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Research Article
Variationally Asymptotically Stable Difference Systems

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We characterize the h-stability in variation and asymptotic equilibrium in variation for nonlinear difference systems via n_∞-summable similarity and comparison principle. Furthermore we study the asymptotic equivalence between nonlinear difference systems and their variational difference systems by means of asymptotic equilibria of two systems.

1. Introduction

Conti [1] introduced the notion of t_∞-similarity in the set of all $m \times m$ continuous matrices $A(t)$ defined on $\mathbb{R}_+ = [0, \infty)$ and showed that t_∞-similarity is an equivalence relation preserving strict, uniform, and exponential stability of linear homogeneous differential systems. Choi et al. [2] studied the variational stability of nonlinear differential systems using the notion of t_∞-similarity. Trench [3] introduced a definition called t_∞-quasisimilarity that is not symmetric or transitive, but still preserves stability properties. Their approach included most types of stability.

As a discrete analog of Conti's definition of t_∞-similarity, Trench [4] defined the notion of summable similarity on pairs of $m \times m$ matrix functions and showed that if A and B are summably similar and the linear system $\Delta x(n) = A(n)x(n)$, $n = 0, 1, \dots$, is uniformly, exponential or strictly stable or has linear asymptotic equilibrium, then the linear system $\Delta y(n) = B(n)y(n)$ has also the same properties. Also, Choi and Koo [5] introduced the notion of n_∞-similarity in the set of all $m \times m$ invertible matrices and showed that two concepts of global h-stability and global h-stability in variation are equivalent by using the concept of n_∞-similarity and Lyapunov functions. Furthermore, they showed that h-stability of the perturbed system can be derived from h-stability in variation of the nonlinear system in [6]. Note that the n_∞-similarity is not symmetric or transitive relation but still preserves h-stability which included the most types of stability. For the variational stability in difference systems, see [6]. Also, see [7–9] for the asymptotic property of difference systems and Volterra difference systems, respectively.

In this paper, we study the variational stability for nonlinear difference systems using the notion of n_∞-summable similarity and show that asymptotic equilibrium for linear difference system is preserved by n_∞-summable similarity. Furthermore, we obtain the asymptotic equivalence between nonlinear difference system and its variational difference system using the comparison principle and asymptotic equilibria.

2. Preliminaries

Let $\mathbb{N}(n_0) = \{n_0, n_0+1, \ldots, n_0+k, \ldots\}$, where n_0 is a nonnegative integer and $\mathbb{R}^m$ the m-dimensional real Euclidean space. We consider the nonlinear difference system

$$x(n+1) = f(n,x(n)), \tag{2.1}$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, and $f(n,0) = 0$. We assume that $f_x = \partial f/\partial x$ exists and is continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$. Let $x(n) = x(n,n_0,x_0)$ be the unique solution of (2.1) satisfying the initial condition $x(n_0,n_0,x_0) = x_0$. Also, we consider its associated variational systems

$$v(n+1) = f_x(n,0)v(n), \tag{2.2}$$
$$z(n+1) = f_x(n,x(n,n_0,x_0))z(n). \tag{2.3}$$

The fundamental matrix solution $\Phi(n,n_0,0)$ of (2.2) is given by

$$\Phi(n,n_0,0) = \frac{\partial x(n,n_0,0)}{\partial x_0} \tag{2.4}$$

and the fundamental matrix solution $\Phi(n,n_0,x_0)$ of (2.3) is given by Lakshmikantham and Trigiante [10],

$$\Phi(n,n_0,x_0) = \frac{\partial x(n,n_0,x_0)}{\partial x_0}. \tag{2.5}$$

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^m$. Δ is the forward difference operator with unit spacing, that is, $\Delta u(n) = u(n+1) - u(n)$. Let $V : N(n_0) \times \mathbb{R}^m \to \mathbb{R}_+$ be a function with $V(n,0) = 0$, for all $n \geq n_0$, and continuous with respect to the second argument. We denote the total difference of the function V along the solutions x of (2.1) by

$$\Delta V_{(2.1)}(n,x) = V(n+1,x(n+1,n,x)) - V(n,x(n,n,x)). \tag{2.6}$$

When we study the asymptotic stability, it is not easy to work with nonexponential types of stability. Medina and Pinto [11–13] extended the study of exponential stability to a variety of reasonable systems called h-systems. They introduced the notion of h-stability for difference systems as well as for differential systems. To study the various stability notions of nonlinear difference systems, the comparison principle [10] and the variation of constants formula by Agarwal [14, 15] play a fundamental role.

Now, we recall some definitions of stability notions in [12–14].

Definition 2.1. The zero solution of system (2.1) (or system (2.1)) is said to be

(SS) *strongly stable* if for each $\varepsilon > 0$, there is a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $x(n,n_0,x_0)$ of system (2.1) which satisfies the inequality $|x(n_1,n_0,x_0)| < \delta$ for some $n_1 \geq n_0$ exists and satisfies the inequality $|x(n,n_0,x_0)| < \varepsilon$, for all $n \in \mathbb{N}(n_0)$.

