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Connections between Hyers-Ulam stability and uniform exponential stability of 2-periodic linear nonautonomous systems

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Abstract

We prove that the system $\dot{\theta}(t) = \Lambda(t)\theta(t)$, $t \in \mathbb{R}_+$, is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions; we take the exact solutions of the Cauchy problem $\dot{\phi}(t) = \Lambda(t)\phi(t) + e^{\gamma t}\xi(t)$, $t \in \mathbb{R}_+$, $\phi(0) = \theta_0$ as the approximate solutions of $\dot{\theta}(t) = \Lambda(t)\theta(t)$, where γ is any real number, ξ is a 2-periodic, continuous, and bounded vectorial function with $\xi(0) = 0$, and $\Lambda(t)$ is a 2-periodic square matrix of order l .

MSC: 34D20

Keywords: Hyers-Ulam stability; uniform exponential stability; nonautonomous system; periodic system

1 Introduction

The stability theory is an important branch of the qualitative theory of differential equations. In 1940, Ulam [1] queried a problem regarding the stability of differential equations for homomorphism as follows: *when can an approximate homomorphism from a group G_1 to a metric group G_2 be approximated by an exact homomorphism?*

Hyers [2] brilliantly gave a partial answer to this question assuming that G_1 and G_2 are Banach spaces. Later on, Aoki [3] and Rassias [4] extended and improved the results obtained in [2]. In particular, Rassias [4] relaxed the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$. To the best of our knowledge, papers by Obłozza [5, 6] published in the late 1990s were among the first contributions dealing with the Hyers-Ulam stability of differential equations.

Since then, many authors have studied the Hyers-Ulam stability of various classes of differential equations. Properties of solutions to different classes of equations were explored by using a wide spectrum of approaches; see, e.g., [7–26] and the references cited therein. Alsina and Ger [7] proved Hyers-Ulam stability of a first-order differential equation $y'(x) = y(x)$, which was then extended to the Banach space-valued linear differential equation of the form $y'(x) = \lambda y(x)$ by Takahasi *et al.* [24]. Zada *et al.* [26] generalized the concept of Hyers-Ulam stability of the nonautonomous w -periodic linear differential matrix system $\dot{\theta}(t) = \Lambda(t)\theta(t)$, $t \in \mathbb{R}$ to its dichotomy (for dichotomy in autonomous case; see, e.g., [27, 28]). We conclude by mentioning that Barbu *et al.* [10] proved that Hyers-

Ulam stability and the exponential dichotomy of linear differential periodic systems are equivalent.

Very recently, Li and Zada [19] gave connections between Hyers-Ulam stability and uniform exponential stability of the first-order linear discrete system

$$\theta_{n+1} = \Lambda_n \theta_n, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

where \mathbb{Z}_+ is the set of all nonnegative integers and (Λ_n) is an ω -periodic sequence of bounded linear operators on Banach spaces. They proved that system (1.1) is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions. The natural question now is: is it possible to extend the results of [19] to continuous nonautonomous systems over Banach spaces? The purpose of this paper is to develop a new method and give an affirmative answer to this question in finite dimensional spaces. We consider the first-order linear nonautonomous system $\dot{\theta}(t) = \Lambda(t)\theta(t)$, $t \in \mathbb{R}_+$, where $\Lambda(t)$ is a square matrix of order l . We proved that the 2-periodic system $\dot{\theta}(t) = \Lambda(t)\theta(t)$ is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions. Our result can be extended to any q -periodic system, because we choose 2 as the period in our approach.

2 Notation and preliminaries

Throughout the paper, \mathbb{R} is the set of all real numbers, \mathbb{R}_+ denotes the set of all nonnegative real numbers, \mathbb{Z}_+ stands for the set of all nonnegative integers, \mathbb{C}^l denotes the l -dimensional space of all l -tuples complex numbers, $\|\cdot\|$ is the norm on \mathbb{C}^l , $\mathcal{L}(\mathbb{Z}_+, \mathbb{C}^l)$ is the space of all \mathbb{C}^l -valued bounded functions with ‘sup’ norm, and let $\mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$ be the set of all continuous, bounded, and 2-periodic vectorial functions f with the property that $f(0) = 0$.

