y}^{(2)}=-g^{(1)} g^{(1) *}$,
$2 f_{x y}^{(4)}=-D_{x} D_{y} f^{(2)} \cdot f^{(2)}-g^{(1)} g^{(3) *}-g^{(3)} g^{(1) *}$,

### 2.1 One-soliton solution

It is obvious that Eqs. (6a)-(6f) possess a particular solution

$$
\begin{equation*}
g^{(1)}=e^{\xi_{1}}, \quad f^{(2)}=e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}, \quad g^{(3)}=f^{(4)}=\cdots=0, \tag{7}
\end{equation*}
$$

where

$$
\xi_{1}=k_{1} x+p_{1} y+w_{1} t+\xi_{1}^{0}, \quad w_{1}=i k_{1}^{2}, \quad e^{\theta_{13}}=-\frac{1}{2\left(k_{1}+k_{1}^{*}\right)\left(p_{1}+p_{1}^{*}\right)} .
$$

If we take $\varepsilon=1$, then it is easy to see that $g_{1}=e^{\xi_{1}}, f_{1}=1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}$ is a solution of Eqs. (4a)(4c). Thus, through transformations (3), we can obtain the one-soliton solution of system (2) as

$$
\begin{equation*}
u=\frac{g_{1}}{f_{1}}=\frac{e^{\xi_{1}}}{1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}}, \quad Q=-2\left(\ln f_{1}\right)_{x} . \tag{8}
\end{equation*}
$$

A specific one-soliton solution with choices of the involved parameters is plotted in Figure 1.

### 2.2 Two-soliton solution

If we choose the solution to Eq. (6a) in the form

$$
\begin{equation*}
g^{(1)}=e^{\xi_{1}}+e^{\xi_{2}}, \quad \xi_{j}=k_{j} x+p_{j} y+w_{j} t+\xi_{j}^{0}, \quad w_{j}=i k_{j}^{2}(j=1,2), \tag{9}
\end{equation*}
$$



Figure 2 The plots of expression (15) with special parameters: $k_{1}=-1, k_{2}=3, p_{1}=2, p_{2}=4, y=1$.
then substituting (9) into Eq. (6e), we get

$$
\begin{align*}
& f^{(2)}=e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}+e^{\xi_{1}+\xi_{2}^{*}+\theta_{14}}+e^{\xi_{2}+\xi_{1}^{*}+\theta_{23}}+e^{\xi_{2}+\xi_{2}^{*}+\theta_{24}} \\
& e^{\theta_{j, 2+l}}=-\frac{1}{2\left(k_{j}+k_{l}^{*}\right)\left(p_{j}+p_{l}^{*}\right)}, \quad(j, l=1,2) \tag{10}
\end{align*}
$$

Substituting (9) and (10) into Eq. (6b), we have

$$
\begin{equation*}
g^{(3)}=e^{\xi_{1}+\xi_{2}+\xi_{1}^{*}+\theta_{12}+\theta_{13}+\theta_{23}}+e^{\xi_{1}+\xi_{2}+\xi_{2}^{*}+\theta_{12}+\theta_{14}+\theta_{24}}, \quad e^{\theta_{12}}=2\left(k_{1}-k_{2}\right)\left(p_{2}-p_{1}\right) . \tag{11}
\end{equation*}
$$

With the help of (9)-(11), solving Eq. (6f), we can obtain

$$
\begin{equation*}
f^{(4)}=e^{\xi_{1}+\xi_{2}+\xi_{1}^{*}+\xi_{2}^{*}+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}, \quad e^{\theta_{34}}=2\left(k_{1}^{*}-k_{2}^{*}\right)\left(p_{2}^{*}-p_{1}^{*}\right) . \tag{12}
\end{equation*}
$$

By similar steps as before a direct calculation yields

$$
g^{(5)}=f^{(6)}=\cdots=0 .
$$

Taking $\varepsilon=1$, then we have

$$
\begin{align*}
f_{2}= & 1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}+e^{\xi_{1}+\xi_{2}^{*}+\theta_{14}}+e^{\xi_{2}+\xi_{1}^{*}+\theta_{23}}+e^{\xi_{2}+\xi_{2}^{*}+\theta_{24}} \\
& +e^{\xi_{1}+\xi_{2}+\xi_{1}^{*}+\xi_{2}^{*}+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}  \tag{13}\\
g_{2}= & e^{\xi_{1}}+e^{\xi_{2}}+e^{\xi_{1}+\xi_{2}+\xi_{1}^{*}+\theta_{12}+\theta_{13}+\theta_{23}}+e^{\xi_{1}+\xi_{2}+\xi_{2}^{*}+\theta_{12}+\theta_{14}+\theta_{24}} \tag{14}
\end{align*}
$$