Definition 2.2. Linear system (2.1) with $f(n,x(n)) = A(n)x(n)$ is said to be

(RS) *restrictively stable* if it is stable and its adjoint system $y(n) = A^T(n)y(n+1)$ is also stable.

Strong stability implies uniform stability which, in turn, leads to stability. For linear homogeneous systems, restrictive stability and strong stability are equivalent. Thus restrictive stability implies uniform stability which, in turn, gives stability [14].

Definition 2.3. System (2.1) is called an *h-system* if there exist a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ and a constant $c \geq 1$, such that

$$|x(n,n_0,x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0 \tag{2.7}$$

for $|x_0|$ small enough (here $h^{-1}(n) = 1/h(n)$).

Moreover, system (2.1) is said to be

(hS) *h-stable* if h is a bounded function in the definition of h-system,

(GhS) *globally h-stable* if system (2.1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^m$ is a region which includes the origin,

(hSV) *h-stable in variation* if system (2.3) is hS,

(GhSV) *globally h-stable in variation* if system (2.3) is GhS.

The various notions about h-stability given by Definition 2.3 include several types of known stability properties such as uniform stability, uniform Lipschitz stability, and exponential asymptotic stability. See [5, 11–13].

Definition 2.4. One says that (2.1) has *asymptotic equilibrium* if

(i) there exist $\xi \in \mathbb{R}^m$ and $r > 0$ such that any solution $x(n,n_0,x_0)$ of (2.1) with $|x_0| < r$ satisfies

$$x(n) = \xi + o(1) \quad \text{as } n \longrightarrow \infty, \tag{2.8}$$

(ii) corresponding to each $\xi \in \mathbb{R}^m$, there exists a solution of (2.1) satisfying (2.8), and (2.1) has *asymptotic equilibrium in variation* if system (2.3) has asymptotic equilibrium.

Two difference systems $x(n+1) = f(n,x(n))$ and $y(n+1) = g(n,y(n))$ are said to be *asymptotically equivalent* if, for every solution $x(n)$, there exists a solution $y(n)$ such that

$$x(n) = y(n) + o(1) \quad \text{as } n \longrightarrow \infty, \tag{2.9}$$

and conversely, for every solution $y(n)$, there exists a solution $x(n)$ such that the above asymptotic relation holds.

The problem of asymptotic equivalence in difference equations has been initiated by H. Poincaré (1885) and O. Perron (1921), and it shows an asymptotic relationship between equations. In [16], Pinto studied asymptotic equivalence between difference systems by using the concept of dichotomy. Also, Medina and Pinto in [17] investigated this problem by replacing the dichotomy conditions and the Lipschitz condition by a global domination of the fundamental matrix of the linear difference system and a general majoration on the perturbing term, respectively. Moreover, Medina in [18] established asymptotic equivalence by using the general discrete inequality combined with the Schauder's fixed point theorem. Also, Galescu and Talpalaru [8], Morchało [19], and Zafer [20] studied the asymptotic equivalence for difference systems.

Conti [1] defined two $m\times m$ matrix functions A and B on $\mathbb{R}_+$ to be t_∞-*similar* if there is an $m\times m$ matrix function S defined on $\mathbb{R}_+$ such that $S'(t)$ is continuous, $S(t)$ and $S^{-1}(t)$ are bounded on $\mathbb{R}_+$, and

$$\int_0^\infty |S'+SB-AS|\,dt<\infty. \tag{2.10}$$

Now, we introduce the notion of n_∞-summable similarity which is the corresponding t_∞-similarity for the discrete case.

Let $\mathfrak{M}$ denote the set of all $m\times m$ invertible matrix-valued functions defined on $\mathbb{N}(n_0)$ and let $\mathfrak{S}$ be the subset of $\mathfrak{M}$ consisting of those nonsingular bounded matrix-valued functions S such that $S^{-1}(n)$ is also bounded.

Definition 2.5. A matrix-valued function $A\in\mathfrak{M}$ is n_∞-*summably similar* to a matrix-valued function $B\in\mathfrak{M}$ if there exists an $m\times m$ matrix $F(n)$ absolutely summable over $\mathbb{N}(n_0)$, that is,

$$\sum_{l=n_0}^{\infty}|F(l)|<\infty, \tag{2.11}$$

such that

$$S(n+1)B(n)-A(n)S(n)=F(n) \tag{2.12}$$

for some $S\in\mathfrak{S}$.

Example 2.6. Let A and B be matrix-valued functions defined on $\mathbb{N}(0)$ by

$$A(n)=\begin{pmatrix} e^{-n} & 0\\ 0 & 1\end{pmatrix},\qquad B(n)=\begin{pmatrix} \dfrac{e^{-n}}{2\sqrt{2}} & 0\\ 0 & 1\end{pmatrix}. \tag{2.13}$$

If we put

$$S(n)=\begin{pmatrix}\dfrac{2+\sum_{l=0}^{n-2}e^{-l(l+1)}}{2+\sum_{l=0}^{n-3}e^{-l(l+1)}} & 0\\ 0 & 1\end{pmatrix},\quad n\in\mathbb{N}(0), \tag{2.14}$$

where $\sum_{l=0}^{-3}=\sum_{l=0}^{-2}=-1$ and $\sum_{l=0}^{-1}=0$, then $S(n)$ and $S^{-1}(n)$ are bounded nonsingular matrices.