Let \mathcal{H} be a square matrix of order $l \geq 1$ which has complex entries and let Υ denote the spectrum of \mathcal{H} , i.e., $\Upsilon := \{\lambda : \lambda \text{ is an eigenvalue of } \mathcal{H}\}$. We have the following lemmas.

Lemma 2.1 *If $\|\mathcal{H}^n\| < \infty$ for any $n \in \mathbb{Z}_+$, then $|\lambda| \leq 1$ for any $\lambda \in \Upsilon$.*

Proof Suppose to the contrary that $|\lambda| > 1$. By the definition of eigenvalue, there exists a nonzero vector $\theta \in \mathbb{C}^l$ such that $\mathcal{H}\theta = \lambda\theta$, which implies that $\mathcal{H}^n\theta = \lambda^n\theta$ for any $n \in \mathbb{Z}_+$, and thus $\|\mathcal{H}^n\| \geq \|\mathcal{H}^n\theta\|/\|\theta\| = |\lambda|^n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, $|\lambda| \leq 1$. The proof is complete. □

Lemma 2.2 *If $\|\sum_{j=0}^P \mathcal{H}^j\| < \infty$ for any $P \in \mathbb{Z}_+$, then 1 does not belong to Υ .*

Proof If $1 \in \Upsilon$, then $\mathcal{H}\theta = \theta$ for some nonzero vector θ in \mathbb{C}^l and $\mathcal{H}^k\theta = \theta$ for all $k = 1, 2, \dots, P$. Therefore, we conclude that

$$\sup_{P \in \mathbb{Z}_+} \left\| \sum_{j=0}^P \mathcal{H}^j \right\| = \sup_{P \in \mathbb{Z}_+} \sup_{\theta \neq 0} \frac{\|(I + \mathcal{H} + \dots + \mathcal{H}^P)(\theta)\|}{\|\theta\|} \geq \sup_{P \in \mathbb{Z}_+, \theta \neq 0} \frac{P\|\theta\|}{\|\theta\|} = \infty,$$

and so 1 does not belong to Υ . This completes the proof. □

Let \mathcal{S} be a square matrix of order $l \geq 1$ which has complex entries. We have the following two corollaries.

Corollary 2.3 *If $\| \sum_{j=0}^P (e^{i\gamma} S)^j \| < \infty$ for any $\gamma \in \mathbb{R}$ and any $P \in \mathbb{Z}_+$, then $e^{-i\gamma}$ is not an eigenvalue of S .*

Proof Let $\mathcal{H} = e^{i\gamma} S$. By virtue of Lemma 2.2, 1 is not an eigenvalue of $e^{i\gamma} S$, and thus $I - e^{i\gamma} S$ is an invertible matrix or $e^{i\gamma} (e^{-i\gamma} I - S)$ is an invertible matrix, i.e., $e^{-i\gamma}$ is not an eigenvalue of S . The proof is complete. □

Corollary 2.4 *If $\| \sum_{j=0}^P (e^{i\gamma} S)^j \| < \infty$ for any $\gamma \in \mathbb{R}$ and any $P \in \mathbb{Z}_+$, then $|\lambda| < 1$ for any eigenvalue λ of S .*

Proof By virtue of

$$I - (e^{i\gamma} S)^P = (I - e^{i\gamma} S)(I + e^{i\gamma} S + \dots + (e^{i\gamma} S)^{P-1}) \quad \text{for any } P \in \mathbb{Z}_+ \text{ and any } \gamma \in \mathbb{R},$$

we deduce that

$$\| (e^{i\gamma} S)^P \| \leq 1 + \| (I - e^{i\gamma} S) \| \| (I + e^{i\gamma} S + \dots + (e^{i\gamma} S)^{P-1}) \| \leq 1 + (1 + \| S \|)K.$$