Hence, the two-soliton solution of system (2) is

$$
\begin{equation*}
u=\frac{g_{2}}{f_{2}}, \quad Q=-2\left(\ln f_{2}\right)_{x} . \tag{15}
\end{equation*}
$$

Figure 2 shows the plots of the two-soliton solution by choosing specific parameters.

We now consider the asymptotic analysis on the two-soliton solution of (2). Let us define the soliton wave variables as

$$
\xi_{1 R}=k_{1 R} x+p_{1 R} y-2 k_{1 R} k_{1 I} t, \quad \xi_{1 I}=k_{1 I} x+p_{1 I} y+\left(k_{1 R}^{2}-k_{1 I}^{2}\right) t
$$

and

$$
\xi_{2 R}=k_{2 R} x+p_{2 R} y-2 k_{2 R} k_{2 I} t, \quad \xi_{2 I}=k_{2 I} x+p_{2 I} y+\left(k_{2 R}^{2}-k_{2 I}^{2}\right) t,
$$

where $k_{1 R}, k_{1 I}, k_{2 R}, k_{2 I}, p_{1 R}, p_{1 I}, p_{2 R}$, and $p_{2 I}$ all are real parameters.
Before collision (limit $t \rightarrow-\infty$ ):
(1) Soliton $1\left(\xi_{1}+\xi_{1}^{*} \sim 0, \xi_{2}+\xi_{2}^{*} \rightarrow-\infty\right)$ :

$$
\begin{align*}
& u \rightarrow u^{1-}=\frac{e^{\xi_{1}}}{1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}}=\frac{1}{2} e^{i \xi_{11}} e^{-\frac{\theta_{13}}{2}} \operatorname{sech}\left(\xi_{1 R}+\frac{\theta_{13}}{2}\right),  \tag{16a}\\
& Q \rightarrow Q^{1-}=-2\left[\ln \left(1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{13}}\right)\right]_{x} . \tag{16b}
\end{align*}
$$

(2) Soliton $2\left(\xi_{2}+\xi_{2}^{*} \sim 0, \xi_{1}+\xi_{1}^{*} \rightarrow+\infty\right)$ :

$$
\begin{align*}
\begin{aligned}
u \rightarrow u^{2-} & =\frac{e^{\xi_{2}+\theta_{12}+\theta_{23}}}{1+e^{\xi_{2}+\xi_{2}^{*}+\theta_{12}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}} \\
& =\frac{1}{2} e^{i \xi_{22}} e^{\frac{\theta_{12}+\theta_{23}-\theta_{14}-\theta_{24}-\theta_{34}}{2}} \operatorname{sech}\left(\xi_{2 R}+\frac{\theta_{12}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}{2}\right), \\
Q \rightarrow Q^{2-} & =-2\left[\ln \left(1+e^{\xi_{2}+\xi_{2}^{*}+\theta_{12}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}\right)\right]_{x} .
\end{aligned} .
\end{align*}
$$

After collision (limit $t \rightarrow+\infty$ ):
(1) Soliton $1\left(\xi_{1}+\xi_{1}^{*} \sim 0, \xi_{2}+\xi_{2}^{*} \rightarrow+\infty\right)$ :

$$
\begin{align*}
\begin{aligned}
u \rightarrow u^{1+} & =\frac{e^{\xi_{1}+\theta_{12}+\theta_{14}}}{1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{34}}} \\
& =\frac{1}{2} e^{i \xi_{11}} e^{\frac{\theta_{12}+\theta_{14}-\theta_{13}-\theta_{23}-\theta_{34}}{2}} \operatorname{sech}\left(\xi_{1 R}+\frac{\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{34}}{2}\right), \\
Q \rightarrow Q^{1+} & =-2\left[\ln \left(1+e^{\xi_{1}+\xi_{1}^{*}+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{34}}\right)\right]_{x} .
\end{aligned} .
\end{align*}
$$