Moreover, we have

$$\begin{aligned}
&S(n+1)B(n)-A(n)S(n)\\
&\quad=\begin{pmatrix}\dfrac{2+\sum_{l=0}^{n-1}e^{-l(l+1)}}{2+\sum_{l=0}^{n-2}e^{-l(l+1)}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dfrac{e^{-n}}{2\sqrt{2}} & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}e^{-n} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dfrac{2+\sum_{l=0}^{n-2}e^{-l(l+1)}}{2+\sum_{l=0}^{n-3}e^{-l(l+1)}} & 0\\ 0 & 1\end{pmatrix}\\
&\quad=\begin{pmatrix}p(n) & 0\\ 0 & 0\end{pmatrix}=F(n),
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
p(n)&=\left(\frac{2+\sum_{l=0}^{n-1}e^{-l(l+1)}}{2+\sum_{l=0}^{n-2}e^{-l(l+1)}}\right)\frac{e^{-n}}{2\sqrt{2}}-\left(\frac{2+\sum_{l=0}^{n-2}e^{-l(l+1)}}{2+\sum_{l=0}^{n-3}e^{-l(l+1)}}\right)e^{-n},\\
F(n)&=\begin{pmatrix}p(n) & 0\\ 0 & 0\end{pmatrix}.
\end{aligned} \tag{2.16}$$

Thus we have

$$\begin{aligned}
\sum_{n=0}^{\infty}|F(n)|&\le\sum_{n=0}^{\infty}e^{-n}\left|\left(1+\frac{e^{-n(n-1)}}{2+\sum_{l=0}^{n-2}e^{-l(l+1)}}\right)-\left(1+\frac{e^{-(n-1)(n-2)}}{2+\sum_{l=0}^{n-3}e^{-l(l+1)}}\right)\right|\\
&\le\sum_{n=0}^{\infty}e^{-n^2}+\sum_{n=0}^{\infty}e^{-n(n-2)}<\infty.
\end{aligned} \tag{2.17}$$

This implies that A and B are n_∞-summably similar.

Remark 2.7. We can easily show that the n_∞-summable similarity is an equivalence relation by the same method of Trench in [4]. Also if A and B are n_∞-summably similar with $F(n)=0$, then we say that they are *kinematically similar*.

3. *h*-stability in variation for nonlinear difference systems

For the linear difference systems, Medina and Pinto [13] showed that

$$\text{GhSV}\Longleftrightarrow\text{GhS}\Longleftrightarrow\text{hS}\Longleftrightarrow\text{hSV}. \tag{3.1}$$

Also, the associated variational system inherits the property of hS from the original nonlinear system. That is, (2.2) is hS when (2.1) is hS in [13, Theorem 2]. Our purpose is to characterize the global stability in variation via n_∞-summable similarity and Lyapunov functions. To do this, we need the following lemmas.

Lemma 3.1 [13]. *The linear difference system*

$$y(n+1) = A(n)y(n), \qquad y(n_0) = y_0, \tag{3.2}$$

where $A(n)$ is an $m\times m$ matrix, is an h-system if and only if there exist a constant $c \geq 1$ and a positive function h defined on $\mathbb{N}(n_0)$ such that for every $y_0 \in \mathbb{R}^m$,

$$|\Phi(n,n_0,y_0)| \leq ch(n)h^{-1}(n_0), \tag{3.3}$$

for $n \geq n_0$, where Φ is a fundamental matrix solution of (3.2).

Lemma 3.2. *If two matrix-valued functions A and B in the set $\mathfrak{M}$ are n_∞-summably similar, then for $n \geq n_0$, one has*

$$X^{-1}(n)S(n)Y(n) = X^{-1}(n_0)S(n_0)Y(n_0) + \sum_{l=n_0}^{n-1} X^{-1}(l+1)F(l)Y(l), \tag{3.4}$$

where X and Y are fundamental matrix solutions of the linear homogeneous difference system (3.2) with the coefficient matrix functions $A(n)$ and $B(n)$, respectively.

