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Boundary value problems of the nonlinear multiple base points impulsive fractional differential equations with constant coefficients

Yunsong Miao and Fang Li*

*Correspondence:
fangli860@gmail.com
School of Mathematics, Yunnan Normal University, Kunming, 650092, P.R. China

Abstract

In this paper, the nonlinear multiple base points boundary value problems of the impulsive fractional differential equations are studied. By using the fixed point theorem and the Mittag-Leffler functions, the sufficient conditions for the existence of the solutions to the given problems are formulated. An example is presented to illustrate the result.

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1 Introduction

The application of fractional calculus is very broad, including characterization of mechanics and electricity, earthquake analysis, the memory of many kinds of material, electronic circuits, electrolysis chemical, etc. ([1–5]). In recent years, there has been a significant development in solving differential equations involving fractional derivatives ([6–14] and the references therein).

In the left and right fractional derivatives ${}^cD^{\alpha}_{a^+}x$ and ${}^cD^{\alpha}_{b^-}x$, a is called a left base point and b a right base point. Both a and b are called base points of the fractional derivatives. A fractional differential equation (FDE) containing more than one base point is called a multiple base points FDE ([10]). In this paper, we study the following boundary value problem (BVP) of nonlinear multiple base points fractional differential equations with impulses:

$$\begin{cases} {}^cD^{\alpha}_{*}x(t)+\lambda x(t)=f\bigl(t,x(t)\bigr), \quad t\in J\setminus\{t_1,t_2,\dots,t_{m-1}\}, J=[0,1], & (1.1)\\ \Delta x(t_k)=I_k, \quad k=1,2,\dots,m-1, & (1.2)\\ x(0)+I^{\gamma}_{0^+}x(\eta)=0, \qquad x(1)+{}^cD^{\delta}_{t_m^+}x(1)=0, \quad \eta\in(0,t_1), & (1.3)\end{cases}$$

where $\alpha,\gamma,\delta\in(0,1)$, $\alpha>\gamma$, $\alpha>\delta$, $\lambda>0$. ${}^cD^{\cdot}_{*}$ is the standard Caputo fractional derivative at the base points $t=t_k$ $(k=0,1,2,\dots,m)$, that is, ${}^cD^{\cdot}_{*}|_{(t_k,t_{k+1}]}x(t)={}^cD^{\cdot}_{t_k^+}x(t)$ for all $t\in(t_k,\ t_{k+1}]$,

$I_{0^+}^{\gamma}$ denotes the fractional integral of order γ, $f: J \times \mathbb{R} \to \mathbb{R}$ is an appropriate function to be specified later. The impulsive moments $\{t_k\}$ are given such that $0 < t_1 < \cdots < t_{m-1} < 1$, $\Delta x(t_k)$ represents the jump of function x at t_k, which is defined by $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$, $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, the constant I_k denotes the size of the jump.

In 1954, Barrett ([6]) applied the method of successive approximations to derive the existence of solutions to the fractional differential equations of order $\alpha \in (0,1)$ with constant coefficients. Recently, as mentioned in [13, 14] and the references therein, the existence results of the impulsive fractional differential equations with anti-periodic boundary conditions involving the Caputo differential operator of order $\alpha \in (0,1)$ are obtained by the Mittag-Leffler functions. Inspired by the work of the above papers, the aim of the present paper is to establish some simple criteria for the existence of solutions of BVP (1.1)-(1.3).

The paper is organized as follows. In Section 2, we present some basic concepts, the notations about the fractional calculus and the properties of the Mittag-Leffler functions. In Section 3, we present the definition of solution for (1.1)-(1.3). In Section 4, by applying Krasnoselskii's fixed point theorem, we verify the existence of solutions for problem (1.1)-(1.3). We give an example to illustrate the result in Section 5.

2 Preliminaries

In this paper, we denote by $L^p(J,\mathbb{R})$ the Banach space of all Lebesgue measurable functions $l: J \to \mathbb{R}$ with the norm $\|l\|_{L^p} = (\int_J |l(t)|^p\,dt)^{\frac{1}{p}} < \infty$ and by $\mathrm{AC}([a,b],\mathbb{R})$ the space of all the absolutely continuous functions defined on $[a,b]$.

Definition 2.1 ([2, 3]) The fractional integral of order q with the lower limit a for a function $g(t) \in L^1([a,+\infty),\mathbb{R})$ is defined as

$$\left(I_{a^+}^q g\right)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} g(s)\,ds, \quad t > a, q > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([2, 3]) If $g(t) \in \mathrm{AC}^n([a,b],\mathbb{R})$, then the Riemann-Liouville fractional derivative $({}^L D_{a^+}^q g)(t)$ of order q exists almost everywhere on $[a,b]$ and can be written as

$$\left({}^L D_{a^+}^q g\right)(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} g(s)\,ds, \quad t > a, n-1 < q < n.$$

Definition 2.3 ([2, 3]) If $g(t) \in \mathrm{AC}^n([a,b],\mathbb{R})$, then the Caputo derivative $({}^c D_{a^+}^q g)(t)$ of order q exists almost everywhere on $[a,b]$ and can be written as

$$\left({}^c D_{a^+}^q g\right)(t) = \left({}^L D_{a^+}^q \left[g(s) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(s-a)^k \right]\right)(t), \quad t > a, n-1 < q < n,$$

moreover, if $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = 0$, then $({}^c D_{a^+}^q g)(t) = ({}^L D_{a^+}^q g)(t)$.

Remark 2.4 ([2, 3]) The Caputo fractional derivative of order q for a function $g \in C^n([a,b],\mathbb{R})$ is defined by

$$\left({}^cD^q_{a^+}g\right)(t) = \frac{1}{\Gamma(n-q)}\int_a^t \frac{g^{(n)}(s)}{(t-s)^{q-n+1}}\,ds, \quad t > a, n-1 < q < n.$$

Definition 2.5 ([2, 3]) For $\alpha, \beta > 0$, $z \in \mathbb{C}$, the classical Mittag-Leffler function $E_\alpha(z)$ and the generalized Mittag-Leffler functions $E_{\alpha,\beta}(z)$ are defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \qquad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$
$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\frac{(\rho)_k}{k!},$$

where $(\rho)_0 = 1$ and $(\rho)_k = \rho(\rho+1)\cdots(\rho+k-1)$ for $k \in \mathbb{N}$.

Clearly, $E_{\alpha,1}(z) = E_\alpha(z)$.