(2) Soliton $2\left(\xi_{2}+\xi_{2}^{*} \sim 0, \xi_{1}+\xi_{1}^{*} \rightarrow-\infty\right)$ :

$$
\begin{align*}
& u \rightarrow u^{2+}=\frac{e^{\xi_{2}}}{1+e^{\xi_{2}+\xi_{2}^{*}+\theta_{24}}}=\frac{1}{2} e^{i \xi_{2 I}} e^{-\frac{\theta_{24}}{2}} \operatorname{sech}\left(\xi_{2 R}+\frac{\theta_{24}}{2}\right),  \tag{19a}\\
& Q \rightarrow Q^{2-}=-2\left[\ln \left(1+e^{\xi_{2}+\xi_{2}^{*}+\theta_{24}}\right)\right]_{x} . \tag{19b}
\end{align*}
$$

From these expressions it is easy to observe that the amplitude of the colliding solitons in the $u$ change from $\frac{1}{2} e^{-\frac{\theta_{13}}{2}}$ and $\frac{1}{2} e^{-\frac{\theta_{12}+\theta_{23}-\theta_{14}-\theta_{24}-\theta_{34}}{2}}$ to $\frac{1}{2} e^{-\frac{\theta_{12}+\theta_{14}-\theta_{13}-\theta_{23}-\theta_{34}}{2}}$ and $\frac{1}{2} e^{-\frac{\theta_{24}}{2}}$ due to collision, respectively. In addition to this change in the amplitudes, the phase shift is given by

$$
\begin{equation*}
\Phi=\frac{\theta_{12}+\theta_{14}+\theta_{23}+\theta_{34}}{2} \tag{20}
\end{equation*}
$$

### 2.3 N -soliton solution

In general, the $N$-solition solution of system (2) can be expressed as

$$
\begin{equation*}
u=\frac{g_{n}}{f_{n}}, \quad Q=-2\left(\ln f_{n}\right)_{x} \tag{21}
\end{equation*}
$$

with

$$
\begin{align*}
& f_{n}=\sum_{\mu=0,1} A_{1}(\mu) e^{\sum_{j=1}^{2 n} \mu_{j} \xi_{j}+\sum_{1 \leq j<l}^{2 n} \mu_{j} \mu_{l} \theta_{j l}}, \\
& g_{n}=\sum_{\mu=0,1} A_{2}(\mu) e^{\sum_{j=1}^{2 n} \mu_{j} \xi_{j}+\sum_{1 \leq j l l}^{2 n} \mu_{j} \mu_{l} \theta_{j l}}, \tag{22}
\end{align*}
$$

where

$$
\begin{align*}
& \xi_{n+j}=\xi_{j}^{*} \quad(j=1,2, \ldots, n), \quad e^{\theta_{j(n+l)}=-\frac{1}{2\left(k_{j}+k_{l}^{*}\right)\left(p_{j}+p_{l}^{*}\right)}} \quad(j, l=1,2, \ldots, n),  \tag{23a}\\
& e^{\theta_{j l}}=2\left(k_{j}-k_{l}\right)\left(p_{l}-p_{j}\right) \quad(j<l=2,3, \ldots, n),  \tag{23b}\\
& e^{\theta_{(n+j)}(n+l)}=2\left(k_{j}^{*}-k_{l}^{*}\right)\left(p_{l}^{*}-p_{j}^{*}\right)=e^{\theta_{j l}^{*}} \quad(j<l=2,3, \ldots, n), \tag{23c}
\end{align*}
$$

and $A_{1}(u)$ and $A_{2}(u)$ take all possible combinations of $\mu_{j}=0,1(j=1,2, \ldots, n)$ and satisfy the following conditions:

$$
\begin{equation*}
\sum_{j=1}^{n} \mu_{j}=\sum_{j=1}^{n} \mu_{n+j}, \quad \sum_{j=1}^{n} \mu_{j}=\sum_{j=1}^{n} \mu_{n+j}+1 . \tag{24}
\end{equation*}
$$

Thus, we can obtain the $N$-soliton solution of system (2) via the above results.