It follows from Lemmas 2.1 and 2.2 that the absolute value of each eigenvalue λ of $e^{i\gamma} S$ is less than or equal to one and $e^{-i\gamma}$ is in the resolvent set of S , respectively. Thus, we have, for any eigenvalue λ of S , $|\lambda| < 1$. This completes the proof. □

Definition 2.5 Let ϵ be a positive real number. If there exists a constant $L \geq 0$ such that, for every differentiable function ϕ satisfying the relation $\| \dot{\phi}(t) - \Lambda(t)\phi(t) \| \leq \epsilon$ for any $t \in \mathbb{R}_+$, there exists an exact solution $\theta(t)$ of $\dot{\theta}(t) = \Lambda(t)\theta(t)$ such that

$$\| \phi(t) - \theta(t) \| \leq L\epsilon, \tag{2.1}$$

then the system $\dot{\theta}(t) = \Lambda(t)\theta(t)$ is said to be Hyers-Ulam stable.

Remark 2.6 If $\phi(t)$ is an approximate solution of $\dot{\theta}(t) = \Lambda(t)\theta(t)$, then $\dot{\phi}(t) \approx \Lambda(t)\phi(t)$. Hence, letting g be an error function, then $\phi(t)$ is the exact solution of $\dot{\phi}(t) = \Lambda(t)\phi(t) + g(t)$.

On the basis of Remark 2.6, Definition 2.5 can be modified as follows.

Definition 2.7 Let ϵ be a positive real number. If there exists a constant $L \geq 0$ such that, for every differentiable function ϕ satisfying $\| g(t) \| \leq \epsilon$ for any $t \in \mathbb{R}_+$, there exists an exact solution $\theta(t)$ of $\dot{\theta}(t) = \Lambda(t)\theta(t)$ such that (2.1) holds, then the system $\dot{\theta}(t) = \Lambda(t)\theta(t)$ is said to be Hyers-Ulam stable.

3 Main results

Let us consider the time dependent 2-periodic system

$$\dot{\theta}(t) = \Lambda(t)\theta(t), \quad \theta(t) \in \mathbb{C}^l \text{ and } t \in \mathbb{R}_+, \tag{Λ(t)}$$

where $\Lambda(t + 2) = \Lambda(t)$ for all $t \in \mathbb{R}_+$.

Definition 3.1 Let $\mathcal{B}(t)$ be the fundamental solution matrix of $(\Lambda(t))$. The system $(\Lambda(t))$ is said to be uniformly exponentially stable if there exist two positive constants M and α such that

$$\|\mathcal{B}(t)\mathcal{B}^{-1}(s)\| \leq Me^{-\alpha(t-s)} \quad \text{for all } t \geq s.$$

It follows from [11] that system $(\Lambda(t))$ is uniformly exponentially stable if and only if the spectrum of the matrix $\mathcal{B}(2)$ lies inside of the circle of radius one.

Consider now the Cauchy problem

$$\begin{cases} \dot{\phi}(t) = \Lambda(t)\phi(t) + e^{i\gamma t}\xi(t), & t \in \mathbb{R}_+, \\ \phi(0) = \theta_0. \end{cases} \quad (\Lambda(t), \gamma, \theta_0)$$

The solution of the Cauchy problem $(\Lambda(t), \gamma, \theta_0)$ is given by

$$\phi(t) = \mathcal{B}(t)\mathcal{B}^{-1}(0)\theta_0 + \int_0^t \mathcal{B}(t)\mathcal{B}^{-1}(s)e^{i\gamma s}\xi(s) ds.$$

For $I := [0, 2]$ and $i \in \{1, 2\}$, we define the functions $\pi_i : I \rightarrow \mathbb{C}$ by

$$\pi_1(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2 - t, & 1 \leq t \leq 2, \end{cases} \quad \text{and} \quad \pi_2(t) = t(2 - t). \tag{3.1}$$

Let us denote by \mathcal{M}_i the set $\{\xi \in \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l) : \xi(t) = \mathcal{B}(t)\pi_i(t), i \in \{1, 2\}\}$. We are now in a position to state our main results.