Lemma 2.6 ([2]) *Let $\nu, \beta, \alpha > 0$. The usual derivatives of $E_\alpha(z)$, $E_{\alpha,\beta}(z)$ and the Riemann-Liouville integration of $E_\alpha(-\lambda t^\alpha)$ are expressed by*

(1) $(\frac{d}{dz})^n[E_{\alpha,\beta}(z)] = n!E^{n+1}_{\alpha,\beta+\alpha n}(z)$, $n \in \mathbb{N}$;
(2) $(\frac{d}{dz})^n[E_\alpha(z)] = n!E^{n+1}_{\alpha,1+\alpha n}(z)$, $n \in \mathbb{N}$;
(3) $(\frac{d}{dt})^n[t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)] = t^{\beta-n-1}E_{\alpha,\beta-n}(-\lambda t^\alpha)$, $n \geq 1$;
(4) $[I^{\beta}_{0^+}(s^{\nu-1}E_{\alpha,\nu}(-\lambda s^\alpha))](t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\nu-1}E_{\alpha,\nu}(-\lambda s^\alpha)\,ds = t^{\beta+\nu-1}E_{\alpha,\beta+\nu}(-\lambda t^\alpha)$.

As mentioned in ([14]), $E_\alpha(-\lambda t^\alpha)$ and $E_{\alpha,\alpha}(-\lambda t^\alpha)$ can be represented by

$$E_\alpha\left(-\lambda t^\alpha\right) = \int_0^\infty e^{-\lambda t^\alpha \theta}\phi(\theta)\,d\theta, \tag{2.1}$$
$$E_{\alpha,\alpha}\left(-\lambda t^\alpha\right) = \alpha\int_0^\infty \theta e^{-\lambda t^\alpha \theta}\phi(\theta)\,d\theta, \tag{2.2}$$

where

$$\phi(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad 0 < \alpha < 1, \theta > 0.$$

Moreover,

$$\int_0^\infty \theta^\xi\phi(\theta)\,d\theta = \frac{\Gamma(\xi+1)}{\Gamma(\alpha\xi+1)} \quad (\xi \geq 0). \tag{2.3}$$

Lemma 2.7 *For $\lambda > 0$, $\alpha, \beta, \theta_1, \theta_2 \in (0,1)$, $\alpha \geq \theta_2$, the generalized Mittag-Leffler functions have the following properties*:

(1) $\frac{d}{dt}[E_\alpha(-\lambda t^\alpha)] = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$;
(2) $E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) = \frac{1}{\Gamma(\beta)}\int_0^1 E_{\alpha,\alpha}(-\lambda t^\alpha u^\alpha)u^{\alpha-1}(1-u)^{\beta-1}\,du$;
(3) $E_{\alpha,\beta}(-\lambda t^\alpha) = \frac{1}{\Gamma(\beta)} - \lambda t^\alpha E_{\alpha,\alpha+\beta}(-\lambda t^\alpha)$;
(4) $E_{\alpha,\theta_1+1}(-\lambda t^\alpha) = \frac{1}{\Gamma(\theta_1)}\int_0^1 E_\alpha(-\lambda t^\alpha u^\alpha)(1-u)^{\theta_1-1}\,du$;

(5) $[{}^cD_{a^+}^{\theta_2}E_\alpha(-\lambda(s-a)^\alpha)](t) = -\lambda(t-a)^{\alpha-\theta_2}E_{\alpha,\alpha-\theta_2+1}(-\lambda(t-a)^\alpha)$.
In particular, when $\alpha = \theta_2$, $[{}^cD_{a^+}^{\alpha}E_\alpha(-\lambda(s-a)^\alpha)](t) = -\lambda E_\alpha(-\lambda(t-a)^\alpha)$.

Proof We denote the beta function by $\mathbb{B}(\cdot,\cdot)$. From Lemma 2.6(2),

$$\begin{aligned}\frac{d}{dt}\big[E_\alpha(-\lambda t^\alpha)\big] &= -\lambda\alpha t^{\alpha-1}E^2_{\alpha,1+\alpha}(-\lambda t^\alpha)\\ &= -\lambda\alpha t^{\alpha-1}\sum_{k=0}^\infty\frac{(-\lambda t^\alpha)^k(1+k)}{\Gamma(\alpha k+1+\alpha)}\\ &= -\lambda t^{\alpha-1}\sum_{k=0}^\infty\frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+\alpha)}\\ &= -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha).\end{aligned}$$

From [14], the second result holds. Moreover,

$$\begin{aligned}E_{\alpha,\beta}(-\lambda t^\alpha) &= \sum_{k=0}^\infty\frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+\beta)} = \frac{1}{\Gamma(\beta)} - \lambda t^\alpha\sum_{k=1}^\infty\frac{(-\lambda t^\alpha)^{k-1}}{\Gamma(\alpha k+\beta)}\\ &= \frac{1}{\Gamma(\beta)} - \lambda t^\alpha E_{\alpha,\alpha+\beta}(-\lambda t^\alpha),\\ E_{\alpha,\theta_1+1}(-\lambda t^\alpha) &= \sum_{k=0}^\infty\frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+\theta_1+1)} = \frac{1}{\Gamma(\theta_1)}\sum_{k=0}^\infty\frac{(-\lambda t^\alpha)^k\mathbb{B}(\alpha k+1,\theta_1)}{\Gamma(\alpha k+1)}\\ &= \frac{1}{\Gamma(\theta_1)}\int_0^1\sum_{k=0}^\infty\frac{(-\lambda t^\alpha u^\alpha)^k}{\Gamma(\alpha k+1)}(1-u)^{\theta_1-1}\,du\\ &= \frac{1}{\Gamma(\theta_1)}\int_0^1 E_\alpha(-\lambda t^\alpha u^\alpha)(1-u)^{\theta_1-1}\,du.\end{aligned}$$

Applying Remark 2.4 and the fact $\int_a^t(t-s)^{m_1-1}(s-a)^{m_2-1}\,ds = (t-a)^{m_1+m_2-1}\mathbb{B}(m_1,m_2)$, we have

$$\begin{aligned}\big[{}^cD_{a^+}^{\theta_2}E_\alpha(-\lambda(s-a)^\alpha)\big](t) &= \frac{1}{\Gamma(1-\theta_2)}\int_a^t(t-s)^{-\theta_2}\frac{d}{ds}\left(\sum_{k=0}^\infty\frac{(-\lambda(s-a)^\alpha)^k)}{\Gamma(\alpha k+1)}\right)ds\\ &= \frac{1}{\Gamma(1-\theta_2)}\sum_{k=1}^\infty\frac{(-\lambda)^k}{\Gamma(\alpha k)}\int_a^t(t-s)^{-\theta_2}(s-a)^{\alpha k-1}\,ds\\ &= -\lambda(t-a)^{\alpha-\theta_2}\sum_{k=1}^\infty\frac{(-\lambda)^{k-1}(t-a)^{\alpha(k-1)}}{\Gamma(\alpha k+1-\theta_2)}\\ &= -\lambda(t-a)^{\alpha-\theta_2}E_{\alpha,\alpha-\theta_2+1}(-\lambda(t-a)^\alpha).\end{aligned}$$ □