## 3 Generalized double Wronskian solution

In what follows, we derive the generalized double Wronskian determinant solution of system (2) by applying the Wronskian technique. To use this technique, we adopt the compact notation introduced by Freeman and Nimmo [20, 21], who set

$$
\begin{aligned}
W^{N, M}(\phi ; \psi) & =\left|\phi, \partial_{x} \phi, \ldots, \partial_{x}^{N-1} \phi ; \psi, \partial_{x} \psi, \ldots, \partial_{x}^{M-1} \psi\right| \\
& =|\widehat{N-1} ; \widehat{M-1}|
\end{aligned}
$$

where $\phi=\left(\phi_{1}, \phi_{2}, \ldots, \phi_{N+M}\right)^{T}, \psi=\left(\psi_{1}, \psi_{2}, \ldots, \psi_{N+M}\right)^{T}$, and $\partial_{x}^{(j)}$ denotes the $j$ th partial derivative with respect to $x$.

Theorem 3.1 The bilinear system (4a)-(4c) possesses the following generalized double Wronskian solution:

$$
\begin{align*}
& f=W^{N+1, N+1}(\phi ; \psi)=|\widehat{N} ; \widehat{N}|,  \tag{25a}\\
& g=2 W^{N+2, N}(\phi ; \psi)=2|\widehat{N+1} ; \widehat{N-1}|,  \tag{25b}\\
& g^{*}=-2 W^{N, N+2}(\phi ; \psi)=-2|\widehat{N-1} ; \widehat{N+1}|, \tag{25c}
\end{align*}
$$

where the entries $\phi_{j}$ and $\psi_{j}(j=1,2, \ldots, N+1)$ satisfy the following conditions:

$$
\begin{align*}
& \phi_{x}=-A \phi, \quad \psi_{x}=A \psi  \tag{26a}\\
& \phi_{j, y}=\frac{1}{2} \phi_{j, x}, \quad \psi_{j, y}=\frac{1}{2} \psi_{j, x},  \tag{26b}\\
& \phi_{j, t}=2 i \phi_{j, x x}, \quad \psi_{j, t}=-2 i \psi_{j, x x},  \tag{26c}\\
& \phi_{N+1+j}=-\psi_{j}^{*}, \quad \psi_{N+1+j}=-\phi_{j}^{*}, \tag{26d}
\end{align*}
$$

with $A=\left(a_{i j}\right)$ is an $(N+1) \times(N+1)$ arbitrary real matrix independent of $x, y, t$.

For convenience of proof, we first give the following lemma.

## Lemma 3.1

$$
\begin{equation*}
|M, a, b||M, c, d|-|M, a, c||M, b, d|+|M, a, d||M, b, c|=0, \tag{27}
\end{equation*}
$$

where $M$ is an $N \times(N-2)$ matrix, and $a, b, c, d$ represent $N$ column vectors.
Lemma 3.1 has been proved by Chen et al. [15].

Proof of Theorem 3.1 The derivatives of $f$ can be easily computed:

$$
\begin{aligned}
f_{x}= & |\widehat{N-1}, N+1 ; \widehat{N}|+|\widehat{N} ; \widehat{N-1}, N+1|, \\
f_{x x}= & |\widehat{N-2}, N, N+1 ; \widehat{N}|+|\widehat{N-1}, N+2 ; \widehat{N}|+2|\widehat{N-1}, N+1 ; \widehat{N-1}, N+1| \\
& +|\widehat{N} ; \widehat{N-2}, N, N+1|+|\widehat{N} ; \widehat{N-1}, N+2|, \\
f_{y}= & \frac{1}{2}(|\widehat{N-1}, N+1 ; \widehat{N}|+|\widehat{N} ; \widehat{N-1}, N+1|), \\
f_{x y}= & \frac{1}{2}(|\widehat{N-2}, N, N+1 ; \widehat{N}|+|\widehat{N-1}, N+2 ; \widehat{N}|+2|\widehat{N-1}, N+1 ; \widehat{N-1}, N+1| \\
& +|\widehat{N} ; \widehat{N-2}, N, N+1|+|\widehat{N} ; \widehat{N-1}, N+2|), \\
f_{t}= & 2 i(-|\widehat{N-2}, N, N+1 ; \widehat{N}|+|\widehat{N-1}, N+2 ; \widehat{N}| \\
& +|\widehat{N} ; \widehat{N-2}, N, N+1|-|\widehat{N} ; \widehat{N-1}, N+2|, \\
g_{x}= & 2(|\widehat{N, N+2} ; \widehat{N-1}|+|\widehat{N+1} ; \widehat{N-2}, N|, \\
g_{x x}= & 2(|\widehat{N-1}, N+1, N+2 ; \widehat{N-1}|+|\widehat{N}, N+3 ; \widehat{N-1}|+|\widehat{N}, N+2 ; \widehat{N-2}, N| \\
& +|\widehat{N}, N+2 ; \widehat{N-2}, N|+|\widehat{N+1} ; \widehat{N-3}, N-1, N|+|\widehat{N+1} ; \widehat{N-2}, N+1|, \\
g_{t}= & 4 i(-|\widehat{N-1}, N+1, N+2 ; \widehat{N-1}|+|\widehat{N}, N+3 ; \widehat{N-1}| \\
& +|\widehat{N+1} ; \widehat{N-3}, N-1, N|-|\widehat{N+1} ; \widehat{N-2}, N+1|) .
\end{aligned}
$$