Proof. Note that $A(n) = X(n+1)X^{-1}(n)$ and $B(n) = Y(n+1)Y^{-1}(n)$. Since A and B are n_∞-summably simliar, we can rewrite (2.12) as

$$F(n) = S(n+1)Y(n+1)Y^{-1}(n) - X(n+1)X^{-1}(n)S(n), \tag{3.5}$$

for some $S \in \mathfrak{S}$ and $m\times m$ matrix $F(n)$ with an absolutely summable property over $\mathbb{N}(n_0)$. Thus we easily obtain

$$\begin{aligned} &X^{-1}(n+1)F(n)Y(n) \\ &\quad = X^{-1}(n+1)S(n+1)Y(n+1) - X^{-1}(n)S(n)Y(n) = \Delta\big(X^{-1}(n)S(n)Y(n)\big). \end{aligned} \tag{3.6}$$

Summing this difference equation (3.6) from $l = n_0$ to $l = n-1$ yields the difference equation (3.4). This completes the proof. □

Lemma 3.3. *Assume that $f_x(n,0)$ is n_∞-summably similar to $f_x(n,x(n,n_0,x_0))$ for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and $\sum_{n=n_0}^{\infty}(h(n)/h(n+1))|F(n)| < \infty$. Then (2.3) is an h-system provided (2.2) is an h-system with the positive function $h(n)$ defined on $\mathbb{N}(n_0)$.*

Proof. It follows from Lemma 3.1 that there exist a constant $c \geq 1$ and a positive function h defined on $\mathbb{N}(n_0)$ such that for every $x_0 \in \mathbb{R}^m$,

$$|\Phi(n,n_0,0)| \leq ch(n)h^{-1}(n_0) \tag{3.7}$$

for all $n \geq n_0 \geq 0$, where $\Phi(n,n_0,0)$ is a fundamental matrix solution of (2.2). Let $\Phi(n,n_0,x_0)$ denote a fundamental matrix solution of (2.3). Since $\Phi(n,n_0,0)$ and $\Phi(n,n_0,x_0)$ are fundamental matrix solutions of the variational systems (2.2) and (2.3), respectively, they satisfy

$$\begin{aligned}\Phi(n+1,n_0,0) &= f_x(n,0)\Phi(n,n_0,0),\\ \Phi(n+1,n_0,x_0) &= f_x(n,x(n))\Phi(n,n_0,x_0).\end{aligned} \tag{3.8}$$

Note that

$$\Phi(n,n_0,x_0) = \Phi(n,l,x(l,n_0,x_0))\Phi(l,n_0,x_0) \tag{3.9}$$

for all $n \geq n_0 \geq 0$. Then we have

$$\Phi(n,n_0,x_0) = S^{-1}(n)\left[\Phi(n,n_0,0)S(n_0) + \sum_{l=n_0}^{n-1}\Phi(n,l+1,0)F(l)\Phi(l,n_0,x_0)\right], \tag{3.10}$$

in view of Lemma 3.2. Then, from Lemma 3.1 and the boundedness of $S(n)$ and $S^{-1}(n)$, there are positive constants c_1 and c_2 such that

$$|\Phi(n,n_0,x_0)| \leq c_1c_2h(n)h^{-1}(n_0) + c_1c_2\sum_{l=n_0}^{n-1}h(n)h^{-1}(l+1)|F(l)||\Phi(l,n_0,x_0)|. \tag{3.11}$$

It follows that

$$|\Phi(n,n_0,x_0)|h^{-1}(n) \leq c_1c_2h^{-1}(n_0) + c_1c_2\sum_{l=n_0}^{n-1}\frac{h(l)}{h(l+1)}|F(l)|h^{-1}(l)|\Phi(l,n_0,x_0)|. \tag{3.12}$$

Applying the discrete Bellman's inequality [14], we have

$$\begin{aligned}|\Phi(n,n_0,x_0)| &\leq dh(n)h^{-1}(n_0)\prod_{l=n_0}^{n-1}\left(1+\frac{h(l)}{h(l+1)}|F(l)|\right)\\ &\leq dh(n)h^{-1}(n_0)\exp\left(\sum_{l=n_0}^{n-1}\frac{h(l)}{h(l+1)}|F(l)|\right)\\ &\leq ch(n)h^{-1}(n_0),\end{aligned} \tag{3.13}$$

where $c = d\exp(\sum_{l=n_0}^{\infty}(h(l)/h(l+1))|F(l)|)$ and $d = c_1c_2$.

Therefore

$$|\Phi(n,n_0,x_0)| \le ch(n)h^{-1}(n_0), \quad n \ge n_0 \ge 0, \tag{3.14}$$

for some positive constant $c \ge 1$. This implies that (2.3) is an h-system. □

COROLLARY 3.4. *Under the same conditions of Lemma 3.3, (2.1) is hSV.*

Letting $h(n)$ be bounded on $\mathbb{N}(n_0)$, we obtain the following result [13, Theorem 4] as a corollary of Lemma 3.3.

COROLLARY 3.5. *If (2.2) is hS and for some* $\delta > 0$,

$$\sum_{l=n_0}^{\infty} \frac{h(l)}{h(l+1)} |f_x(l,x(n,n_0,x_0)) - f_x(l,0)| < \infty, \quad n_0 \ge 0 \tag{3.15}$$

for $|x_0| \le \delta$*, holds, then (2.3) is also hS.*

Proof. Setting $F(n) = f_x(n,x(n,n_0,x_0)) - f_x(n,0)$ and $S(n) = I$, for $n \ge n_0 \ge 0$, we can easily see that $f_x(n,x(n,n_0,x_0))$ and $f_x(n,0)$ are n_∞-summably similar. Thus all conditions of Lemma 3.3 are satisfied, and hence (2.3) is hS. □

Remark 3.6. If $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$, then $h(n)/h(n+1)$ is not bounded in general.