Lemma 2.8 ([3]) *If* $0<\alpha<2$, β *is an arbitrary real number*, $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$, *then*

$$\big|E_{\alpha,\beta}(z)\big| \le \frac{\mathcal{C}}{1+|z|},\qquad \mu\le\big|\arg(z)\big|\le\pi,\qquad |z|\ge 0,$$

where $\mathcal{C}$ *is a positive constant.*

Lemma 2.9 *Let $\alpha, \beta \in (0,1)$, $\lambda > 0$. Then the functions E_α, $E_{\alpha,\alpha}$ and $E_{\alpha,\alpha+\beta}$ are nonnegative and have the following properties*:

(i) *For any $t \in J$, $E_\alpha(-\lambda t^\alpha) \le 1$, $E_{\alpha,\alpha}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\alpha)}$, $E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\alpha+\beta)}$, $E_{\alpha,\beta}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\beta)}$, moreover, $E_\alpha(0) = 1$. In particular,*

$$E_{\alpha,\alpha-\delta}\bigl(-\lambda t^\alpha\bigr) \le \frac{1}{\Gamma(\alpha-\delta)}, \qquad \bigl|E_{\alpha,\alpha-\delta}\bigl(-\lambda t^\alpha\bigr)\bigr| \le \mathcal{C}. \tag{2.4}$$

(ii) *For any $t_1, t_2 \in J$,*

$$\begin{aligned}
&\bigl|E_\alpha\bigl(-\lambda t_2^\alpha\bigr) - E_\alpha\bigl(-\lambda t_1^\alpha\bigr)\bigr| = O\bigl(|t_2-t_1|^\alpha\bigr), \quad \textit{as } t_2 \to t_1,\\
&\bigl|E_{\alpha,\alpha}\bigl(-\lambda t_2^\alpha\bigr) - E_{\alpha,\alpha}\bigl(-\lambda t_1^\alpha\bigr)\bigr| = O\bigl(|t_2-t_1|^\alpha\bigr), \quad \textit{as } t_2 \to t_1,\\
&\bigl|E_{\alpha,\alpha-\delta}\bigl(-\lambda t_2^\alpha\bigr) - E_{\alpha,\alpha-\delta}\bigl(-\lambda t_1^\alpha\bigr)\bigr| = O\bigl(|t_2-t_1|^\alpha\bigr), \quad \textit{as } t_2 \to t_1.
\end{aligned}$$

Proof (i) From (2.1), we get $E_\alpha(-\lambda t^\alpha) = \int_0^\infty e^{-\lambda t^\alpha \theta}\phi(\theta)\,d\theta \le \int_0^\infty \phi(\theta)\,d\theta = 1$.

By (2.2), we find $E_{\alpha,\alpha}(-\lambda t^\alpha) = \alpha\int_0^\infty \theta e^{-\lambda t^\alpha\theta}\phi(\theta)\,d\theta \le \frac{1}{\Gamma(\alpha)}$.

Using Lemma 2.7(2), one sees

$$E_{\alpha,\alpha+\beta}\bigl(-\lambda t^\alpha\bigr) = \frac{1}{\Gamma(\beta)}\int_0^1 E_{\alpha,\alpha}\bigl(-\lambda t^\alpha u^\alpha\bigr)u^{\alpha-1}(1-u)^{\beta-1}\,du \le \frac{1}{\Gamma(\alpha+\beta)}.$$

Noting $E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) > 0$ and Lemma 2.7(3), we have $E_{\alpha,\beta}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\beta)}$. Taking $\beta = \alpha - \delta$ in $E_{\alpha,\beta}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\beta)}$, we obtain $E_{\alpha,\alpha-\delta}(-\lambda t^\alpha) \le \frac{1}{\Gamma(\alpha-\delta)}$. By Lemma 2.8, we get $|E_{\alpha,\alpha-\delta}(-\lambda t^\alpha)| \le \mathcal{C}$.

(ii) For $0 \le t_1 < t_2 \le 1$, using the Lagrange mean value theorem and the fact $|t_2^\alpha - t_1^\alpha| \le (t_2-t_1)^\alpha$, (2.1), (2.2) and (2.3), we find

$$\begin{aligned}
\bigl|E_\alpha\bigl(-\lambda t_2^\alpha\bigr) - E_\alpha\bigl(-\lambda t_1^\alpha\bigr)\bigr| &= \int_0^\infty \bigl|e^{-\lambda t_2^\alpha\theta} - e^{-\lambda t_1^\alpha\theta}\bigr|\phi(\theta)\,d\theta \le \lambda(t_2-t_1)^\alpha\int_0^\infty \theta\phi(\theta)\,d\theta\\
&= \frac{\lambda(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} := O\bigl(|t_2-t_1|^\alpha\bigr), \quad \text{as } t_2 \to t_1,\\
\bigl|E_{\alpha,\alpha}\bigl(-\lambda t_2^\alpha\bigr) - E_{\alpha,\alpha}\bigl(-\lambda t_1^\alpha\bigr)\bigr| &= \alpha\int_0^\infty \bigl|e^{-\lambda t_2^\alpha\theta} - e^{-\lambda t_1^\alpha\theta}\bigr|\theta\phi(\theta)\,d\theta\\
&\le \frac{2\lambda\alpha(t_2-t_1)^\alpha}{\Gamma(2\alpha+1)} := O\bigl(|t_2-t_1|^\alpha\bigr), \quad \text{as } t_2 \to t_1,
\end{aligned}$$