Noting that

$$
\begin{equation*}
\left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)^{2}|\widehat{N} ; \widehat{N}|\right\}|\widehat{N} ; \widehat{N}|=\left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)|\widehat{N} ; \widehat{N}|\right\}^{2} \tag{28a}
\end{equation*}
$$

from condition (26a) we have

$$
\begin{align*}
&|\widehat{N} ; \widehat{N}|(|\widehat{N-2}, N, N+1 ; \widehat{N}|+|\widehat{N-1}, N+2 ; \widehat{N}|-2|\widehat{N-1}, N+1 ; \widehat{N-1}, N+1| \\
&+|\widehat{N} ; \widehat{N-2}, N, N+1|+|\widehat{N} ; \widehat{N-1}, N+2|) \\
& \quad=(|\widehat{N-1}, N+1 ; \widehat{N}|-|\widehat{N} ; \widehat{N-1}, N+1|)^{2} . \tag{28b}
\end{align*}
$$

Under condition (26d), we also have

$$
g^{*}=-2 W^{N, N+2}(\phi ; \psi)=-2|\widehat{N-1} ; \widehat{N+1}| .
$$

Substituting $f, g, g^{*}$ and the derivatives of $f$ into the left-hand side of (4c) and using (28a)(28b), we get

$$
\begin{align*}
& 2\left(f_{x y}-f_{x} f_{y}\right)+g g^{*} \\
& \quad=4(|\widehat{N} ; \widehat{N}||\widehat{N-1}, N+1 ; \widehat{N-1}, N+1|-|\widehat{N-1}, N+1 ; \widehat{N}||\widehat{N} ; \widehat{N-1}, N+1| \\
& \quad-|\widehat{N+1} ; \widehat{N-1}||\widehat{N-1} ; \widehat{N+1}|) . \tag{29}
\end{align*}
$$

By Lemma 3.1 the right-hand side of (29) is equal to zero. So the proof of (4c) is finished.
Using the identities, which are similar to (28a)-(28b),

$$
\begin{align*}
& \left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)^{2}|\widehat{N} ; \widehat{N}|\right\}|\widehat{N+1} ; \widehat{N-1}| \\
& \quad=\left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)|\widehat{N} ; \widehat{N}|\right\}\left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)|\widehat{N+1} ; \widehat{N-1}|\right\} \\
& \quad=\left\{\left(\sum_{i=1}^{N+1} a_{i i}-\sum_{i=1}^{N+1} a_{i i}\right)^{2}|\widehat{N+1} ; \widehat{N-1}|\right\}|\widehat{N} ; \widehat{N}|, \tag{30a}
\end{align*}
$$