For example, letting $h(n) = \exp(-\sum_{s=n_0}^{n-1} s)$, $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$ but $\lim_{n\to\infty}(h(n)/h(n+1)) = \lim_{n\to\infty}\exp(n) = \infty$. Thus if $h(n)/h(n+1)$ is bounded, then the condition $(h(n)/h(n+1))|F(n)| \in l_1(\mathbb{N}(n_0))$ in Lemma 3.3 can be replaced by $|F(n)| \in l_1(\mathbb{N}(n_0))$.

THEOREM 3.7. *Assume that $f_x(n,0)$ is n_∞-summably similar to $f_x(n,x(n,n_0,x_0))$ for $n \ge n_0 \ge 0$ and every $x_0 \in \mathbb{R}^m$ with $(h(n)/h(n+1))|F(n)| \in l_1(\mathbb{N}(n_0))$. Then (2.1) is GhS if and only if there exists a function $V(n,z)$ defined on $\mathbb{N}(n_0)\times\mathbb{R}^m$ such that the following properties hold:*

(i) *$V(n,z)$ is defined on $\mathbb{N}(n_0)\times\mathbb{R}^m$ and continuous with respect to the second argument;*

(ii) *$|x-y| \le V(n,x-y)| \le c|x-y|$, for $(n,x,y) \in \mathbb{N}(n_0)\times\mathbb{R}^m\times\mathbb{R}^m$;*

(iii) *$|V(n,z_1) - V(n,z_2)| \le c|z_1 - z_2|$, for $n \in \mathbb{N}(n_0)$, $z_1,z_2 \in \mathbb{R}^m$;*

(iv) *$\Delta V(n,x-y)/V(n,x-y) \le \Delta h(n)/h(n)$, for $(n,x,y) \in \mathbb{N}(n_0)\times\mathbb{R}^m\times\mathbb{R}^m$ with $x \ne y$.*

Proof. Define the function V by

$$V(n,x-y) = \sup_{\tau\in\mathbb{N}(0)} |x(n+\tau,n,x) - x(n+\tau,n,y)|\, h^{-1}(n+\tau)h(n). \tag{3.16}$$

Then, this theorem can be easily proved by following the proof of Theorem 2.1 in [6] and and Theorem 3.2 in [12]. □

Note that Theorem 3.2 in [12] was improved by Theorem 2.1 in [6] and our Theorem 3.7 as we replace the fundamental matrix $\Phi(n+1,n_0,x_0)$ by $\Phi(n,n_0,x_0)$ in [12, Theorems 3.1 and 3.2]. See [6, Remark 2.1].

4. Asymptotic equilibrium of linear difference systems

We consider two linear systems

$$x(n+1) = A(n)x(n), \tag{4.1}$$

$$y(n+1) = B(n)y(n), \tag{4.2}$$

where A and B are nonsingular $m \times m$ matrix-valued functions defined on $\mathbb{N}(n_0)$.

Lemma 4.1 [4, Theorem 1]. *Equation (4.1) has asymptotic equilibrium if and only if* $\lim_{n\to\infty} X(n)$ *exists and is invertible, where* $X(n)$ *is a fundamental matrix solution of (4.1).*

Lemma 4.2. *If (4.1) has asymptotic equilibrium, then (4.1) is strongly stable.*

Proof. It follows from Lemma 4.1 that

$$\lim_{n\to\infty} X(n)X^{-1}(n) = \lim_{n\to\infty} X(n) \lim_{n\to\infty} X^{-1}(n) = X_\infty \lim_{n\to\infty} X^{-1}(n) = I, \tag{4.3}$$

where $X_\infty = \lim_{n\to\infty} X(n)$ is invertible. Then we obtain

$$\lim_{n\to\infty} X^{-1}(n) = X_\infty^{-1}. \tag{4.4}$$

Hence there exists a positive constant M such that

$$|X(n)| \le M, \quad |X^{-1}(n)| \le M, \quad n \ge n_0. \tag{4.5}$$

This implies that (4.1) is strongly stable by [14, Theorem 5.5.1]. □

Example 4.3. We give an example which shows the converse of Lemma 4.2 is not true in general. We consider the difference system

$$x(n+1) = A(n)x(n) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x(n), \quad n \ge 0, \tag{4.6}$$

where $A(n) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the invertible 2×2 matrix.

Then we easily see that a fundamental matrix solution $X(n)$ of (4.6) is given by

$$X(n) = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix} = X^{-1}(n), \quad n \ge 0, \tag{4.7}$$

and there exists a positive constant $M \ge 2$ such that

$$|X(n)| \le M, \quad |X^{-1}(n)| \le M, \quad n \ge 0. \tag{4.8}$$

Thus (4.6) is strongly stable. But, since $\lim_{n\to\infty} X(n)$ does not exist, (4.6) does not have asymptotic equilibrium.