by Lemma 2.7(3), Lemma 2.9(i) and Lemma 2.7(2), one has

$$\begin{aligned}
&\bigl|E_{\alpha,\alpha-\delta}\bigl(-\lambda t_2^\alpha\bigr) - E_{\alpha,\alpha-\delta}\bigl(-\lambda t_1^\alpha\bigr)\bigr|\\
&\quad= \lambda\bigl|t_2^\alpha E_{\alpha,2\alpha-\delta}\bigl(-\lambda t_2^\alpha\bigr) - t_1^\alpha E_{\alpha,2\alpha-\delta}\bigl(-\lambda t_1^\alpha\bigr)\bigr|\\
&\quad\le \lambda\bigl[|t_2-t_1|^\alpha E_{\alpha,2\alpha-\delta}\bigl(-\lambda t_2^\alpha\bigr) + t_1^\alpha\bigl|E_{\alpha,2\alpha-\delta}\bigl(-\lambda t_2^\alpha\bigr) - E_{\alpha,2\alpha-\delta}\bigl(-\lambda t_1^\alpha\bigr)\bigr|\bigr]\\
&\quad\le \frac{\lambda}{\Gamma(2\alpha-\delta)}|t_2-t_1|^\alpha\\
&\qquad+ \frac{\lambda}{\Gamma(\alpha-\delta)}\int_0^1 \bigl|E_{\alpha,\alpha}\bigl(-\lambda t_2^\alpha u^\alpha\bigr) - E_{\alpha,\alpha}\bigl(-\lambda t_1^\alpha u^\alpha\bigr)\bigr|u^{\alpha-1}(1-u)^{\alpha-\delta-1}\,du\\
&\quad:= O\bigl(|t_2-t_1|^\alpha\bigr), \quad \text{as } t_2 \to t_1.
\end{aligned} \tag{2.5}$$

□

Lemma 2.10 ([2]) *The solution to the Cauchy problem*

$$\begin{cases} {}^cD^{\alpha}_{a^+}x(t) + \lambda x(t) = f(t), \\ x(a) = b_1, \quad b_1 \in \mathbb{R}, \end{cases}$$

with $0 < \alpha < 1$ *has the form*

$$x(t) = b_1 E_\alpha\bigl(-\lambda(t-a)^\alpha\bigr) + \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr) f(s)\,ds.$$

Theorem 2.11 (Krasnoselskii's fixed point theorem) *Let* $\mathcal{M}$ *be a closed convex and nonempty subset of a Banach space* X. *Let* $\mathcal{A}$, $\mathcal{B}$ *be two operators such that* (i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ *whenever* $x, y \in \mathcal{M}$, (ii) $\mathcal{A}$ *is compact and continuous,* (iii) $\mathcal{B}$ *is a contraction mapping. Then there exists a* $z \in \mathcal{M}$ *such that* $z = \mathcal{A}z + \mathcal{B}z$.

3 Solutions for BVP

Setting $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m-1$, $J_m = [t_m, 1]$, and we define $X = \{x : [0,1] \to \mathbb{R} : x|_{J_k} \in C(J_k, \mathbb{R})$ and there exist $x(t_k^+)$ and $x(t_k^-)$, with $x(t_k^-) = x(t_k), k = 1, \dots, m-1\}$ with the norm

$$\|x\|_1 := \sup_{k=0,1,\dots,m} \sup_{t \in J_k} \bigl|x(t)\bigr|.$$

Obviously, X is a real Banach space.

In this paper, we consider the following assumption.

(H1) $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies $f(\cdot, x) : J \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}$ and $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$, and there exists a function $\mu \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ $(0 < q_1 < \min\{\frac{\alpha}{2}, \alpha - \delta\})$ such that $|f(t,x)| \le \mu(t)$.

Definition 3.1 A function $x : J \to \mathbb{R}$ is said to be a solution of (1.1)-(1.3) if

(1) $x \in \mathrm{AC}(J_k, \mathbb{R})$;
(2) x satisfies the equation ${}^cD^{\alpha}_{t_k^+}x(t) + \lambda x(t) = f(t, x(t))$ on J_k;
(3) for $k = 1, 2, \dots, m-1$, $\Delta x(t_k) = I_k$, and $x(0) + I^{\gamma}_{0^+}x(\eta) = 0$, $x(1) + {}^cD^{\delta}_{t_m^+}x(1) = 0$.

Next, we present the following lemmas.

Lemma 3.2 *For any* $\tau_2, \tau_1 \in J_k$ $(k = 0, 1, 2, \dots, m)$ *and* $\tau_2 < \tau_1$,

$$\int_{t_k}^{\tau_2} \bigl[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\bigr]\mu(s)\,ds \to 0, \quad \text{as } \tau_2 \to \tau_1.$$

Proof It follows from the Hölder inequality that

$$\begin{aligned} &\left|\int_{t_k}^{\tau_2} \bigl[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\bigr]\mu(s)\,ds\right| \\ &\quad \le \|\mu\|_{L^{\frac{1}{q_1}}} \left[\int_{t_k}^{\tau_2} \bigl|(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\bigr|^{\frac{1}{1-q_1}}\,ds\right]^{1-q_1} \end{aligned}$$

$$= (1-\alpha)\|\mu\|_{L^{\frac{1}{q_1}}}\left(\int_{t_k}^{\tau_2}\left|\int_{\tau_2}^{\tau_1}(\zeta-s)^{\alpha-2}\,d\zeta\right|^{\frac{1}{1-q_1}}ds\right)^{1-q_1}$$
$$\le \overline{M}\left[\int_{t_k}^{\tau_2}\left((\tau_2-s)^{\theta}-(\tau_1-s)^{\theta}\right)ds\right]^{1-q_1}$$
$$= \frac{\overline{M}}{(1+\theta)^{1-q_1}}\left[(\tau_1-\tau_2)^{1+\theta}-(\tau_1-t_k)^{1+\theta}+(\tau_2-t_k)^{1+\theta}\right]^{1-q_1}$$
$$\to 0, \quad \text{as } \tau_2 \to \tau_1,$$

where $\overline{M} > 0$ is a constant and $\theta = \frac{\alpha-1-q_1}{1-q_1} \in (-1, 0)$. □

For $y > q_1$ and $t_{i-1} \in J$ ($i = 1, \dots, m+1$), from the Hölder inequality, we have

$$\int_{t_{i-1}}^{t_i}(t_i-s)^{y-1}\mu(s)\,ds \le \left(\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{y-1}{1-q_1}}\,ds\right)^{1-q_1}\|\mu\|_{L^{\frac{1}{q_1}}} = \zeta_y(t_i-t_{i-1})^{y-q_1}, \tag{3.1}$$

where $\zeta_y = (\frac{1-q_1}{y-q_1})^{1-q_1}\|\mu\|_{L^{\frac{1}{q_1}}}$.

