A parallel hybrid accelerated extragradient

fixed point, and split null point problems

algorithm for pseudomonotone equilibrium,

# RESEARCH

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# Abstract

This paper provides iterative construction of a common solution associated with the classes of equilibrium problems (EP) and split convex feasibility problems. In particular, we are interested in the EP defined with respect to the pseudomonotone bifunction, the fixed point problem (FPP) for a finite family of k-demicontractive operators, and the split null point problem. From the numerical standpoint, combining various classical iterative algorithms to study two or more abstract problems is a fascinating field of research. We, therefore, propose an iterative algorithm that combines the parallel hybrid extragradient algorithm with the inertial extrapolation technique. The analysis of the proposed algorithm comprises theoretical results concerning strong convergence under a suitable set of constraints and numerical results.

MSC: 47H05; 47H10; 47J25; 49M30; 54H25

**Keywords:** Parallel hybrid algorithm; Inertial extrapolation technique; Pseudomonotone equilibrium problem; Fixed point problem; Null point problem

# **1** Introduction

The class of convex feasibility problems (CFP) has been widely studied in the current literature as it encompasses a variety of problems arising in mathematical and physical sciences. Numerous iterative algorithms have been studied to obtain an approximate solution for the CFP in Hilbert spaces. However, the class of projection algorithms is prominent among various iterative algorithms to solve the CFP. It is remarked that the class of CFP is closely related to the theory of convex optimization and hence monotone operator theory. As a consequence, CFP found valuable applications in the field of partial differential equations, image recovery problem, approximation theory, signal and image processing through projection algorithms, control problems, evolution equations and inclusions, see for instance [6, 15, 16] and the references cited therein.

The class of CFP has been generalized in several ways. One of the elegant modifications and generalizations of the CFP is the split convex feasibility problems (SCFP) proposed by Censor and Elfving [12]. The mathematical formulation of the SCFP opens up an inter-

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esting framework to model the medical image reconstruction problem and the intensitymodulated radiation therapy [8, 11]. As a consequence, the SCFP has been studied extensively in the current literature with possible real-world applications, see for example [9, 11, 13, 14, 19] and the references cited therein. Recall that a SCFP deals with a model aiming to find a point

$$\bar{x} \in C \subseteq \mathcal{H}_1 \tag{1}$$

such that

$$\bar{y} = \hbar \bar{x} \in Q \subseteq \mathcal{H}_2,\tag{2}$$

where  $\hbar: \mathcal{H}_1 \to \mathcal{H}_2$  is a bounded linear operator between two real Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ .

Since the introduction of SCFP, various important instances of SCFP have been introduced and analyzed such as the split variational inequality problem [13], the split common null point problem (SCNPP) [9], the split common FPP [14], and the split equilibrium problem [7]. We are interested in studying the SCNPP, one of the important instances of SCFP, defined as follows:

Given two multivalued operators  $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$  and  $A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ , the SCNPP problem deals with a model aiming to find a point

$$\bar{x} \in \mathcal{H}_1$$
 such that  $0 \in A_1(\bar{x})$  and  $0 \in A_2(\bar{h}\bar{x})$ . (3)

In 2012, Byrne et al. [9] suggested the following iterative schemes to solve the SCNPP (3) associated with two maximal monotone operators  $A_1$  and  $A_2$ :

$$x_{k+1} = J_m^{A_1} \left( x_k + \delta \hbar \left( J_m^{A_2} - \operatorname{Id} \right) \hbar x_k \right), \quad k \in \mathbb{N},$$
(4)

and

(

$$\begin{cases} x_0, \quad \nu \in \mathcal{H}_1; \\ x_{k+1} = \beta_k \nu + (1 - \beta_k) J_m^{A_1} (x_k + \delta \hbar^* (J_m^{A_2} - \mathrm{Id}) \hbar x_k), \quad k \in \mathbb{N}, \end{cases}$$
(5)

where  $\hbar^*$  denotes the adjoint operator of  $\hbar$ , Id denotes the identity operator and  $J_m^{A_1}, J_m^{A_2}$  denote the corresponding resolvents of  $A_1, A_2$ , respectively. The set of solutions of the SCNPP (3) is denoted by  $\Omega := \{\bar{x} \in A_1^{-1}(0) : \hbar \bar{x} \in A_2^{-1}(0)\}$ . It is remarked that the scheme (4) exhibits weak convergence, while the scheme (5) exhibits strong convergence under suitable sets of constraints.

In 1994, Blum and Oettli [7] proposed, in a mathematical formulation, an EP with respect to a (monotone) bifunction g defined on a nonempty subset C of a real Hilbert space  $\mathcal{H}_1$  that aims to find a point  $\bar{x} \in C$  such that

$$g(\bar{x},\bar{y}) \ge 0 \quad \text{for all } \bar{y} \in C.$$
 (6)

The set of equilibrium points or solutions of the problem (6) is denoted by EP(g).

In 2006, Tada and Takahashi [29] suggested a hybrid algorithm for the analysis of monotone EP and FPP in Hilbert spaces. Nevertheless, the iterative algorithm proposed in [29] fails for the case of pseudomonotone EP. In order to address this issue, Anh [2] suggested a hybrid extragradient method, based on the seminal work of Korpelevich [23], in Hilbert spaces. Inspired by the work of Anh [2], Hieu et al. [20] suggested a parallel hybrid extragradient framework to address pseudomonotone EP together with the FPP. In 2015, Takahashi et al. [30] constructed a common solution of the zero point problem and FPP. Therefore, it is natural to study the pseudomonotone EP and the SCNPP with the FPP associated with a more general class of demicontractive operators.

It is remarked that the computational performance of an iterative algorithm can be enhanced by employing different techniques. The parallel architecture of an iterative algorithm reduces the computational cost, whereas the inertial extrapolation technique [26] provides fast convergence characteristics of the algorithm. The latter technique has successfully been combined with different classical iterative algorithms, see for example [1, 3–5, 10, 18, 21, 22, 25, 32] and the references cited therein. We, therefore, study the convergence analysis of a variant of parallel hybrid extragradient iterative algorithm embedded with the inertial extrapolation technique in Hilbert spaces.

The rest of the paper is organized as follows: Sect. 2 contains some relevant preliminary concepts and results for (split) monotone operator theory, EP theory, and fixed point theory. Section 3 comprises strong convergence results, whereas Sect. 4 provides numerical results concerning the viability of the proposed algorithm with respect to various real world applications.

### 2 Preliminaries

We first define some necessary notions from fixed point theory. Let  $T : C \to C$  be an operator defined on a nonempty subset *C* of a real Hilbert space  $\mathcal{H}_1$ , then *T* is known as nonexpansive if  $||Tx - Ty|| \le ||x - y||$  for all  $x, y \in C$ . Further, *T* is known as firmly nonexpansive if

$$||Tx - Ty||^2 \le ||x - y||^2 - ||(Id - T)x - (Id - T)y||^2$$
 for all  $x, y \in C$ .

Moreover, the operator *T* is defined as  $\Bbbk$ -demicontractive if  $Fix(T) \neq \emptyset$ , and there exists  $\Bbbk \in [0, 1)$  such that

$$||Tx - y||^2 \le ||x - y||^2 + ||x - Tx||^2, \quad \forall x \in C, y \in Fix(T),$$

where Fix(T) = { $x \in C : x - Tx = 0$ } denotes the set of all fixed points of the operator T. Note that the operator Id – T is said to be demiclosed at the origin if for any sequence  $(x_k)$  in a nonempty closed and convex subset C of  $\mathcal{H}_1$  converges weakly to some x and if  $((Id - T)x_k)$  converges strongly to 0, then (Id - T)(x) = 0. It is remarked that, for each  $x \in \mathcal{H}_1$ , there exists unique  $P_C x \in C$  satisfying  $||x - P_C x|| \le ||x - z||$  for all  $z \in C$ . Such an operator  $P_C : \mathcal{H}_1 \to C$  is coined as metric projection and satisfies  $\langle x - P_C x, P_C x - y \rangle \ge 0$ , for all  $x \in \mathcal{H}_1$  and  $y \in C$ .

We now state a brief introductory material covering monotone operator theory from the celebrated monograph of Bauschke and Combettes [6].