The following lemma comes from [4, Theorem 4].

Lemma 4.4. *Assume that two matrix-valued functions A and B are n_∞-summably similar. If (4.1) is strongly stable, then (4.2) is also strong stable.*

Proof. From [4, Theorem 1], we see that $|X(n)X^{-1}(m)|$ is bounded for each $n, m \geq n_0$. Thus it suffices to show that $|Y(n)Y^{-1}(m)|$ is also bounded for each $n, m \geq n_0$. First, it follows from Lemma 3.3 that

$$|Y(n)Y^{-1}(m)| = |Y(n,m)| \leq d\exp\left(\sum_{l=n_0}^{\infty}\frac{h(l)}{h(l+1)}|F(l)|\right) \leq M, \tag{4.9}$$

for each $n \geq m \geq n_0$ and by letting $h(n) = 1^n$. Next, we show that $|Y(n)Y^{-1}(m)|$ is also bounded for each $n_0 \leq n \leq m$. Summing (3.6) from $l = n$ to $l = m-1$ yields

$$X^{-1}(n)S(n)Y(n) = X^{-1}(m)S(m)Y(m) - \sum_{l=n}^{m-1}X^{-1}(l+1)F(l)Y(l). \tag{4.10}$$

Then we have

$$Y(n)Y^{-1}(m) = S^{-1}(n)X(n)X^{-1}(m)S(m) - S^{-1}(n)\sum_{l=n}^{m-1}X(n)X^{-1}(l+1)F(l)Y(l)Y^{-1}(m), \tag{4.11}$$

for each $n_0 \leq n \leq m$. From this and the strong stability of (4.1), there exist two positive constants α and β such that

$$\begin{aligned} |S^{-1}(n)X(n)X^{-1}(m)S(m)| &\leq \alpha, \quad n \leq m,\\ |S^{-1}(n)X(n)X^{-1}(l+1)| &\leq \beta, \quad n \leq l \leq m-1. \end{aligned} \tag{4.12}$$

Thus we obtain

$$\begin{aligned} |Y(n)Y^{-1}(m)| &\leq \alpha + \beta\sum_{l=n}^{m-1}|F(l)||Y(l)Y^{-1}(m)|\\ &= v_{m,n}, \quad n_0 \leq n \leq m, \end{aligned} \tag{4.13}$$

where $v_{m,n} = \alpha + \beta\sum_{l=n}^{m-1}|F(l)||Y(l)Y^{-1}(m)|$. Since

$$v_{m,n+1} - v_{m,n} = -\beta|F(n)||Y(n)Y^{-1}(m)| \geq -\beta|F(n)|v_{m,n}, \quad n_0 \leq n \leq m, \tag{4.14}$$

we have

$$v_{m,n+1} \geq (1 - \beta|F(n)|)v_{m,n}, \quad n_0 \leq n \leq m. \tag{4.15}$$

Since $\sum_{n=n_0}^{\infty}|F(n)| < \infty$, we can choose $m_0 \geq n_0$ so large that $\beta|F(n)| < 1/2$ for each $n \geq m_0$. Then we have

$$\frac{1}{1-\beta|F(n)|} \leq 1 + 2\beta|F(n)|, \quad n \geq m_0 \geq n_0. \tag{4.16}$$

Thus (4.13) implies that

$$v_{m,n} \leq v_{m,n+1}(1 + 2\beta|F(n)|), \quad v_{n,n} = \alpha,\ n \geq m_0 \geq n_0. \tag{4.17}$$

It follows from the easy calculation that

$$v_{m,n} \le \alpha \prod_{l=n}^{m-1} \left(1 + 2\beta |F(l)|\right) \le \alpha \exp\left(\sum_{l=n}^{m-1} 2\beta |F(l)|\right) \le M, \tag{4.18}$$

where $M = \alpha \exp(\sum_{l=n_0}^{\infty} 2\beta |F(l)|)$. In view of inequality (4.13), we have

$$|Y(n)Y^{-1}(m)| \le M, \quad n_0 \le m_0 \le n \le m. \tag{4.19}$$

Also, we can easily see that this estimation holds for each $n, m \ge n_0$. This completes the proof. □

We remark that for linear homogeneous systems, restrictive stability and strong stability are equivalent [14, Theorem 5.5.2]. Also the linear difference system is restrictively stable if and only if it is reducible to zero [14, Theorem 5.5.3]. Lemma 4.4 can be easily proved by using the notion of reducibility in [14]. The linear difference system (4.1) is *reducible (reducible to zero)* if there exists an $m \times m$ matrix $L(n)$ which, together with its inverse $L^{-1}(n)$, is defined and bounded on $\mathbb{N}(n_0)$ such that $L^{-1}(n+1)A(n)L(n)$ is a constant (identity) matrix on $\mathbb{N}(n_0)$.