For brevity, we define

$$\left(Q_k^{\varsigma}x\right)(t) := \int_{t_k}^{t}(t-s)^{\varsigma-1}E_{\alpha,\varsigma}\left(-\lambda(t-s)^{\alpha}\right)f\left(s,x(s)\right)ds,$$

then, for $t \in (t_k, t_{k+1}]$, from (3.1) and Lemma 2.9(i), we obtain

$$\left|\left(Q_k^{\alpha}x\right)(t)\right| \le \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}\mu(s)}{\Gamma(\alpha)}\,ds \le \zeta_\alpha\frac{(t-t_k)^{\alpha-q_1}}{\Gamma(\alpha)}, \tag{3.2}$$

$$\left|\left(Q_k^{\alpha-\delta}x\right)(t)\right| \le \mathcal{C}\int_{t_k}^{t}(t-s)^{\alpha-\delta-1}\mu(s)\,ds \le \mathcal{C}\zeta_{\alpha-\delta}(t-t_k)^{\alpha-\delta-q_1}, \tag{3.3}$$

which means that $(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})f(s,x(s))$ and $(t-s)^{\alpha-\delta-1}E_{\alpha,\alpha-\delta}(-\lambda(t-s)^{\alpha})f(s,x(s))$ are Lebesgue integrable with respect to $s \in [t_k, t_{k+1}]$ for all $t \in [t_k, t_{k+1}]$ and $x \in X$.

Lemma 3.3 *For any $k = 0, 1, 2, \dots, m$, $(Q_k^{\alpha}x)(t) \in C(J_k, \mathbb{R})$, $(Q_k^{\alpha-\delta}x)(t) \in C(J_k, \mathbb{R})$.*

Proof For any $h > 0$, $t_k < t < t+h < t_{k+1}$, by (H1), Lemma 2.9(i), (ii), Lemma 3.2 and (3.1), we get

$$\left|\left(Q_k^{\alpha}x\right)(t+h)-\left(Q_k^{\alpha}x\right)(t)\right|$$
$$\le \int_{t_k}^{t}\left|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right|E_{\alpha,\alpha}\left(-\lambda(t+h-s)^{\alpha}\right)\left|f\left(s,x(s)\right)\right|ds$$
$$+ \int_{t_k}^{t}(t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-\lambda(t+h-s)^{\alpha}\right)-E_{\alpha,\alpha}\left(-\lambda(t-s)^{\alpha}\right)\right|\left|f\left(s,x(s)\right)\right|ds$$
$$+ \int_{t}^{t+h}(t+h-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(t+h-s)^{\alpha}\right)\left|f\left(s,x(s)\right)\right|ds$$
$$\le \int_{t_k}^{t}\frac{|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|}{\Gamma(\alpha)}\mu(s)\,ds + O\left(h^{\alpha}\right)\int_{t_k}^{t}(t-s)^{\alpha-1}\mu(s)\,ds$$

$$
\begin{aligned}
&+\int_t^{t+h}\frac{(t+h-s)^{\alpha-1}}{\Gamma(\alpha)}\mu(s)\,ds \\
&\to 0, \quad \text{as } h\to 0.
\end{aligned}
$$

Similarly, noting (2.4) and (2.5), we find $(Q_k^{\alpha-\delta}x)(t)\in C(J_k,\mathbb{R})$. □

Lemma 3.4 *Assume that* (H1) *holds. Then* $(Q_k^\alpha x)(t)\in \mathrm{AC}([t_k,t_{k+1}],\mathbb{R})$, *for* $x\in X$, $k=0,1,\ldots,m$.

Proof For every finite collection $\{(a_i,b_i)\}_{1\le i\le n}$ on $[t_k,t_{k+1}]$ with $\sum_{i=1}^n(b_i-a_i)\to 0$, noting (3.1), Lemma 3.2 and Lemma 2.9(ii), we have

$$
\begin{aligned}
&\sum_{i=1}^n\bigl|(Q_k^\alpha x)(b_i)-(Q_k^\alpha x)(a_i)\bigr| \\
&\quad\le\sum_{i=1}^n\biggl|\int_{a_i}^{b_i}(b_i-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda(b_i-s)^\alpha\bigr)f\bigl(s,x(s)\bigr)\,ds\biggr| \\
&\qquad+\sum_{i=1}^n\int_{t_k}^{a_i}\bigl|\bigl[(b_i-s)^{\alpha-1}-(a_i-s)^{\alpha-1}\bigr]E_{\alpha,\alpha}\bigl(-\lambda(b_i-s)^\alpha\bigr)f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\qquad+\sum_{i=1}^n\int_{t_k}^{a_i}(a_i-s)^{\alpha-1}\bigl|E_{\alpha,\alpha}\bigl(-\lambda(b_i-s)^\alpha\bigr)-E_{\alpha,\alpha}\bigl(-\lambda(a_i-s)^\alpha\bigr)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\quad\le\sum_{i=1}^n\int_{a_i}^{b_i}\frac{(b_i-s)^{\alpha-1}\mu(s)}{\Gamma(\alpha)}\,ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\int_{t_k}^{a_i}\bigl[(a_i-s)^{\alpha-1}-(b_i-s)^{\alpha-1}\bigr]\mu(s)\,ds \\
&\qquad+\sum_{i=1}^n\int_{t_k}^{a_i}(a_i-s)^{\alpha-1}\mu(s)\,ds\cdot O\bigl(|b_i-a_i|^\alpha\bigr) \\
&\quad\le\frac{\zeta_\alpha}{\Gamma(\alpha)}\sum_{i=1}^n(b_i-a_i)^{\alpha-q_1}+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\int_{t_k}^{a_i}\bigl[(a_i-s)^{\alpha-1}-(b_i-s)^{\alpha-1}\bigr]\mu(s)\,ds \\
&\qquad+\zeta_\alpha\sum_{i=1}^n O\bigl(|b_i-a_i|^\alpha\bigr) \\
&\quad\to 0.
\end{aligned}
$$

Hence, $(Q_k^\alpha x)(t)$ is absolutely continuous on $[t_k,t_{k+1}]$. Furthermore, for almost all $t\in[t_k,t_{k+1}]$, $[{}^cD_{t_k^+}^\alpha(Q_k^\alpha x)(s)](t)$ and $[{}^cD_{t_k^+}^\delta(Q_k^\alpha x)(s)](t)$ exist. □

Lemma 3.5 *Assume that* (H1) *holds. Then, for* $x\in X$, $k=0,1,\ldots,m$,

$$
\begin{aligned}
&\bigl[{}^cD_{t_k^+}^\alpha\bigl(Q_k^\alpha x\bigr)(s)\bigr](t)=f\bigl(t,x(t)\bigr)-\lambda\bigl(Q_k^\alpha x\bigr)(t),\quad \text{a.e. } t\in J_k, \\
&\bigl[{}^cD_{t_k^+}^\delta\bigl(Q_k^\alpha x\bigr)(s)\bigr](t)=\bigl(Q_k^{\alpha-\delta}x\bigr)(t),\quad \text{a.e. } t\in J_k.
\end{aligned}
$$