For a set-valued operator  $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ , the following sets dom $(A_1) = \{x \in \mathcal{H}_1 | A_1 x \neq \emptyset\}$ , ran $(A_1) = \{u \in \mathcal{H}_1 | (\exists x \in \mathcal{H}_1) u \in A_1 x\}$ , gra $(A_1) = \{(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1 | u \in A_1 x\}$ , and zer $(A_1) = \{x \in \mathcal{H}_1 | u \in A_1 x\}$ , and zer $(A_1) = \{x \in \mathcal{H}_1 | u \in A_1 x\}$ , and zer $(A_1) = \{x \in \mathcal{H}_1 | u \in A_1 x\}$ , and zer $(A_1) = \{x \in \mathcal{H}_1 | u \in A_1 x\}$ , respectively.  $\{x \in \mathcal{H}_1 | 0 \in A_1 x\}$  define and denote the domain, range, graph, and zeros of  $A_1$ , respectively. The inverse operator  $A_1^{-1}$  of  $A_1$  can be defined as

$$x \in A_1^{-1}(y)$$
 if and only if  $y \in A_1(x)$ .

The set-valued operator  $A_1$  is said to be monotone if  $\langle x - y, u - v \rangle \ge 0$  for all  $(x, u), (y, v) \in$ gra( $A_1$ ). A monotone operator  $A_1$  is coined as maximal monotone operator if there is no proper monotone extension of  $A_1$ , equivalently if ran(Id +  $mA_1$ ) =  $\mathcal{H}_1$  for all m > 0. An important notion associated with the monotone operator  $A_1$  is the well-defined resolvent operator  $J_m^{A_1} = (\text{Id} + mA_1)^{-1}$ . Such an operator is single-valued and satisfies nonexpansiveness as well as Fix( $J_m^{A_1}$ ) =  $A_1^{-1}(0)$  for all m > 0.

The rest of this section is organized with celebrated results required in the sequel.

**Assumption 2.1** Let  $g : C \times C \to \mathbb{R} \cup \{+\infty\}$  be a bifunction satisfying the following assumptions:

(A1): *g* is pseudomonotone, i.e.,  $g(x, y) \le 0 \Rightarrow g(x, y) \ge 0$  for all  $x, y \in C$ ;

(A2): g is Lipschitz-type continuous, i.e., there exist two nonnegative constants  $d_1, d_2$  such that

$$g(x, y) + g(y, z) \ge g(x, z) - d_1 ||x - y||^2 - d_2 ||y - z||^2$$
 for all  $x, y, z \in C$ ;

(A3): *g* is weakly continuous on  $C \times C$  implies that, if  $x, y \in C$  and  $(x_k), (y_k)$  are two sequences in *C* converging weakly to *x* and *y*, respectively, then  $g(x_k, y_k)$  converges to g(x, y);

(A4): For each fixed  $x \in C$ , g(x, .) is convex and subdifferentiable on C.

It is remarked that the monotonicity of g, i.e.,  $g(x, y) + g(y, x) \le 0$ , for all  $x, y \in C$  implies the pseudomonotonicity but the converse is not true in general. The following lemmas are helpful in proving the strong convergence results in the next section.

**Lemma 2.2** Let  $x, y \in \mathcal{H}_1$  and  $\beta \in \mathbb{R}$ , then

- 1.  $||x + y||^2 \le ||x||^2 + 2\langle y, x + y \rangle;$
- 2.  $||x y||^2 \le ||x||^2 ||y||^2 2\langle x y, y \rangle;$
- 3.  $\|\beta x + (1-\beta)y\|^2 = \beta \|x\|^2 + (1-\beta)\|y\|^2 \beta(1-\beta)\|x-y\|^2$ .

**Lemma 2.3** ([24]) Let C be a nonempty closed and convex subset of a real Hilbert space  $\mathcal{H}_1$ . For every  $x, y, z \in \mathcal{H}_1$  and  $\gamma \in \mathbb{R}$ , the set

$$D = \{ v \in C : ||y - v||^2 \le ||x - v||^2 + \langle z, v \rangle + \gamma \}$$

is closed and convex.

**Lemma 2.4** ([31]) Let C be a nonempty closed and convex subset of a real Hilbert space  $\mathcal{H}_1$ , and let  $h: C \to \mathbb{R}$  be a convex and subdifferentiable function on C. Then  $\bar{x}$  is the solution of convex problem min{ $h(x): x \in C$ } if and only if  $0 \in \partial h(\bar{x}) + N_C(\bar{x})$ , where  $\partial h(\cdot)$  denotes the subdifferential of h and  $N_C(\bar{x})$  is the normal cone of C at  $\bar{x}$ . Algorithm 1 Accelerated projection based parallel hybrid extragradient algorithm (Alg. 1)

**Initialization:** Choose arbitrarily  $x_0, x_1 \in C$ , set  $k \ge 1$  and nonincreasing sequence  $\alpha_k, \beta_k \subset (0, 1), 0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2}), \Theta_k \subset [0, 1), m_k, n_k \in (0, \infty)$ , and  $\delta \in (0, \frac{2}{\|\hbar\|^2})$  such that  $\|\hbar\|^2 = L$  is the spectral radius of  $\hbar^*\hbar$ .

**Iterative Steps:** Given  $x_k \in \mathcal{H}_1$ , calculate  $b_k$ ,  $\bar{v}_k$ ,  $\bar{w}_k$ , and  $y_k$  as follows:

Step 1. Compute

$$\begin{cases} b_k = x_k + \Theta_k (x_k - x_{k-1}); \\ u_k^i = \arg\min\{\gamma g_i(b_k, y) + \frac{1}{2} \| b_k - y \|^2 : u \in C\}, & i = 1, 2, \dots, M; \\ v_k^i = \arg\min\{\gamma g_i(u_k^i, y) + \frac{1}{2} \| b_k - y \|^2 : u \in C\}, & i = 1, 2, \dots, M; \\ i_k = \arg\max\{\|v_k^i - x_k\| : i = 1, 2, \dots, M\}, & \bar{v}_k = v_k^{i_k}; \\ w_k^j = (1 - \alpha_k)\bar{v}_k + \alpha_k S_j \bar{v}_k; \\ j_k = \arg\max\{\|w_k^j - x_k\| : j = 1, 2, \dots, N\}, & \bar{w}_k = w_k^{j_k}; \\ y_k = (1 - \beta_k)\bar{w}_k + \beta_k (J_{m_k}^{A_1}(\bar{w}_k + \delta\hbar^*(J_{n_k}^{A_2} - \mathrm{Id})\hbar\bar{w}_k)); \end{cases}$$

If  $y_k = \bar{w}_k = \bar{v}_k = b_k = x_k$  then stop and  $x_k$  is the solution of problem  $\Gamma$ . Otherwise, **Step 2.** Compute

$$\begin{split} C_{k+1} &= \big\{ z \in C_k : \|y_k - z\|^2 \le \|x_k - z\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \langle x_k - z, x_k - x_{k-1} \rangle \big\},\\ x_{k+1} &= P_{C_{k+1}} x_1, \quad \forall k \ge 1, \end{split}$$

Set *k* =: *k* + 1 and go back to **Step 1**.

#### 3 Algorithm and convergence analysis

We enlist standard necessary hypotheses for the main result of this section. Note that, for a finite family of pseudomonotone bifunctions  $g_i$ , we can compute the same Lipschitz coefficients ( $d_1$ ,  $d_2$ ) by employing Assumption 2.1(A2) as follows:

$$g_i(x,z) - g_i(x,y) - g_i(y,z) \le d_{1,i} ||x - y||^2 + d_{2,i} ||y - z||^2 \le d_1 ||x - y||^2 + d_2 ||y - z||^2$$

where  $d_1 = \max_{1 \le i \le M} \{d_{1,i}\}$  and  $d_2 = \max_{1 \le i \le M} \{d_{2,i}\}$ . Therefore,  $g_i(x, y) + g_i(y, z) \ge g_i(x, z) - d_1 ||x - y||^2 - d_2 ||y - z||^2$ .

Let  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  be two real Hilbert spaces, and let  $C \subseteq \mathcal{H}_1$  be a nonempty, closed, and convex subset. Then

- (H1) Let  $A_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}, A_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$  be two maximal monotone operators, and for m, n > 0, let  $J_m^{A_1}, J_n^{A_2}$  be the resolvents of  $A_1$  and  $A_2$ , respectively;
- (H2) Let  $\hbar: \mathcal{H}_1 \to \mathcal{H}_2$  be a bounded linear operator such that  $\hbar^*$  is the adjoint operator of  $\hbar$ ;
- (H3) Let  $g_i : C \times C \to \mathbb{R} \cup \{+\infty\}$  be a finite family of bifunctions satisfying Assumption 2.1;
- (H4) Let  $S_j : \mathcal{H}_1 \to \mathcal{H}_1$  be a finite family of  $\Bbbk$ -demicontractive operators;
- (H5) Assume that  $\Gamma := \Omega \cap (\bigcap_{i=1}^{M} EP(g_i)) \cap (\bigcap_{i=1}^{N} Fix(S_i)) \neq \emptyset$ .