Corollary 4.5. *Assume that two matrix-valued functions A and B are n_∞-summably similar with $F(n) = 0$. If (4.1) is strongly stable, then (4.2) is also strongly stable.*

Proof. Since (4.1) is strongly stable, there exists an $m \times m$ matrix $L(n)$ which, together with its inverse $L^{-1}(n)$, is defined and bounded on $\mathbb{N}(n_0)$ such that $L^{-1}(n+1)A(n)L(n)$ is the identity matrix on $\mathbb{N}(n_0)$ by Theorem 5.5.5 in [14]. Putting $T(n) = S^{-1}(n)L(n)$, we obtain

$$\begin{aligned} T^{-1}(n+1)B(n)T(n) &= L^{-1}(n+1)S(n+1)B(n)S^{-1}(n)L(n) \\ &= L^{-1}(n+1)S(n+1)S^{-1}(n+1)L(n+1)L^{-1}(n)S(n)S^{-1}(n)L(n) \\ &= I, \end{aligned} \tag{4.20}$$

by the definition of n_∞-similarity between A and B. Thus (4.2) is reducible to zero. It follows from [14, Theorems 5.5.2 and 5.5.3] that (4.2) is strongly stable. □

The following theorem means that asymptotic equilibrium for linear system is preserved by the notion of n_∞-summable similarity.

Theorem 4.6. *Suppose that two matrix-valued functions A and B are n_∞-summably similar with $\lim_{n\to\infty} S(n) = S_\infty < \infty$. If (4.1) has asymptotic equilibrium, then (4.2) also has asymptotic equilibrium.*

Proof. It follows from Lemmas 4.2 and 4.13 that (4.2) is strongly stable. In particular, $Y^{-1}(n)$ is bounded. Also, our assumption on $S(n)$ implies that $\lim_{n\to\infty} S(n) = S_\infty$ is invertible and $\lim_{n\to\infty} S^{-1}(n) = S_\infty^{-1}$. Since $\sum_{n=n_0}^{\infty} |F(n)| < \infty$, we easily see that $Y(n)$ is Cauchy. It follows from the boundedness of $Y^{-1}(n)$ that $\lim_{n\to\infty} Y(n) = Y_\infty$ is invertible. Therefore (4.2) has asymptotic equilibrium by Lemma 4.1. □

By using asymptotic equilibria of linear difference systems, we obtain the asymptotic equivalence between two linear difference systems (4.1) and (4.2).

Theorem 4.7. *In addition to the assumption of Theorem 4.6 assume that* $\lim_{n\to\infty} X(n) = X_\infty$ *exists and* $|\det(X(n))| > \alpha > 0$ *for each* $n \geq n_0$ *and some positive constant* α. *Then (4.1) and (4.2) are asymptotically equivalent.*

Proof. We easily see that (4.1) and (4.2) have asymptotic equilibria by the assumption and Theorem 4.6. Let $x(n,n_0,x_0)$ be any solution of (4.1). Then $\lim_{n\to\infty} x(n) = x_\infty$ exists. For $x_\infty \in \mathbb{R}^m$, the condition on asymptotic equilibrium for (4.2) implies that there exists a solution $y(n,n_0,y_0)$ of (4.2) such that $\lim_{n\to\infty} y(n) = x_\infty$. This implies that

$$y(n) = x(n) + o(1) \quad \text{as } n \to \infty. \tag{4.21}$$

Also, the converse asymptotic relationship holds. □

Next, we study the asymptotic equivalence between homogeneous linear system and nonhomogeneous system by means of asymptotic equilibrium of homogeneous system. So we consider the perturbation of (4.1)

$$x(n+1) = A(n)x(n) + g(n), \tag{4.22}$$

where $g(n)$ is a vector function on $\mathbb{N}(n_0)$.

Lemma 4.8. *Assume that (4.1) has asymptotic equilibrium and* $|g(n)| \in l_1(\mathbb{N}(n_0))$. *Then (4.22) has also asymptotic equilibrium.*

Proof. Let $y(n,n_0,y_0)$ be any solution (4.22). Then the solution $y(n)$ of (4.22) is given by

$$y(n) = \Psi(n,n_0)y_0 + \Psi(n,n_0)\sum_{s=n_0}^{n-1}\Psi^{-1}(s+1,n_0)g(s), \tag{4.23}$$

where $\Psi(n,n_0)$ is a fundamental matrix solution of (4.1). Putting $p(n) = \sum_{s=n_0}^{n-1}\Psi^{-1}(s+1,n_0)g(s)$, we easily see that $p(n)$ is Cauchy by the fact that $|g(n)| \in l_1(\mathbb{N}(n_0))$ and the boundedness of $\Psi^{-1}(n)$. Thus $y(n)$ converges to a vector $\xi \in \mathbb{R}^m$.