Proof According to Lemma 2.6(4), we can see that

$$\begin{aligned}\int_s^t (t-\tau)^{-\alpha}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda(\tau-s)^\alpha\big)\,d\tau &= \int_0^{t-s}(t-s-\tau)^{-\alpha}\tau^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda\tau^\alpha\big)\,d\tau \\ &= \Gamma(1-\alpha)E_\alpha\big(-\lambda(t-s)^\alpha\big), \\ \int_s^t (t-\tau)^{-\delta}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda(\tau-s)^\alpha\big)\,d\tau &= \int_0^{t-s}(t-s-\tau)^{-\delta}\tau^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda\tau^\alpha\big)\,d\tau \\ &= \Gamma(1-\delta)(t-s)^{\alpha-\delta}E_{\alpha,\alpha-\delta+1}\big(-\lambda(t-s)^\alpha\big).\end{aligned}$$

Moreover, noting Lemma 2.6(1) and Lemma 2.7(1), we obtain

$$\begin{aligned}&\big[{}^LD^\alpha_{t_k^+}\big(Q_k^\alpha x\big)(s)\big](t) \\ &\quad= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_k}^t (t-s)^{-\alpha}\left[\int_{t_k}^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda(s-\tau)^\alpha\big)f\big(\tau,x(\tau)\big)\,d\tau\right]ds \\ &\quad= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_k}^t f\big(\tau,x(\tau)\big)\,d\tau\int_\tau^t (t-s)^{-\alpha}(s-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda(s-\tau)^\alpha\big)\,d\tau \\ &\quad= \frac{d}{dt}\int_{t_k}^t E_\alpha\big(-\lambda(t-\tau)^\alpha\big)f\big(\tau,x(\tau)\big)\,d\tau \\ &\quad= f\big(t,x(t)\big)-\lambda\big(Q_k^\alpha x\big)(t), \quad \text{a.e. } t\in[t_k,t_{k+1}],\end{aligned} \tag{3.4}$$

and by Lemma 2.6(3), one gets

$$\begin{aligned}&\big[{}^LD^\delta_{t_k^+}\big(Q_k^\alpha x\big)(s)\big](t) \\ &\quad= \frac{1}{\Gamma(1-\delta)}\frac{d}{dt}\int_{t_k}^t f\big(\tau,x(\tau)\big)\,d\tau\int_\tau^t (t-s)^{-\delta}(s-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda(s-\tau)^\alpha\big)\,ds \\ &\quad= \frac{d}{dt}\int_{t_k}^t (t-\tau)^{\alpha-\delta}E_{\alpha,\alpha-\delta+1}\big(-\lambda(t-\tau)^\alpha\big)f\big(\tau,x(\tau)\big)\,d\tau \\ &\quad= \int_{t_k}^t (t-\tau)^{\alpha-\delta-1}E_{\alpha,\alpha-\delta}\big(-\lambda(t-\tau)^\alpha\big)f\big(\tau,x(\tau)\big)\,d\tau \\ &\quad= \big(Q_k^{\alpha-\delta}x\big)(t), \quad \text{a.e. } t\in[t_k,t_{k+1}].\end{aligned} \tag{3.5}$$

Noting (3.2) and (3.3), we have $(Q_k^\alpha x)(t_k^+)=0$ and $(Q_k^{\alpha-\delta}x)(t_k^+)=0$. Then, from Definition 2.3, with $g(t)$ replaced by $(Q_k^\alpha x)(t)$ and $(Q_k^{\alpha-\delta}x)(t)$, and applying (3.4) and (3.5), we derive

$$\big[{}^cD^\alpha_{t_k^+}\big(Q_k^\alpha x\big)(s)\big](t) = \big[{}^LD^\alpha_{t_k^+}\big(Q_k^\alpha x\big)(s)\big](t) = f\big(t,x(t)\big)-\lambda\big(Q_k^\alpha x\big)(t)$$

and $[{}^cD^\delta_{t_k^+}(Q_k^\alpha x)(s)](t) = (Q_k^{\alpha-\delta}x)(t)$. This completes the proof. □

Lemma 3.6 *Assume that* (H1) *holds. Then* $[I^\gamma_{0^+}(Q_0^\alpha x)(s)](t) = (Q_0^{\alpha+\gamma}x)(t)$.

Proof It follows from (3.2) that $(Q_0^\alpha x)(t)$ is Lebesgue integrable, noting Lemma 2.6(4), we have

$$
\begin{aligned}
&\left[I_{0^+}^\gamma\left(Q_0^\alpha x\right)(s)\right](t)\\
&=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\left(\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda(s-\tau)^\alpha\right)f\left(\tau,x(\tau)\right)d\tau\right)ds\\
&=\frac{1}{\Gamma(\gamma)}\int_0^t f\left(\tau,x(\tau)\right)d\tau\int_0^{t-\tau}(t-\tau-s)^{\gamma-1}s^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda s^\alpha\right)ds\\
&=\int_0^t (t-\tau)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\left(-\lambda(t-\tau)^\alpha\right)f\left(\tau,x(\tau)\right)d\tau=\left(Q_0^{\alpha+\gamma}x\right)(t). \qquad \square
\end{aligned}
$$

As a consequence of Lemmas 3.4-3.6, by directly computation, we get the following result. For brevity, we define

$$
\begin{aligned}
&\widetilde{c}:=-\frac{(Q_0^{\alpha+\gamma}x)(\eta)}{1+\eta^\gamma E_{\alpha,\gamma+1}(-\lambda\eta^\alpha)},\\
&(P_0x)(t):=\widetilde{c}E_\alpha\left(-\lambda t^\alpha\right),\\
&(P_ix)(t):=\left[(P_{i-1}x)(t_i)+\left(Q_{i-1}^\alpha x\right)(t_i)+I_i\right]E_\alpha\left(-\lambda(t-t_i)^\alpha\right),\quad i=1,\dots,m-1,\\
&(P_mx)(t):=-\frac{[(Q_m^\alpha x)(1)+(Q_m^{\alpha-\delta}x)(1)]E_\alpha(-\lambda(t-t_m)^\alpha)}{E_\alpha(-\lambda(1-t_m)^\alpha)-\lambda(1-t_m)^{\alpha-\delta}E_{\alpha,\alpha-\delta+1}(-\lambda(1-t_m)^\alpha)}.
\end{aligned}
$$