**Theorem 3.1** If  $\Gamma \neq \emptyset$ , then the sequence  $(x_k)$  generated by Algorithm 1 converges strongly to an element in  $\Gamma$ , provided the following conditions hold:

- (C1)  $\sum_{k=1}^{\infty} \Theta_k \|x_k x_{k-1}\| < \infty;$
- (C2)  $0 < a^* < \liminf_{k \to \infty} \alpha_k \le \limsup_{k \to \infty} \alpha_k \le b^* < 1 \text{ and } \alpha_k \in (0, 1 \mathbb{k});$
- (C3)  $\liminf_{k\to\infty}\beta_k > 0;$
- (C4)  $\liminf_{k\to\infty} m_k > 0$ ,  $\liminf_{k\to\infty} n_k > 0$ .

*Remark* 3.2 We remark here that the condition (C1) is easily implementable in a numerical computation since the values of  $||x_k - x_{k-1}||$  are known before choosing  $\Theta_k$ . The parameter  $\Theta_k$  can be taken as  $0 \le \Theta_k \le \widehat{\Theta_k}$ ,

$$\widehat{\Theta_k} = \begin{cases} \min\{\frac{v_k}{\|x_k - x_{k-1}\|}, \Theta\} & \text{if } x_k \neq x_{k-1}; \\ \Theta & \text{otherwise,} \end{cases}$$

where  $\{v_k\}$  is a positive sequence such that  $\sum_{k=1}^{\infty} v_k < \infty$  and  $\Theta \in [0, 1)$ .

We use the following result for the analysis of Algorithm 1.

**Lemma 3.3** ([27]) Suppose that  $\bar{x} \in EP(g_i)$ , and  $x_k$ ,  $b_k$ ,  $u_k^i$ ,  $w_k^i$ ,  $i \in \{1, 2, ..., M\}$  are defined in Step 1 of Algorithm 1. Then we have

$$\|v_k^i - \bar{x}\|^2 \le \|b_k - \bar{x}\|^2 - (1 - 2\gamma d_1) \|u_k^i - b_k\|^2 - (1 - 2\gamma d_2) \|u_k^i - v_k^i\|^2.$$

*Proof of Theorem* 3.1 The proof is divided into the following steps.

*Step* 1. We show that the sequence  $(x_k)$  defined in Algorithm 1 is well defined.

We know that  $\Gamma$  is closed and convex. Moreover, from Lemma 2.3 we have that  $C_{k+1}$  is closed and convex for each  $k \ge 1$ . Hence the projection  $P_{C_{k+1}}x_1$  is well defined. For any  $\bar{x} \in \Gamma$ , observe that

$$\|b_{k} - \bar{x}\|^{2} = \|(x_{k} - \bar{x}) + \Theta_{k}(x_{k} - x_{k-1})\|^{2}$$
  
$$\leq \|x_{k} - \bar{x}\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} + 2\Theta_{k}\langle x_{k} - \bar{x}, x_{k} - x_{k-1}\rangle.$$
(7)

Further

$$\begin{split} |\bar{w}_{k} - \bar{x}||^{2} &= \left\| (1 - \alpha_{k})\bar{v}_{k} + \alpha_{k}S_{j}\bar{v}_{k} - \bar{x} \right\|^{2} \\ &\leq (1 - \alpha_{k})\|\bar{v}_{k} - \bar{x}\|^{2} + \alpha_{k}\|S_{j}\bar{v}_{k} - \bar{x}\|^{2} - \alpha_{k}(1 - \alpha_{k})\|(\mathrm{Id} - S_{j})\bar{v}_{k}\|^{2} \\ &\leq (1 - \alpha_{k})\|\bar{v}_{k} - \bar{x}\|^{2} + \alpha_{k}\|\bar{v}_{k} - \bar{x}\|^{2} + \alpha_{k}\mathbb{K}\|S_{j}\bar{v}_{k} - \bar{v}_{k}\|^{2} \\ &- \alpha_{k}(1 - \alpha_{k})\|(\mathrm{Id} - S_{j})\bar{v}_{k}\|^{2} \\ &\leq \|\bar{v}_{k} - \bar{x}\|^{2} - \alpha_{k}(1 - \mathbb{K} - \alpha_{k})\|(\mathrm{Id} - S_{j})\bar{v}_{k}\|^{2} \\ &\leq \|x_{k} - \bar{x}\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} + 2\Theta_{k}\langle x_{k} - \bar{x}, x_{k} - x_{k-1}\rangle. \end{split}$$
(8)

Furthermore,

$$\|y_k - \bar{x}\|^2 = \|(1 - \beta_k)(\bar{w}_k - \bar{x}) + \beta_k (J_{m_k}^{A_1}(\bar{w}_k + \delta \hbar^* (J_{n_k}^{A_2} - \mathrm{Id})\hbar \bar{w}_k) - \bar{x})\|^2$$

$$\leq (1 - \beta_k) \|\bar{w}_k - \bar{x}\|^2 + \beta_k \|J_{m_k}^{A_1}(\bar{w}_k + \delta\hbar^* (J_{n_k}^{A_2} - \mathrm{Id})\hbar\bar{w}_k) - \bar{x}\|^2.$$
(9)

Since  $J_{m_k}^{A_1}$  is nonexpansive, therefore the expression  $\|J_{m_k}^{A_1}(\bar{w}_k + \delta \hbar^* (J_{n_k}^{A_2} - \text{Id})\hbar \bar{w}_k) - \bar{x}\|^2$  simplifies as follows:

$$\begin{split} \left\| J_{m_{k}}^{A_{1}} \left( \bar{w}_{k} + \delta \hbar^{*} \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} \right) - J_{m_{k}}^{A_{1}} \bar{x} \right\|^{2} \\ &\leq \left\| \bar{w}_{k} + \delta \hbar^{*} \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} - \bar{x} \right\|^{2} \\ &\leq \left\| \bar{w}_{k} - \bar{x} \right\|^{2} + \delta^{2} \left\| \hbar^{*} \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} \right\|^{2} + 2\delta \langle \bar{w}_{k} - \bar{x}, \hbar^{*} \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} \rangle \\ &\leq \left\| \bar{w}_{k} - \bar{x} \right\|^{2} + \delta^{2} \left\| \hbar \right\|^{2} \left\| \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} \right\|^{2} + 2\delta \langle \hbar \bar{w}_{k} - \hbar \bar{x}, \left( J_{n_{k}}^{A_{2}} - \mathrm{Id} \right) \hbar \bar{w}_{k} \rangle \rangle. \end{split}$$
(10)

Using  $J_{n_k}^{A_2}$  as firmly nonexpansive, we simplify the expression  $\lambda_k = 2\delta \langle \hbar \bar{w}_k - \hbar \bar{x}, (J_{n_k}^{A_2} - \text{Id})\hbar \bar{w}_k \rangle$  as follows:

$$\lambda_{k} = 2\delta \langle \hbar \bar{w}_{k} - \hbar \bar{x} + (f_{n_{k}}^{A_{2}}(\hbar \bar{w}_{k}) - \hbar \bar{w}_{k}) - (f_{n_{k}}^{A_{2}}(\hbar \bar{w}_{k}) - \hbar \bar{w}_{k}), f_{n_{k}}^{A_{2}}(\hbar \bar{w}_{k}) - \hbar \bar{w}_{k} \rangle$$

$$= 2\delta (\langle f_{n_{k}}^{A_{2}}(\hbar \bar{w}_{k}) - \hbar \bar{x}, f_{n_{k}}^{A_{2}}(\hbar \bar{w}_{k}) - \hbar \bar{w}_{k} \rangle - \| (f_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2})$$

$$\leq -2\delta \| (f_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2}. \tag{11}$$