Theorem 3.2 *Let the exact solution $\phi(t)$ of the Cauchy problem $(\Lambda(t), \gamma, \theta_0)$ be an approximate solution of system $(\Lambda(t))$ with the error term $e^{i\gamma t}\xi(t)$, where $\gamma \in \mathbb{R}$ and $\xi \in \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$. Then the following two statements hold.*

- (1) *If system $(\Lambda(t))$ is uniformly exponentially stable, then system $(\Lambda(t))$ is Hyers-Ulam stable.*
- (2) *If $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$, $\xi \in \mathcal{M} \subset \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$, and system $(\Lambda(t))$ is Hyers-Ulam stable, then system $(\Lambda(t))$ is uniformly exponentially stable.*

Proof (1) Let $\epsilon > 0$ and $\phi(t)$ be the approximate solution of $(\Lambda(t))$ such that $\sup_{t \in \mathbb{R}_+} \|\dot{\phi}(t) - \Lambda(t)\phi(t)\| = \sup_{t \in \mathbb{R}_+} \|e^{i\gamma t}\xi(t)\|$, $\phi(0) = \theta_0$, and $\sup_{t \in \mathbb{R}_+} \|\xi(t)\| \leq \epsilon$, and let $\theta(t)$ be the exact solution of $(\Lambda(t))$. Then

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} \|\phi(t) - \theta(t)\| &= \sup_{t \in \mathbb{R}_+} \left\| \mathcal{B}(t)\mathcal{B}^{-1}(0)\theta_0 + \int_0^t \mathcal{B}(t)\mathcal{B}^{-1}(s)e^{i\gamma s}\xi(s) ds - \mathcal{B}(t)\mathcal{B}^{-1}(0)\theta_0 \right\| \\ &= \int_0^t \mathcal{B}(t)\mathcal{B}^{-1}(s)e^{i\gamma s}\xi(s) ds \leq \int_0^t \|\mathcal{B}(t)\mathcal{B}^{-1}(s)\| \|\xi(s)\| ds \\ &\leq \int_0^t Me^{-\alpha(t-s)} \|\xi(s)\| ds = Me^{-\alpha t} \int_0^t e^{\alpha s} \|\xi(s)\| ds \\ &\leq Me^{-\alpha t} \int_0^t e^{\alpha s} \epsilon ds = \epsilon \frac{M}{\alpha} (1 - e^{-\alpha t}) \leq \frac{M}{\alpha} \epsilon = L\epsilon, \end{aligned}$$

where $M > 0$, $\alpha > 0$, and $L := M/\alpha$. Hence, $\sup_{t \in \mathbb{R}_+} \|\phi(t) - \theta(t)\| \leq L\epsilon$, which implies that system $(\Lambda(t))$ is Hyers-Ulam stable.

(2) The proof of the second part is more tricky. Let $a \in \mathbb{C}^l$ and $\xi_1 \in \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$ be such that

$$\xi_1(s) = \begin{cases} \mathcal{B}(s)sa, & \text{if } s \in [0, 1), \\ \mathcal{B}(s)(2-s)a, & \text{if } s \in [1, 2]. \end{cases}$$

Then we have, for each $s \in \mathbb{R}_+$, $\xi_1(s) = \mathcal{B}(s)\pi_1(s)a$, where π_1 is defined by (3.1). Thus, for any positive integer $n \geq 1$,

$$\phi_{\xi_1}(2n) = \int_0^{2n} \mathcal{B}(2n)\mathcal{B}^{-1}(\tau)e^{i\gamma\tau} \xi_1(\tau) d\tau = \sum_{k=0}^{n-1} \int_{2k}^{2k+2} \mathcal{B}(2n)\mathcal{B}^{-1}(\tau)e^{i\gamma\tau} \xi_1(\tau) d\tau.$$

Let $\tau = 2k + s$. We know that $\mathcal{B}^{-1}(2k + s) = \mathcal{B}^{-1}(2k)\mathcal{B}^{-1}(s)$, and so

$$\begin{aligned} \phi_{\xi_1}(2n) &= \sum_{k=0}^{n-1} \int_0^1 \mathcal{B}(2n)\mathcal{B}^{-1}(2k + s)e^{2i\gamma k} e^{i\gamma s} \xi_1(s) ds \\ &= \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a \int_0^2 e^{i\gamma s} \pi_1(s) ds. \end{aligned}$$