Lemma 3.7 *A function x is a solution of* (1.1)-(1.3) *if and only if x is a solution of the following equation*:

$$
x(t)=\begin{cases}
(P_0x)(t)+(Q_0^\alpha x)(t), & \textit{for } t\in J_0,\\
(P_1x)(t)+(Q_1^\alpha x)(t), & \textit{for } t\in J_1,\\
\cdots\\
(P_{m-1}x)(t)+(Q_{m-1}^\alpha x)(t), & \textit{for } t\in J_{m-1},\\
(P_mx)(t)+(Q_m^\alpha x)(t), & \textit{for } t\in J_m.
\end{cases}
\tag{3.6}
$$

Proof (Necessity) For $t\in J_0$, it follows from Lemma 2.10 that $x(t)=a_0E_\alpha(-\lambda t^\alpha)+(Q_0^\alpha x)(t)$. Obviously, $x(0)=a_0$. Moreover, from Lemma 2.6(4) (taking $\beta:=\gamma$, $\nu:=1$) and Lemma 3.6, we have

$$
I_{0^+}^\gamma x(\eta)=a_0\eta^\gamma E_{\alpha,\gamma+1}\left(-\lambda\eta^\alpha\right)+\left(Q_0^{\alpha+\gamma}x\right)(\eta).
$$

Using the condition $x(0)+I_{0^+}^\gamma x(\eta)=0$, we obtain $a_0=\widetilde{c}$, then, for $t\in J_0$,

$$
x(t)=(P_0x)(t)+\left(Q_0^\alpha x\right)(t).
$$

For $t\in J_1$, $x(t)=a_1E_\alpha(-\lambda(t-t_1)^\alpha)+(Q_1^\alpha x)(t)$, since $x(t_1^+)=a_1=(P_0x)(t_1)+(Q_0^\alpha x)(t_1)+I_1$, then, for $t\in J_1$,

$$
x(t)=(P_1x)(t)+\left(Q_1^\alpha x\right)(t).
$$

Repeating the above process, we find

$$x(t) = (P_k x)(t) + \big(Q_k^\alpha x\big)(t), \quad t \in J_k, k = 0,1,\dots,m-1.$$

For $t \in J_m = [t_m, 1]$, $x(t) = a_m E_\alpha(-\lambda(t-t_m)^\alpha) + (Q_m^\alpha x)(t)$.

Noting Lemma 2.7(5) and Lemma 3.5, we get

$${}^cD_{t_m^+}^\delta x(t) = -\lambda a_m (t-t_m)^{\alpha-\delta} E_{\alpha,\alpha-\delta+1}\big(-\lambda(t-t_m)^\alpha\big) + \big(Q_m^{\alpha-\delta} x\big)(t).$$

From $x(1) + {}^cD_{t_m^+}^\delta x(1) = 0$, one can obtain

$$a_m = -\frac{(Q_m^\alpha x)(1) + (Q_m^{\alpha-\delta} x)(1)}{E_\alpha(-\lambda(1-t_m)^\alpha) - \lambda(1-t_m)^{\alpha-\delta} E_{\alpha,\alpha-\delta+1}(-\lambda(1-t_m)^\alpha)}.$$

Now, $x(t) = (P_m x)(t) + (Q_m^\alpha x)(t)$.

(Sufficiency) Let $x(t)$ satisfy (3.6). Noting Lemma 2.7(5) and Lemma 3.5, $({}^cD_{t_k^+}^\alpha x)(t)$ exists and ${}^cD_{t_k^+}^\alpha x(t) + \lambda x(t) = f(t, x(t))$ for $t \in J_k$ $(k = 0,1,\dots,m)$. Moreover, for $k = 1,2,\dots,m-1$,

$$\begin{aligned} x\big(t_k^+\big) - x\big(t_k^-\big) &= (P_k x)(t_k) + \big(Q_k^\alpha x\big)(t_k) - (P_{k-1}x)(t_k) - \big(Q_{k-1}^\alpha x\big)(t_k) \\ &= (P_{k-1}x)(t_k) + \big(Q_{k-1}^\alpha x\big)(t_k) + I_k - (P_{k-1}x)(t_k) - \big(Q_{k-1}^\alpha x\big)(t_k) \\ &= I_k. \end{aligned}$$

The boundary conditions of (1.3) are clearly satisfied, that is, $x(t)$ satisfies (1.1)-(1.3). □

4 Existence result

In this section, we deal with the existence of solution for the problem (1.1)-(1.3). To this end, we consider the following assumption.

(H2) There exists a function $\psi \in L^{\frac{1}{q_2}}(J, \mathbb{R}^+)$ $(q_2 \in (0,\alpha))$ such that

$$\big|f(t,x) - f(t,y)\big| \le \psi(t)|x-y|.$$

For convenience, we introduce the following notation:

$$\begin{aligned} &c_\alpha = \frac{1}{\Gamma(\alpha)}\left(\frac{1-q_1}{\alpha-q_1}\right)^{1-q_1} \|\mu\|_{L^{\frac{1}{q_1}}}, \qquad M_\alpha = \frac{1}{\Gamma(\alpha)}\left(\frac{1-q_2}{\alpha-q_2}\right)^{1-q_2} \|\psi\|_{L^{\frac{1}{q_2}}}, \\ &T_0 = \frac{c_{\alpha+\gamma}}{1+\eta^\gamma E_{\alpha,\gamma+1}(-\lambda\eta^\alpha)}, \\ &T_i = T_{i-1} + c_\alpha + |I_i|, \quad i = 1,2,\dots,m-1, \\ &T_m = \frac{c_\alpha + \mathcal{C}\zeta_{\alpha-\delta}}{|E_\alpha(-\lambda(1-t_m)^\alpha) - \lambda(1-t_m)^{\alpha-\delta} E_{\alpha,\alpha-\delta+1}(-\lambda(1-t_m)^\alpha)|}. \end{aligned}$$

Clearly, $T_0 < T_1 < \cdots < T_{m-1}$.

Theorem 4.1 *Assume that* (H1) *and* (H2) *are satisfied, then the problem* (1.1)-(1.3) *has at least a solution* $x \in X$ *if* $M_\alpha < 1$.