Utilizing (10), (11), and Lemma 3.3, we then obtain from (9) that

$$\begin{aligned} \|y_{k} - \bar{x}\|^{2} &\leq (1 - \beta_{k}) \|\bar{w}_{k} - \bar{x}\|^{2} + \beta_{k} (\|\bar{w}_{k} - \bar{x}\|^{2} + \delta^{2} \|\hbar\|^{2} \|(f_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k})\|^{2} \\ &- 2\delta \|(f_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}\|^{2}), \\ &\leq (1 - \beta_{k}) \|\bar{w}_{k} - \bar{x}\|^{2} + \beta_{k} (\|\bar{w}_{k} - \bar{x}\|^{2} - \delta(2 - \delta \|\hbar\|^{2}) \|(f_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}\|^{2}) \\ &\leq \|\bar{w}_{k} - \bar{x}\|^{2} \\ &\leq \|x_{k} - \bar{x}\|^{2} + \Theta_{k}^{2} \|x_{k} - x_{k-1}\|^{2} + 2\Theta_{k} \langle x_{k} - \bar{x}, x_{k} - x_{k-1} \rangle. \end{aligned}$$
(12)

It follows from (12) that

$$\|y_k - \bar{x}\| \le \|x_k - \bar{x}\|^2 + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \|x_k - \bar{x}\| \|x_k - x_{k-1}\|.$$
(13)

The above estimate (13) infers that  $\Gamma \subset C_{k+1}$ . Hence, we conclude that Algorithm 1 is well defined.

*Step* 2. We show that the limit  $\lim_{k\to\infty} ||x_k - x_1||$  exists.

Note that, for  $x_{k+1} = P_{C_{k+1}}x_1$ , we have  $||x_{k+1} - x_1|| \le ||x^* - x_1||$  for all  $x^* \in C_{k+1}$ . In particular  $||x_{k+1} - x_1|| \le ||\bar{x} - x_1||$  for all  $\bar{x} \in \Gamma \subset C_{k+1}$  This proves that the sequence  $(||x_k - x_1||)$  is bounded. On the other hand, from  $x_k = P_{C_k}x_1$  and  $x_{k+1} = P_{C_{k+1}}x_1 \in C_{k+1}$ , we have

$$||x_k - x_1|| \le ||x_{k+1} - x_1||.$$

This implies that  $(||x_k - x_1||)$  is nondecreasing, and hence

$$\lim_{k \to \infty} \|x_k - x_1\| \quad \text{exists.} \tag{14}$$

Step 3. We show that  $\bar{x_*} \in \Gamma$ .

First, observe that

$$\begin{aligned} \|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_1 + x_1 - x_k\|^2 \\ &= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_1 \rangle \\ &= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k + x_k - x_1 \rangle \\ &= \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k \rangle \\ &\leq \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2. \end{aligned}$$

Taking  $\limsup$  on both sides of the above estimate and utilizing (14), we have

 $\limsup_{k \to \infty} \|x_{k+1} - x_k\|^2 = 0$ . That is,

$$\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0.$$
(15)

By the definition of  $(b_k)$  and (C1), we have

$$\lim_{k \to \infty} \|b_k - x_k\| = \lim_{k \to \infty} \Theta_k \|x_k - x_{k-1}\| = 0.$$
(16)

Consider the following triangular inequality:

$$||b_k - x_{k+1}|| \le ||b_k - x_k|| + ||x_k - x_{k+1}||.$$

From (15) and (16), we have

$$\lim_{k \to \infty} \|b_k - x_{k+1}\| = 0.$$
(17)

Since  $x_{k+1} \in C_{k+1}$ , therefore, we have

$$\|y_k - x_{k+1}\| \le \|x_k - x_{k+1}\| + 2\Theta_k \|x_k - x_{k-1}\| + 2\langle x_k - x_{k+1}, x_k - x_{k-1}\rangle.$$

Utilizing (15) and (C1), the above estimate implies that

$$\lim_{k \to \infty} \|y_k - x_{k+1}\| = 0.$$
(18)

From (15), (18), and the following triangular inequality

 $||y_k - x_k|| \le ||y_k - x_{k+1}|| + ||x_{k+1} - x_k||,$ 

we get

$$\lim_{k \to \infty} \|y_k - x_k\| = 0.$$
<sup>(19)</sup>

Consider the following re-arranged variant of the estimate (12) by applying Lemma 3.3:

$$(1-2\gamma d_1) \|u_k^i - b_k\|^2 - (1-2\gamma d_2) \|u_k^i - v_k^i\|^2$$

$$\leq (\|x_k - \bar{x}\| + \|y_k - \bar{x}\|)\|x_k - y_k\| + \Theta_k^2 \|x_k - x_{k-1}\|^2 + 2\Theta_k \|x_k - \bar{x}\|\|x_k - x_{k-1}\|.$$

Letting  $k \to \infty$ , using (C1) and (19), we have

$$(1 - 2\gamma d_1) \lim_{k \to \infty} \left\| u_k^i - b_k \right\|^2 - (1 - 2\gamma d_2) \lim_{k \to \infty} \left\| u_k^i - v_k^i \right\|^2 = 0.$$
(20)

This implies that

$$\lim_{k \to \infty} \|u_k^i - b_k\|^2 = \lim_{k \to \infty} \|u_k^i - v_k^i\|^2 = 0.$$
 (21)

Again, consider the following re-arranged variant of the estimate (13):

$$a^{*}(1-b^{*}) \| (\mathrm{Id} - S_{j})\bar{\nu}_{k} \|^{2} \leq (\|x_{k} - \bar{x}\| + \|y_{k} - \bar{x}\|) \|x_{k} - y_{k}\| + \Theta_{k}^{2} \|x_{k} - x_{k-1}\|^{2} + 2\Theta_{k} \|x_{k} - \bar{x}\| \|x_{k} - x_{k-1}\|.$$

Letting  $k \to \infty$  and utilizing (C1), (C2), and (19), we have

$$\lim_{k \to \infty} \left\| (\mathrm{Id} - S_j) \bar{\nu}_k \right\| = 0.$$
<sup>(22)</sup>

This implies that

$$\lim_{k \to \infty} \|\bar{w}_k - \bar{v}_k\| = \lim_{k \to \infty} a^* \| (\mathrm{Id} - S_j) \bar{v}_k \| = 0.$$
(23)

Utilizing (16), (21), (23), and the following triangle inequalities, we have

- (i)  $\|\bar{v}_k b_k\| \le \|\bar{v}_k \bar{u}_k\| + \|\bar{u}_k b_k\| \to 0;$ (ii)  $\|\bar{v}_k - x_k\| \le \|\bar{v}_k - b_k\| + \|b_k - x_k\| \to 0;$ (iii)  $\|\bar{w}_k - b_k\| \le \|\bar{w}_k - \bar{v}_k\| + \|\bar{v}_k - b_k\| \to 0;$
- (iv)  $\|\bar{w}_k x_k\| \le \|\bar{w}_k b_k\| + \|b_k x_k\| \to 0.$

From (10), (11), and Lemma 2.2, we have

$$\begin{aligned} \|y_{k} - \bar{x}\|^{2} &= \|(1 - \beta_{k})\bar{w}_{k} + \beta_{k}\left(f_{m_{k}}^{A_{1}}\left(\bar{w}_{k} + \delta\hbar^{*}\left(f_{n_{k}}^{A_{2}} - \mathrm{Id}\right)\hbar\bar{w}_{k}\right)\right) - \bar{x})\|^{2} \\ &\leq (1 - \beta_{k})\|\bar{w}_{k} - \bar{x}\|^{2} + \beta_{k}\left(\|\bar{w}_{k} - \bar{x}\|^{2} - \delta\left(2 - \delta\|\hbar\|^{2}\right)\right)\left\|\left(f_{n_{k}}^{A_{2}} - \mathrm{Id}\right)\hbar\bar{w}_{k}\right\|^{2} \\ &\leq \|\bar{w}_{k} - \bar{x}\|^{2} - \beta_{k}\delta\left(2 - \delta\|\hbar\|^{2}\right)\left\|\left(f_{n_{k}}^{A_{2}} - \mathrm{Id}\right)\hbar\bar{w}_{k}\right\|^{2} \\ &\leq \|x_{k} - \bar{x}\|^{2} + 2\Theta_{k}\langle x_{k} - x_{k-1}, b_{k} - \bar{x}\rangle \\ &- \beta_{k}\delta\left(2 - \delta\|\hbar\|^{2}\right)\left\|\left(f_{n_{k}}^{A_{2}} - \mathrm{Id}\right)\hbar\bar{w}_{k}\right\|^{2}. \end{aligned}$$

$$(24)$$

Rearranging the above estimate, we have

$$\beta_{k}\delta(2-\delta\|\hbar\|^{2})\|(I_{n_{k}}^{A_{2}}-\mathrm{Id})\hbar\bar{w}_{k}\|^{2} \leq (\|x_{k}-\bar{x}\|+\|y_{k}-\bar{x}\|)\|x_{k}-y_{k}\|+2\Theta_{k}\langle x_{k}-x_{k-1},b_{k}-\bar{x}\rangle.$$
(25)