Define

$$\mathcal{A}_1 := \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{C}_1(\gamma) := \int_0^2 e^{i\gamma s} \pi_1(s) ds.$$

It is not difficult to verify that $\mathcal{C}_1(\gamma) \neq 0$ for any $\gamma \in \mathcal{A}_1$, and hence

$$\phi_{\xi_1}(2n)(\mathcal{C}_1(\gamma))^{-1} = \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a \quad \text{for all } \gamma \in \mathcal{A}_1. \tag{3.2}$$

Again, let $\xi_2 \in \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$ be given on $[0, 2]$ such that $\xi_2(s) = \mathcal{B}(s)\pi_2(s)a$, where π_2 is defined as in (3.1). With a similar approach to above, we have

$$\phi_{\xi_2}(2n)(\mathcal{C}_2(\gamma))^{-1} = \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a, \quad \gamma \in \mathcal{A}_2 := \{2k\pi : k \in \mathbb{Z}\}, \tag{3.3}$$

where

$$\mathcal{C}_2(\gamma) := \int_0^2 e^{i\gamma s} \pi_2(s) ds.$$

By virtue of the fact that system $(\Lambda(t))$ is Hyers-Ulam stable, we conclude that ϕ_{ξ_1} and ϕ_{ξ_2} are bounded functions, *i.e.*, there exist two positive constants \mathcal{K}_1 and \mathcal{K}_2 such that

$$\|\phi_{\xi_1}(2n)\| \leq \mathcal{K}_1 \quad \text{and} \quad \|\phi_{\xi_2}(2n)\| \leq \mathcal{K}_2 \quad \text{for all } n = 1, 2, \dots$$

It follows from (3.2) and (3.3) that

$$\left\| \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a \right\| \leq \frac{\mathcal{K}}{|\mathcal{C}_1|} := \mathcal{R}_1, \quad \text{if } \gamma \in \mathcal{A}_1, \tag{3.4}$$

and

$$\left\| \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a \right\| \leq \frac{\mathcal{K}_2}{|\mathcal{C}_2|} := \mathcal{R}_2, \quad \text{if } \gamma \in \mathcal{A}_2, \tag{3.5}$$

respectively. Hence, by virtue of (3.4) and (3.5), we have, for any $\gamma \in \mathcal{A}_1 \cup \mathcal{A}_2 = \mathbb{R}$ and each $a \in \mathbb{C}^l$,

$$\left\| \sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a \right\| \leq \mathcal{R}_1 + \mathcal{R}_2. \tag{3.6}$$

Let $n - k = j$. Then

$$\sum_{k=0}^{n-1} e^{2i\gamma k} \mathcal{B}(2n - 2k)a = e^{2i\gamma n} \sum_{j=1}^n e^{-2i\gamma j} \mathcal{B}(2j)a.$$

From (3.6), we obtain

$$\left\| \sum_{j=1}^n e^{-2i\gamma j} (\mathcal{B}(2))^j \right\| \leq L < \infty.$$

Thus, using $\mathcal{S} = \mathcal{B}(2)$ in Corollary 2.4, we deduce that the spectrum of $\mathcal{B}(2)$ lies in the interior of the circle of radius one, *i.e.*, system $(\Lambda(t))$ is uniformly exponentially stable. This completes the proof. □

Corollary 3.3 *Let the exact solution $\phi(t)$ of the Cauchy problem $(\Lambda(t), \gamma, \theta_0)$ be an approximate solution of system $(\Lambda(t))$ with the error term $e^{i\gamma t} \xi(t)$, where $\gamma \in \mathbb{R}$, $\xi \in \mathcal{M} \subset \mathcal{W}_0^2(\mathbb{R}_+, \mathbb{C}^l)$, and $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$. Then system $(\Lambda(t))$ is uniformly exponentially stable if and only if it is Hyers-Ulam stable.*

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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