Proof Define an operator $\mathcal{F}: X \to X$ by

$$(\mathcal{F}x)(t) = \begin{cases} (P_0x)(t) + (Q_0^\alpha x)(t), & t \in J_0, \\ (P_1x)(t) + (Q_1^\alpha x)(t), & t \in J_1, \\ \cdots \\ (P_{m-1}x)(t) + (Q_{m-1}^\alpha x)(t), & t \in J_{m-1}, \\ (P_mx)(t) + (Q_m^\alpha x)(t), & t \in J_m. \end{cases} \tag{4.1}$$

From Lemma 2.9(ii) and Lemma 3.3, we see that $\mathcal{F}: X \to X$ is clearly well defined.
Similar to (3.2) and (3.3), combining with Lemma 2.9(i) and (2.4), one can get

$$\begin{aligned} &\bigl|\bigl(Q_0^{\alpha+\gamma}x\bigr)(t)\bigr| \le c_{\alpha+\gamma}, \qquad \bigl|\bigl(Q_m^{\alpha-\delta}x\bigr)(t)\bigr| \le \mathcal{C}\zeta_{\alpha-\delta}, \\ &\bigl|\bigl(Q_k^\alpha x\bigr)(t)\bigr| \le c_\alpha, \quad k = 0,1,\ldots,m. \end{aligned} \tag{4.2}$$

Setting $B_r = \{x \in X : \|x\|_1 \le r\}$, where $r \ge \max\{T_m, T_{m-1}\} + c_\alpha$, we shall prove $(P_ix)(t) + (Q_i^\alpha y)(t) \in B_r$ for any $x, y \in B_r$ and $t \in J_i$ $(i = 0,1,\ldots,m)$.
By Lemma 2.9(i) and (4.2), we have

$$\bigl|(P_0x)(t) + \bigl(Q_0^\alpha y\bigr)(t)\bigr| \le \frac{c_{\alpha+\gamma}}{1 + \eta^\gamma E_{\alpha,\gamma+1}(-\lambda\eta^\alpha)} + c_\alpha = T_0 + c_\alpha \le r.$$

For $t \in J_1$, one has

$$\begin{aligned} \bigl|(P_1x)(t) + \bigl(Q_1^\alpha y\bigr)(t)\bigr| &\le \bigl|(P_0x)(t_1) + \bigl(Q_0^\alpha x\bigr)(t_1) + I_1\bigr| + \bigl|\bigl(Q_1^\alpha y\bigr)(t)\bigr| \\ &\le T_0 + c_\alpha + |I_1| + c_\alpha = T_1 + c_\alpha \le r. \end{aligned}$$

Repeating the above process, for $t \in J_i$ $(i = 2,\ldots,m-1)$, we find

$$\bigl|(P_ix)(t) + \bigl(Q_i^\alpha y\bigr)(t)\bigr| \le T_i + c_\alpha \le r.$$

For $t \in J_m$, one sees

$$\bigl|(P_mx)(t) + \bigl(Q_m^\alpha y\bigr)(t)\bigr| \le T_m + c_\alpha \le r.$$

Now, we can see that $(P_ix)(t) + (Q_i^\alpha y)(t) \in B_r$ for any $t \in J_i$ $(i = 0,1,\ldots,m)$ and $x, y \in B_r$.
Similar to (3.1), for $t \in J_i$, $i = 0,1,\ldots,m$, one gets

$$\begin{aligned} \bigl|\bigl(Q_i^\alpha x\bigr)(t) - \bigl(Q_i^\alpha y\bigr)(t)\bigr| &\le \int_{t_i}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl(-\lambda(t-s)^\alpha\bigr)\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_i}^t (t-s)^{\alpha-1}\psi(s)\,ds\|x-y\|_1 \le M_\alpha\|x-y\|_1. \end{aligned}$$

This implies that Q_i^α $(i = 0,1,\ldots,m)$ is a contraction mapping.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in X, then there exists $\varepsilon > 0$ such that $\|x_n - x\|_1 \le \varepsilon$ for n sufficiently large. By (H2), we obtain

$$\bigl|f\bigl(t,x_n(t)\bigr) - f\bigl(t,x(t)\bigr)\bigr| \le \psi(t)\varepsilon.$$

Moreover, f satisfies (H1), for almost every $t \in J$, we get $f(t, x_n(t)) \to f(t, x(t))$ as $n \to \infty$. It follows from the Lebesgue dominated convergence theorem that

$$\left\|(P_i x_n) - (P_i x)\right\|_1 \to 0, \quad \text{as } n \to \infty.$$

Now we can see that P_i $(i = 0, 1, \ldots, m)$ is continuous.

Moreover, by Lemma 2.9(ii) and (4.2), $\{P_i x : x \in B_r\}$ is an equicontinuous and uniformly bounded set. Therefore, P_i is a completely continuous operator on $B_r|_{J_i}$ $(i = 0, 1, \ldots, m)$. Now, it follows from Theorem 2.11 that problem (1.1)-(1.3) has at least a solution $x \in B_r$. □

5 Application

In this section, we give an example to illustrate the usefulness of our main result.

Example 5.1 Consider the following impulsive boundary problem of fractional order:

$$\begin{cases} {}^cD_*^{\frac{1}{2}} x(t) + 5x(t) = \frac{1}{6\sqrt[14]{t}} \sin(3 + |x(t)|), & \text{a.e. } t \in (0,1] \setminus \{\frac{1}{4}\}, \\ \Delta x(\frac{1}{4}) = 2, \\ x(0) + I_{0^+}^{\frac{1}{3}} x(\frac{1}{10}) = 0, & x(1) + {}^cD_{\frac{1}{3}^+}^{\frac{1}{4}} x(1) = 0. \end{cases} \tag{5.1}$$

Corresponding to (1.1)-(1.3), we have $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{3}$, $\delta = \frac{1}{4}$, $\lambda = 5$, $m = 2$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{3}$, $\eta = \frac{1}{10}$, $f(t, x(t)) = \frac{1}{6\sqrt[14]{t}} \sin(3 + |x(t)|)$, $I_1 = 2$.

It is easy to see that $|f(t, x(t))| \leq \nu(t)$ and $|f(t, x(t)) - f(t, y(t))| \leq \psi(t)|x(t) - y(t)|$, where $\nu(t) = \psi(t) = \frac{1}{6\sqrt[14]{t}} \in L^{\frac{1}{q}}([0,1])(q = \frac{1}{7})$ and $\|\psi\|_{L^7} = \frac{2^{\frac{1}{7}}}{6}$. By direct computation, we find that

$$M_\alpha = \frac{1}{\Gamma(\alpha)} \left(\frac{1-q}{\alpha - q}\right)^{1-q} \|\psi\|_{L^{\frac{1}{q}}} = \frac{1}{3\sqrt{\pi}} \left(\frac{6}{5}\right)^{\frac{6}{7}} \approx 0.22 < 1.$$

Now, due to the fact that all the assumptions of Theorem 4.1 hold, problem (5.1) has at least a solution.

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