By using (C1), (C3), (19), and  $\delta \in (0, \frac{2}{\|\hbar\|^2})$ , estimate (25) implies that

$$\lim_{k \to \infty} \left\| \left( I_{n_k}^{A_2} - \operatorname{Id} \right) \hbar \bar{w}_k \right\| = 0.$$
<sup>(26)</sup>

Note that  $J_{m_k}^{A_1}$  is firmly nonexpansive, it follows that

$$\|y_{k} - \bar{x}\|^{2} = \|(1 - \beta_{k})\bar{w}_{k} + \beta_{k}(J_{m_{k}}^{A_{1}}(\bar{w}_{k} + \delta\hbar^{*}(J_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k})) - \bar{x}\|^{2}$$

$$\leq (1 - \beta_{k})\|\bar{w}_{k} - \bar{x}\|^{2}$$

$$+ \beta_{k}\|J_{m_{k}}^{A_{1}}(\bar{w}_{k} + \delta\hbar^{*}(J_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}) - \bar{x}\|^{2}.$$
(27)

Utilizing (10) and (11), the expression  $J_{m_k}^{A_1}(\bar{w}_k + \delta \hbar^* (J_{n_k}^{A_2} - \text{Id})\hbar \bar{w}_k)$  from the above estimate simplifies as follows:

$$\|J_{m_{k}}^{A_{1}}(\bar{w}_{k}+\delta\hbar^{*}(J_{n_{k}}^{A_{2}}-\mathrm{Id})\hbar\bar{w}_{k})-J_{m_{k}}^{A_{1}}\bar{x}\|^{2} \leq \|\bar{w}_{k}+\delta\hbar^{*}(J_{n_{k}}^{A_{2}}-\mathrm{Id})\hbar\bar{w}_{k}-\bar{x}\|^{2} \leq \|\bar{w}_{k}-\bar{x}\|^{2}.$$
(28)

Setting  $\xi_k = J_{m_k}^{A_1}(\bar{w}_k + \delta\hbar^*(J_{n_k}^{A_2} - \text{Id})\hbar\bar{w}_k)$  in (27), it follows that

$$\begin{split} \|\xi_{k} - \bar{x}\|^{2} &= \|J_{m_{k}}^{A_{1}} \bar{w}_{k} + \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k}) - J_{m_{k}}^{A_{1}} \bar{x}, \bar{w}_{k} + \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} - \bar{x} \rangle \\ &= \langle \xi_{k} - \bar{x}, \bar{w}_{k} + \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} - \bar{x} \rangle \\ &= \frac{1}{2} (\|\xi_{k} - \bar{x}\|^{2} + \|\bar{w}_{k} + \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} - \bar{x} \|^{2} \\ &- \|\xi_{k} - \bar{w}\|^{2} + \|\bar{w}_{k} + \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2} ) \\ &\leq \frac{1}{2} (\|\xi_{k} - \bar{x}\|^{2} + \|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k} - \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2} ) \\ &= \frac{1}{2} (\|\xi_{k} - \bar{x}\|^{2} + \|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k} - \delta\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2} ) \\ &= \frac{1}{2} (\|\xi_{k} - \bar{x}\|^{2} + \|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k}\|^{2} - \delta^{2} \|\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2} \\ &+ 2\delta (\xi_{k} - \bar{w}_{k}, \hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k})) \\ &\leq \frac{1}{2} (\|\xi_{k} - \bar{x}\|^{2} + \|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k}\|^{2} - \delta^{2} \|\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k} \|^{2} \\ &+ 2\delta |\xi_{k} - \bar{w}_{k}\| \|\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id}) \hbar \bar{w}_{k}\|). \end{split}$$

This implies that

$$\|\xi_{k} - \bar{x}\|^{2} \leq \|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k}\|^{2} + 2\delta \|\xi_{k} - \bar{w}_{k}\| \|\hbar^{*} (J_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}\|.$$
(30)

So, we have

$$\|y_{k} - \bar{x}\|^{2} \leq (1 - \beta_{k}) \|\bar{w}_{k} - \bar{x}\|^{2} + \beta_{k} \|\xi_{k} - \bar{x}\|^{2}$$

$$\leq (1 - \beta_{k}) \|\bar{w}_{k} - \bar{x}\|^{2} + \beta_{k} (\|\bar{w}_{k} - \bar{x}\|^{2} - \|\xi_{k} - \bar{w}_{k}\|^{2} + 2\delta \|\xi_{k} - \bar{w}_{k}\| \|\hbar^{*} (I_{n_{k}}^{A_{2}} - \operatorname{Id}) \hbar \bar{w}_{k}\|).$$
(31)

After simplification, we have

$$\beta_{k} \|\xi_{k} - \bar{w}_{k}\|^{2} \leq \|\bar{w}_{k} - \bar{x}\|^{2} - \|y_{k} - \bar{x}\|^{2} - 2\beta_{k}\delta\|\xi_{k} - \bar{w}_{k}\| \left\|\hbar^{*}(I_{n_{k}}^{A_{2}} - \operatorname{Id})\hbar\bar{w}_{k}\right\|)$$

$$\leq \|x_{k} - \bar{x}\|^{2} - \|y_{k} - \bar{x}\|^{2} - 2\beta_{k}\delta\|\xi_{k} - \bar{w}_{k}\| \|\hbar^{*}(J_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}\|)$$
  
$$\leq (\|x_{k} - \bar{x}\| + \|y_{k} - \bar{x}\|)\|x_{k} - y_{k}\|$$
  
$$- 2\beta_{k}\delta\|\xi_{k} - \bar{w}_{k}\| \|\hbar^{*}(J_{n_{k}}^{A_{2}} - \mathrm{Id})\hbar\bar{w}_{k}\|).$$
(32)

Making use of (19), (26), and (C3), we have the following estimate:

$$\lim_{k \to \infty} \|\xi_k - \bar{w}_k\| = 0. \tag{33}$$

This implies that

$$\lim_{k \to \infty} \|\xi_k - \bar{\nu}_k\| = 0. \tag{34}$$

Reasoning as above, we get from the definition of  $(b_k)$ , (C1), and (34), that

$$\lim_{k \to \infty} \|\xi_k - x_k\| = 0. \tag{35}$$

Since  $(x_k)$  is bounded, then there exists a subsequence  $(x_{k_t})$  of  $(x_k)$  such that  $x_{k_t} \rightarrow \bar{x_*} \in \mathcal{H}_1$ as  $t \rightarrow \infty$ . Therefore  $\xi_{k_t} \rightarrow \bar{x_*}$  and  $\bar{w}_{k_t} \rightarrow \bar{x_*}$  as  $t \rightarrow \infty$ . In order to show that  $\bar{x_*} \in \Omega$ , we assume that  $(r,s) \in \operatorname{gra}(A_1)$ . Since  $\xi_{k_t} = J_{m_{k_t}}^{A_1}(\bar{w}_{k_t} + \delta\hbar^*(J_{n_{k_t}}^{A_2} - \operatorname{Id})\hbar\bar{w}_{k_t})$ , we have

$$\bar{w}_{k_t} + \delta \hbar^* \big( J_{n_{k_t}}^{A_2} - \operatorname{Id} \big) \hbar \bar{w}_{k_t} \in \xi_{k_t} + m_{k_t} A_1(\xi_{k_t}).$$

This implies that

$$\frac{1}{m_{k_t}}(\bar{w}_{k_t}-\xi_{k_t})+\frac{1}{m_{k_t}}\delta\hbar^*(J^{A_2}_{n_{k_t}}-\mathrm{Id})\hbar\bar{w}_{k_t}\in A_1(\xi_{k_t}).$$