namely,

$$
\begin{align*}
&|\widehat{N} ; \widehat{N}|||\widehat{N-1}, N+1, N+2 ; \widehat{N-1}|+|\widehat{N}, N+3 ; \widehat{N-1}|-2| \widehat{N}, N+2 ; \widehat{N-2}, N \mid \\
&+|\widehat{N+1} ; \widehat{N-3}, N-1, N|+|\widehat{N+1} ; \widehat{N-2}, N+1|) \\
&=(|\widehat{N-1}, N+1 ; \widehat{N}|-|\widehat{N} ; \widehat{N-1}, N+1|) \\
& \times(|\widehat{N}, N+2 ; \widehat{N-1}|-|\widehat{N+1} ; \widehat{N-2}, N|),  \tag{30b}\\
&|\widehat{N+1} ; \widehat{N-1}|(|\widehat{N-2}, N, N+1 ; \widehat{N}|+|\widehat{N-1}, N+2 ; \widehat{N \mid}|-2|\widehat{N-1}, N+1 ; \widehat{N-1}, N+1| \\
&+|\widehat{N} ; \widehat{N-2}, N, N+1|+|\widehat{N} ; \widehat{N-1}, N+2|) \\
&=(|\widehat{N}, N+2 ; \widehat{N-1}|-|\widehat{N+1} ; \widehat{N-2}, N|) \\
& \times(|\widehat{N-1}, N+1 ; \widehat{N}|-|\widehat{N} ; \widehat{N-1}, N+1|), \tag{30c}
\end{align*}
$$

we compute the left-hand side of (4a):

$$
\begin{align*}
g_{x x} f & -2 g_{x} f_{x}+g f_{x x}+i g_{t} f-i g f_{t} \\
= & |\widehat{N} ; \widehat{N}||\widehat{N-1}, N+1, N+2 ; \widehat{N-1}|+|\widehat{N} ; \widehat{N}||\widehat{N+1} ; \widehat{N-2}, N+1| \\
& +|\widehat{N+1} ; \widehat{N-1}||\widehat{N-1}, N+2 ; \widehat{N}|+|\widehat{N+1} ; \widehat{N-1}||\widehat{N} ; \widehat{N-2}, N, N+1| \\
& -|\widehat{N}, N+2 ; \widehat{N-1}||\widehat{N-1}, N+1 ; \widehat{N}|-|\widehat{N+1} ; \widehat{N-2}, N||\widehat{N} ; \widehat{N-1}, N+1| . \tag{31}
\end{align*}
$$

By Lemma 3.1 the right-hand side of (31) is equal to zero. Similarly, the bilinear form (4b) can be verified.

Based on conditions (26a)-(26d), we get the general solution

$$
\begin{equation*}
\phi=e^{-A x-\frac{1}{2} A y+2 i A^{2} t} C, \quad \psi=e^{A x+\frac{1}{2} A y-2 i A^{2} t} D \tag{32}
\end{equation*}
$$

where $C=\left(c_{1}, c_{2}, \ldots, c_{N+1}\right)^{T}, D=\left(d_{1}, d_{2}, \ldots, d_{N+1}\right)^{T}$ are constant vectors. Thus, with transformations (3), the corresponding double Wronskian solutions of system (2) can be expressed as

$$
u=2 \frac{W^{N+2, N}(\phi ; \psi)}{W^{N+1, N+1}(\phi ; \psi)}, \quad u^{*}=-2 \frac{W^{N, N+2}(\phi ; \psi)}{W^{N+1, N+1}(\phi ; \psi)}, \quad Q=-2\left(\ln W^{N+1, N+1}(\phi ; \psi)\right)_{x} .
$$

If the coefficient matrix $A$ has the form

$$
A=\left(\begin{array}{cccc}
k_{1} & & & 0  \tag{33}\\
& k_{2} & & \\
& & \ddots & \\
0 & & & k_{N+1}
\end{array}\right), \quad 0<k_{1}<k_{2}<\cdots<k_{N+1},
$$

then we can obtain solitons of system (2), where

$$
\begin{equation*}
\phi_{j}=c_{j} e^{-k_{j} x-\frac{1}{2} k_{j} y+2 i k_{j}^{2} t}, \quad \psi_{j}=d_{j} e^{k_{j} x+\frac{1}{2} k_{j} y-2 i k_{j}^{2} t} \quad(j=1,2, \ldots, N+1) \tag{34}
\end{equation*}
$$

## 4 Conclusion and remarks

In summary, by using the Hirota method and the Wronskian technique, the multiplesoliton solutions and the double Wronskian form satisfying a matrix equation to system (2) have been presented, respectively. As is well known, the Wronskian technique can also be applied to construct rational solutions, positons, negatons, complexitons, and interaction solutions of the nonlinear equations. We also point out that the $N$-soliton solutions of system (2) may be expressed by the Grammian determinants. Therefore, there is a lot of work to be done in these directions, and it should be interesting yet difficult to construct more solution formulas for system (2).

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## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

Both authors contributed equally to the manuscript and approved the final manuscript.

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