From the monotonicity of  $A_1$ , we have

$$\left\langle r-\xi_{k_t},s-\left(\frac{1}{m_{k_t}}(\bar{w}_{k_t}-\xi_{k_t})+\frac{1}{m_{k_t}}\left(\delta\hbar^*\left(J_{n_{k_t}}^{A_2}-\mathrm{Id}\right)\hbar\bar{w}_{k_t}\right)\right)\right\rangle\geq 0.$$

From the above estimate, we also have

$$\langle r - \xi_{k_t}, s \rangle \geq \left\langle r - \xi_{k_t}, \frac{1}{m_{k_t}} (\bar{w}_{k_t} - \xi_{k_t}) + \frac{1}{m_{k_t}} (\delta \hbar^* (J_{n_{k_t}}^{A_2} - \mathrm{Id}) \hbar \bar{w}_{k_t}) \right\rangle$$

$$= \left\langle r - \xi_{k_t}, \frac{1}{m_{k_t}} (\bar{w}_{k_t} - \xi_{k_t}) \right\rangle$$

$$+ \left\langle r - \xi_{k_t}, \frac{1}{m_{k_t}} (\delta \hbar^* (J_{n_{k_t}}^{A_2} - \mathrm{Id}) \hbar \bar{w}_{k_t}) \right\rangle.$$
(36)

Since  $\xi_{k_t} \rightharpoonup \bar{x_*}$ , we obtain

$$\lim_{t\to\infty} \langle r-\xi_{k_t},\nu\rangle = \langle r-\bar{x_*},s\rangle.$$

By making use of (33), (34), and (36), it follows that

$$\langle r-\bar{x_*},s\rangle\geq 0.$$

This implies that  $0 \in A_1(\bar{x_*})$ . Since  $\hbar$  is a bounded linear operator, we have  $\hbar \bar{w}_{k_t} \rightarrow \hbar \bar{x_*}$ as  $t \rightarrow \infty$ . Moreover, from (26) it then follows from the demiclosedness principle that  $0 \in A_2(\bar{x_*})$ , and hence  $\bar{x_*} \in \Omega$ .

Step 4. We show that  $\bar{x_*} \in \bigcap_{i=1}^M EP(g_i)$ .

Observe that the following relation

$$u_k^i = \arg\min\left\{\gamma g_i(b_k, y) + \frac{1}{2} \|b_k - y\|^2 : y \in C\right\}$$

implies via Lemma 2.4 that

$$0 \in \partial_2 \left\{ \gamma g_i(b_k, y) + \frac{1}{2} \|b_k - y\|^2 \right\} (u_k^i) + N_C(u_k^i).$$

This infers the existence of  $\bar{x_*} \in \partial_2 g_i(b_k, u_k^i)$  and  $\bar{p} \in N_C(u_k^i)$  such that

$$\gamma \bar{x_*} + b_k - u_k^i + \bar{p}. \tag{37}$$

Since  $\bar{p} \in N_C(u_k^i)$  and  $\langle \bar{p}, u - u_k^i \rangle \leq 0$  for all  $u \in C$ , by using (37), we have

$$\gamma(\bar{x_*}, u - u_k^i) \ge \langle u_k^i - b_k, u - u_k^i \rangle, \quad \forall u \in C.$$
(38)

Since  $\bar{x_*} \in \partial_2 g_i(b_k, u_k^i)$ ,

$$g_i(b_k, y) - g_i(b_k, u_k^i) \ge \langle \bar{x_*}, u - u_k^i \rangle, \quad \forall u \in C.$$

$$(39)$$

Utilizing (38) and (39), we obtain

$$\gamma\left(g_i(b_k, y) - g_i(b_k, u_k^i)\right) \ge \left\langle u_k^i - b_k, u - u_k^i \right\rangle, \quad \forall u \in C.$$

$$\tag{40}$$

Since  $b_k \rightarrow \bar{x_*}$  and  $||b_k - u_k^i|| \rightarrow 0$  as  $k \rightarrow \infty$ , this implies  $u_k^i \rightarrow \bar{x_*}$ . By using (A3) and (40), letting  $k \rightarrow \infty$ , we deduce that  $g_i(\bar{x_*}, u) \ge 0$  for all  $u \in C$  and  $i \in \{1, 2, ..., M\}$ . Therefore,  $\bar{x_*} \in \bigcap_{i=1}^M EP(g_i)$ .

Step 5. We show that  $\bar{x_*} = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ .

Since  $x_{k_t} \rightarrow \bar{x_*}$  and  $\|\bar{v_k} - x_k\| \rightarrow 0$  as  $t \rightarrow \infty$ , this implies  $\bar{v_k} \rightarrow \bar{x_*}$ . Therefore, utilizing the demiclosedness principle along with estimate (22), we have  $\bar{x_*} \in \bigcap_{j=1}^N \operatorname{Fix}(T_j)$ . Hence  $\bar{x_*} \in \Gamma$ .

Step 6. We show that  $x_k \rightarrow \bar{x} = P_{\Gamma} x_1$ .

Note that  $\bar{x} = P_{\Gamma}x_1$  and  $\bar{x_*} \in \Gamma$  implies that  $x_{k+1} = P_{\Gamma}x_1$  and  $\bar{x} \in \Gamma \in C_{k+1}$ . This infers that  $||x_{k+1} - x_1|| \le ||\bar{x} - x_1||$ . On the other hand, we have

$$\|\bar{x} - x_1\| \le \|\bar{x_*} - x_1\| \le \liminf_{k \to \infty} \|x_k - x_1\| \le \limsup_{k \to \infty} \|x_k - x_1\| \le \|\bar{x} - x_1\|.$$

That is,

$$\|\bar{x_*} - x_1\| = \lim_{k \to \infty} \|x_k - x_1\| = \|\bar{x} - x_1\|.$$

Therefore, we conclude that  $\lim_{k\to\infty} x_k = \bar{x_*} = P_{\Gamma} x_1$ . This completes the proof.

If we take  $A_2 = 0$  in hypothesis (H1), then we have the following results.

**Corollary 3.4** Assume that  $\Gamma := \{x \in A_1^{-1}(0) \cap (\bigcap_{i=1}^M EP(g_i)) \cap (\bigcap_{j=1}^N Fix(S_i)\}) \neq \emptyset$ . Then the sequence  $(x_k)$  defined as

$$\begin{cases} b_{k} = x_{k} + \Theta_{k}(x_{k} - x_{k-1}); \\ u_{k}^{i} = \arg\min\{\gamma g_{i}(b_{k}, y) + \frac{1}{2} \| b_{k} - u \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ v_{k}^{i} = \arg\min\{\gamma g_{i}(u_{k}^{i}, y) + \frac{1}{2} \| b_{k} - u \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ i_{k} = \arg\max\{\|v_{k}^{i} - x_{k}\| : i = 1, 2, ..., M\}, \quad \bar{v}_{k} = v_{k}^{i_{k}}; \\ w_{k}^{j} = (1 - \alpha_{k})\bar{v}_{k} + \alpha_{k}S_{j}\bar{v}_{k}; \\ j_{k} = \arg\max\{\|w_{k}^{j} - x_{k}\| : j = 1, 2, ..., N\}, \quad \bar{w}_{k} = w_{k}^{j_{k}}; \\ y_{k} = (1 - \beta_{k})\bar{w}_{k} + \beta_{k}J_{m_{k}}^{A_{1}}\bar{w}_{k}; \\ C_{k+1} = \{z \in C_{k} : \|y_{k} - z\|^{2} \le \|x_{k} - z\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} \\ + 2\Theta_{k}\langle x_{k} - z, x_{k} - x_{k-1}\rangle\}; \\ x_{k+1} = P_{C_{k+1}}x_{1}, \quad \forall k \ge 1, \end{cases}$$

$$(41)$$

converges strongly to an element in  $\Gamma$  provided that conditions (C1)–(C4) hold.

#### 4 Numerical experiment and results

This section shows the effectiveness of our algorithm by the following example and numerical results.

*Example* 4.1 Let  $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$  with the inner product defined by  $\langle x, y \rangle = xy$  for all  $x, y \in \mathbb{R}$ and induced usual norm  $|\cdot|$ . We define three operators  $\hbar, A_1, A_2 : \mathbb{R} \to \mathbb{R}$  as  $\hbar(x) = 3x$ ,  $A_1x = 2x$ , and  $A_2x = 3x$  for all  $x \in \mathbb{R}$ . It is clear that  $\hbar$  is a bounded linear operator and  $A_1, A_2$  are maximal monotone operators such that  $\Omega := \{\hat{x} \in A_1^{-1}0 : \hbar \hat{x} \in A_2^{-1}0\} = 0$ . For each  $i \in \{1, 2, ..., M\}$ , let the family of pseudomonotone bifunctions  $g_i(x, y) : C \times C \to \mathbb{R}$ on  $C = [0, 1] \subset \mathbb{R}$  be defined by  $g_i(x, y) = T_i(x)(y - x)$ , where

$$T_i(x) = \begin{cases} 0 & 0 \le x \le \mu_i; \\ \sin(x - \mu_i) + \exp(x - \mu_i) - 1, & \mu_i \le x \le 1, \end{cases}$$

where  $0 < \mu_1 < \mu_2 < \cdots < \mu_M < 1$ . Note that  $EP(g_i) = [0, \mu_i]$  if and only if  $0 \le x \le \mu_i$  and  $y \in [0, 1]$ . Consequently,  $\bigcap_{i=1}^M EP(g_i) = [0, \mu_1]$ . For each  $j \in \{1, 2, \dots, N\}$ , let the family of operators  $S_j : \mathbb{R} \to \mathbb{R}$  be defined by

$$S_j(x) = \begin{cases} -\frac{x}{j} & \text{if } x \in [0,\infty); \\ x & \text{if } x \in (-\infty,0). \end{cases}$$

It is also clear that  $S_j$  defines a finite family of  $\frac{1-j^2}{(1+j)^2}$ -demicontractive operators with  $\bigcap_{j=1}^N \operatorname{Fix}(S_j) = \{0\}$ . Hence  $\Gamma = \Omega \cap (\bigcap_{i=1}^M EP(g_i)) \cap (\bigcap_{j=1}^N \operatorname{Fix}(S_j)) = 0$ . In order to compute the numerical values of  $(x_{k+1})$ , we choose:  $\Theta = 0.5$ ,  $\gamma = \frac{1}{8}$ ,  $\alpha_k = \frac{1}{100k+1}$ ,  $\beta_k = \frac{1}{100k+1}$ ,  $\delta = \frac{1}{9}$ ,

Table 1	Numerical results for Example 4 1
Table I	Numerical results for Example 4.

	No. of Iter. $\Theta_k = 0$	Alg. 1, $\Theta_k \neq 0$	CPU (Sec) $\Theta_k = 0$	Alg. 1, $\Theta_k \neq 0$
Choice 1. $x_0 = (5)$ , $x_1 = (2)$ and $N = 20$	91	78	0.088136	0.059103
Choice 1. $x_0 = (5)$ , $x_1 = (2)$ and $N = 5$	95	83	0.088177	0.067473
Choice 2. <i>x</i> <sub>0</sub> = (4.7), <i>x</i> <sub>1</sub> = (1.7) and <i>N</i> = 20	104	90	0.098793	0.091790
Choice 2. <i>x</i> <sub>0</sub> = (4.7), <i>x</i> <sub>1</sub> = (1.7) and <i>N</i> = 5	110	94	0.099405	0.092703
Choice 3. <i>x</i> <sub>0</sub> = (-7), <i>x</i> <sub>1</sub> = (4) and <i>N</i> = 20	95	81	0.074912	0.069149
Choice 3. <i>x</i> <sub>0</sub> = (-7), <i>x</i> <sub>1</sub> = (4) and <i>N</i> = 5	108	89	0.078510	0.071038



*L* = 3, and *m* = 0.01. Since

$$\begin{cases} \min\{\frac{1}{k^2 \|x_k - x_{k-1}\|}, 0.5\} & \text{if } x_k \neq x_{k-1}; \\ 0.5 & \text{otherwise.} \end{cases}$$

Observe that the expression

$$u_{k}^{i} = \arg\min\left\{\gamma T_{i}(b_{k})(y - b_{k}) + \frac{1}{2}(y - x_{k})^{2}, \forall y \in [0, 1]\right\}$$

in Algorithm 1 is equivalent to the following relation  $u_k^i = b_k - \gamma T_i(b_k)$  for all  $i \in \{1, 2, ..., M\}$ . Similarly,  $v_k^i = b_k - \gamma T_i(u_k^i)$  for all  $i \in \{1, 2, ..., M\}$ . Hence, we can compute the intermediate approximation  $\bar{v}_k$  which is farthest from  $b_k$  among  $v_k^i$  for all  $i \in \{1, 2, ..., M\}$ . We compare the parallel hybrid accelerated extragradient algorithm defined in Algorithm 1 (i.e.,  $\Theta_k \neq 0$ ) and its variant with  $\Theta_k = 0$ . The stopping criteria are defined as Error  $=E_k = ||x_k - x_{k-1}|| < 10^{-5}$ . The values of Algorithm 1 and its variant are listed in Table 1.

The error plotting  $E_k$  and  $(x_k)$  of Algorithm 1 with  $\Theta_k \neq 0$  and  $\Theta_k = 0$  for each choice in Table 1 is illustrated in Fig. 1.

We can see from Table 1 and Fig. 1 that Algorithm 1 performs faster and better in view of the error analysis, time consumption, and the number of iterations required for the convergence towards the desired solution in comparison with the variant of Algorithm 1 with  $\Theta_k = 0$ .

### **5** Applications

In this section, we discuss some important instances of the main result in Sect. 3 as applications.

## 5.1 Split feasibility problems

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces and  $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$  be a bounded linear operator. Let *C* and *Q* be nonempty, closed, and convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. The split feasibility problem (SFP) is the problem of finding  $\bar{x} \in C$  such that  $\hbar \bar{x} \in Q$ . We represent the solution set by  $\omega := C \cap \hbar^{-1}(Q) = \{\bar{x} \in C : \hbar \bar{x} \in Q\}$ . This problem is essentially due to Censor and Elfving [12] to solve the inverse problems and their application to medical image reconstruction, radiation therapy, and modeling and simulation in a finite dimensional Hilbert space. Recall the indicator function of *C* 

$$b_C(\bar{x}) := \begin{cases} 0, & \bar{x} \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

The proximal operator of  $b_C$  is the metric projection on C

$$\begin{aligned} \operatorname{prox}_{b_C} &= \arg\min_{\bar{p}\in C} \|\bar{p} - \bar{x}\| \\ &= P_C(\bar{x}). \end{aligned}$$

Let  $P_Q$  be the projection of  $\mathcal{H}_2$  onto a nonempty, closed, and convex subset Q. Take  $f(\bar{x}) = \frac{1}{2} \|\hbar \bar{x} - P_O \hbar \bar{x}\|^2$  and  $g(\bar{x}) = b_C(\bar{x})$ . Then we have the following result.

**Corollary 5.1** Assume that  $\Gamma = \omega \cap (\bigcap_{i=1}^{M} EP(g_i)) \cap (\bigcap_{i=1}^{N} Fix(S_j)) \neq \emptyset$  via hypotheses (H1)–(H5). For given  $x_0, x_1 \in \mathcal{H}_1$ , let the iterative sequence  $(x_k)$  be generated by

$$\begin{cases} b_{k} = x_{k} + \Theta_{k}(x_{k} - x_{k-1}); \\ u_{k}^{i} = \arg\min\{\gamma g_{i}(b_{k}, y) + \frac{1}{2} \| b_{k} - y \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ v_{k}^{i} = \arg\min\{\gamma g_{i}(u_{k}^{i}, y) + \frac{1}{2} \| b_{k} - y \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ i_{k} = \arg\max\{\|v_{k}^{i} - x_{k}\| : i = 1, 2, ..., M\}, \quad \bar{v_{k}} = v_{k}^{i_{k}}; \\ w_{k}^{j} = (1 - \alpha_{k})\bar{v_{k}} + \alpha_{k}S_{j}\bar{v_{k}}; \\ j_{k} = \arg\max\{\|w_{k}^{j} - x_{k}\| : j = 1, 2, ..., N\}, \quad \bar{w_{k}} = w_{k}^{j_{k}}; \\ y_{k} = (1 - \beta_{k})\bar{w}_{k} + \beta_{k}(P_{C}(\bar{w}_{k} + \delta\hbar^{*}(P_{Q} - \operatorname{Id})\hbar\bar{w}_{k})); \\ C_{k+1} = \{z \in C_{k} : \|y_{k} - z\|^{2} \le \|x_{k} - z\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} \\ + 2\Theta_{k}\langle x_{k} - z, x_{k} - x_{k-1}\rangle\}; \\ x_{k+1} = P_{C_{k+1}}x_{1}, \quad \forall k \ge 1, \end{cases}$$

where  $0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$ . Assume that conditions (C1)–(C4) hold, then the sequence  $(x_k)$  generated by (42) converges strongly to an element in  $\Gamma$ .

## 5.2 Split variational inequality problems

Let  $A : C \to \mathcal{H}$  be a nonlinear monotone operator defined on a nonempty, closed, and convex subset *C* of a real Hilbert space  $\mathcal{H}$ . The classical variational inequality problem

aims to find a point  $\bar{x} \in C$  such that

$$\langle A\bar{x}, \bar{y} - \bar{x} \rangle \geq 0 \quad \forall \bar{y} \in C.$$

The solution set of the above problem is denoted by VI(C, A). Let the operator  $\Pi_A \subset \mathcal{H} \times \mathcal{H}$  be defined by

$$\Pi_A \bar{x} = \begin{cases} A(\bar{x}) + N_C(\bar{x}) & \text{if } \bar{x} \in C, \\ \emptyset & \text{if } \bar{x} \notin C, \end{cases}$$

where  $N_C(\bar{x}) := \{z \in \mathcal{H} : \langle \bar{y} - \bar{x}, z \rangle \leq 0 \text{ for all } \bar{y} \in C\}$ . It follows from [28] that  $\Pi_A$  is a maximal monotone such that  $0 \in \Pi_A(\bar{x}) \iff \bar{x} \in VI(C, A) \iff \bar{x} = P_C(\bar{x} - \lambda A(\bar{x}))$ . As an application, we have the following result.

**Corollary 5.2** Let  $\{\mathcal{H}_n\}_{n=1}^2$  be real Hilbert spaces, and let  $\{C_n\}_{n=1}^2$  be nonempty, closed, and convex subsets of  $\mathcal{H}_n$ , respectively. Let  $A_n : C_n \to \mathcal{H}_n$  for n = 1, 2 be single-valued monotone and hemicontinuous operators, and let  $\hbar : \mathcal{H}_1 \to \mathcal{H}_2$  be a bounded linear operator such that  $\Gamma = VI(C_1, A_1) \cap \hbar^{-1}(VI(C_2, A_2)) \cap (\bigcap_{i=1}^M EP(g_i)) \cap (\bigcap_{j=1}^N Fix(S_j)) \neq \emptyset$  via hypotheses  $(\mathcal{H}_1)-(\mathcal{H}_5)$ . For given  $x_0, x_1 \in \mathcal{H}_1$ , let the iterative sequences  $(x_k)$  be generated by

$$b_{k} = x_{k} + \Theta_{k}(x_{k} - x_{k-1});$$

$$u_{k}^{i} = \arg\min\{\gamma g_{i}(b_{k}, y) + \frac{1}{2} \| b_{k} - y \|^{2} : u \in C\}, \quad i = 1, 2, ..., M;$$

$$v_{k}^{i} = \arg\max\{\|v_{k}^{i} - x_{k}\| : i = 1, 2, ..., M\}, \quad \bar{v}_{k} = v_{k}^{i_{k}};$$

$$w_{k}^{j} = (1 - \alpha_{k})\bar{v}_{k} + \alpha_{k}S_{j}\bar{v}_{k};$$

$$j_{k} = \arg\max\{\|w_{k}^{j} - x_{k}\| : j = 1, 2, ..., N\}, \quad \bar{w}_{k} = w_{k}^{j_{k}};$$

$$y_{k} = (1 - \beta_{k})\bar{w}_{k} + \beta_{k}(P_{C_{1}}^{(A_{1},\lambda)}(\bar{w}_{k} + \delta\hbar^{*}(P_{C_{2}}^{(A_{2},\lambda)} - \mathrm{Id})\hbar\bar{w}_{k}));$$

$$C_{k+1} = \{z \in C_{k} : \|y_{k} - z\|^{2} \le \|x_{k} - z\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} + 2\Theta_{k}\langle x_{k} - z, x_{k} - x_{k-1}\rangle\};$$

$$x_{k+1} = P_{C_{k+1}}x_{1}, \quad \forall k \ge 1,$$

$$(43)$$

where  $0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$  and  $P_{C_1}^{(A_1,\lambda)}$ ,  $P_{C_2}^{(A_2,\lambda)}$  denote  $P_C(\text{Id} - \lambda A)$ . Assume that conditions (C1)–(C4) hold, then the sequence  $(x_k)$  generated by (43) converges strongly to an element in  $\Gamma$ .

### 5.3 Split optimization problems

Let  $\phi : \mathcal{H}_1 \to (-\infty, \infty]$  be a proper, convex, and lower semicontinuous (pcls) function, then the set of minimizers associated with  $\phi$  is defined as

$$\operatorname{argmin} \phi := \{ x^* \in \mathcal{H}_1 : \phi(x^*) \le \phi(\bar{z}) \text{ for all } \bar{z} \in \mathcal{H}_1 \}.$$

Recall that  $\partial \phi$  of the pcls function  $\phi$  is a maximal monotone operator and the corresponding resolvent operator of  $\partial \phi$  is called the proximity operator (see [17]). Hence argmin  $\phi = (\partial \phi)^{-1}(0)$ . Hence, we have the following application.

**Corollary 5.3** Let  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  be two Hilbert spaces, and let  $C \subseteq \mathcal{H}_1$  be nonempty, closed, and convex subset of  $\mathcal{H}_1$ . Let  $\phi_1$  and  $\phi_2$  be pcls functions on  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Assume that  $\Gamma = \{x \in \arg \min \phi_1 : \hbar x \in \arg \min \phi_2\} \cap (\bigcap_{i=1}^M EP(g_i)) \cap (\bigcap_{i=1}^N Fix(S_i)) \neq \emptyset$  via hypotheses  $(\mathcal{H}_1)-(\mathcal{H}_5)$ . For given  $x_0, x_1 \in \mathcal{H}_1$ , let the iterative sequence  $(x_k)$  be generated by

$$\begin{cases} b_{k} = x_{k} + \Theta_{k}(x_{k} - x_{k-1}); \\ u_{k}^{i} = \arg\min\{\gamma g_{i}(b_{k}, y) + \frac{1}{2} \| b_{k} - y \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ v_{k}^{i} = \arg\max\{\gamma g_{i}(u_{k}^{i}, y) + \frac{1}{2} \| b_{k} - y \|^{2} : u \in C\}, \quad i = 1, 2, ..., M; \\ i_{k} = \arg\max\{\|v_{k}^{i} - x_{k}\| : i = 1, 2, ..., M\}, \quad \bar{v}_{k} = v_{k}^{i_{k}}; \\ w_{k}^{j} = (1 - \alpha_{k})\bar{v}_{k} + \alpha_{k}S_{j}\bar{v}_{k}; \\ j_{k} = \arg\max\{\|w_{k}^{j} - x_{k}\| : j = 1, 2, ..., N\}, \quad \bar{w}_{k} = w_{k}^{j_{k}}; \\ y_{k} = (1 - \beta_{k})b_{k} + \beta_{k}(f_{m_{k}}^{\partial g_{1}}(b_{k} + \delta\hbar^{*}(f_{m_{k}}^{\partial g_{2}} - \mathrm{Id})\hbar b_{k})); \\ C_{k+1} = \{z \in C_{k} : \|\bar{y}_{k} - z\|^{2} \le \|x_{k} - z\|^{2} + \Theta_{k}^{2}\|x_{k} - x_{k-1}\|^{2} \\ + 2\Theta_{k}\langle x_{k} - z, x_{k} - x_{k-1}\rangle\}; \\ x_{k+1} = P_{C_{k+1}}x_{1}, \quad \forall k \ge 1, \end{cases}$$

$$(44)$$

where  $0 < \gamma < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$ . Assume that conditions (C1)–(C4) hold, then the sequence  $(x_k)$  generated by (44) converges strongly to an element in  $\Gamma$ .

## **6** Conclusions

In this paper, we have analyzed an accelerated projection based parallel hybrid extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems in Hilbert spaces. The convergence analysis of the algorithm is established under the suitable set of conditions. A suitable numerical example has been incorporated to exhibit the effectiveness of the algorithm. Moreover, some well-known instances, as applications, of the main result that can pave a way for an important future research direction are also discussed.

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#### Availability of data and materials

Data sharing not applicable to this article as no data-sets were generated or analysed during the current study.

#### **Competing interests**

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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