Conversely, let ξ be any vector in $\mathbb{R}^m$. Then there exists a solution $y(n,n_0,y_0)$ of (4.22) with the initial point $y_0 = \Psi_\infty^{-1}\xi - p_\infty$ such that

$$\begin{aligned}\lim_{n\to\infty} y(n) &= \lim_{n\to\infty}\left[\Psi(n,n_0)y_0 + \Psi(n,n_0)\sum_{s=n_0}^{n-1}\Psi^{-1}(s+1,n_0)g(s)\right]\\ &= \Psi_\infty[y_0 + p_\infty]\\ &= \Psi_\infty[\Psi_\infty^{-1}\xi - p_\infty + p_\infty]\\ &= \xi,\end{aligned} \tag{4.24}$$

where $\lim_{n\to\infty} p(n) = p_\infty$ and $\lim_{n\to\infty}\Psi(n) = \Psi_\infty$. This completes the proof. □

As a consequence of Lemma 4.8, we easily obtain the following result.

THEOREM 4.9. *Suppose that (4.1) has asymptotic equilibrium and* $|g(n)| \in l_1(\mathbb{N}(n_0))$. *Then (4.1) and (4.22) are asymptotically equivalent.*

Proof. Let $x(n)$ be any solution of (4.1). Then we have $\lim_{n\to\infty} x(n) = x_\infty$ by means of asymptotic equilibrium of (4.1). Setting $y_0 = \Psi_\infty^{-1}x_\infty - p_\infty$ as in Lemma 4.8, there exists a solution $y(n,n_0,y_0)$ of (4.22) such that

$$\begin{aligned}\lim_{n\to\infty}\left[y(n)-x(n)\right] &= \Psi_\infty\left[y_0+p_\infty-x_\infty\right]\\ &= \Psi_\infty\left[\left(\Psi_\infty^{-1}x_\infty-p_\infty\right)+p_\infty-x_\infty\right]\\ &= 0.\end{aligned} \tag{4.25}$$

Conversely, we easily see that the asymptotic relationship also holds by setting $x_0 = y_0 + p_\infty$. This completes the proof. □

Remark 4.10. Note that we can obtain the same result as Theorem 4.9 by putting $y_0 = x_0 - p_\infty$ in the process of the proof. Also, note that the difference system does not have asymptotic equilibrium even though it is asymptotically stable.

We give an example to illustrate Theorem 4.9.

Example 4.11. Consider the homogeneous difference equation

$$x(n+1) = A(n)x(n) = \left(1+a^n\right)x(n) \tag{4.26}$$

and nonhomogeneous difference equation

$$y(n+1) = A(n)y(n)+g(n) = \left(1+a^n\right)y(n)+\alpha^n, \tag{4.27}$$

where $A(n) = 1+a^n$ with the constant a $(0<a<1)$ and $g(n) = \alpha^n$ with $0<\alpha<1$. Then (4.26) and (4.27) are asymptotically equivalent.

Proof. A fundamental matrix solution $\Psi(n,n_0)$ of (4.26) is given by $\prod_{s=n_0}^{n-1}(1+a^s)$. Note that $\Psi(n,n_0)$ is bounded since $1+a^n \le \exp a^n$ for $n \ge n_0 \ge 0$ and is nondecreasing on $\mathbb{N}(n_0)$. Thus $\lim_{n\to\infty}\Psi(n,n_0) = \Psi_\infty$ exists and is a nonzero constant. In fact, this implies that

$$\lim_{n\to\infty}\Psi^{-1}\left(n,n_0\right) = \Psi_\infty^{-1}. \tag{4.28}$$

Hence it follows from Lemma 4.1 that (4.26) has asymptotic equilibrium. Also, the solution $y(n,n_0,y_0)$ of (4.27) is given by

$$y(n) = \prod_{s=n_0}^{n-1}\left(1+a^s\right)y_0 + \sum_{s=n_0}^{n-1}\left[\prod_{\tau=s+1}^{n-1}\left(1+a^\tau\right)\alpha^s\right], \quad n \ge n_0 \ge 0. \tag{4.29}$$

Since $\alpha^n \in l_1(\mathbb{N}(n_0))$ and all conditions of Lemma 4.8 are satisfied, we see that (4.27) has asymptotic equilibrium. Therefore two systems (4.26) and (4.27) are asymptotically equivalent by Theorem 4.9. This completes the proof. □

5. Variationally asymptotic equilibrium of nonlinear difference systems

In this section, we study the asymptotic equilibrium of nonlinear difference system by using n_∞-summable similarity. Furthermore, we show that two concepts of asymptotic equilibrium and asymptotic equilibrium in variation for nonlinear difference systems are equivalent.

Setting $f_x(n,0) = A(n)$ and using the mean value theorem, the nonlinear difference system (2.1) can be written as

$$\begin{aligned} x(n+1) &= A(n)x(n) + f(n,x(n)) - f_x(n,0)x(n) \\ &= A(n)x(n) + G(n,x(n)), x(n_0) = x_0, \end{aligned} \tag{5.1}$$

where $G(n,x) = \int_0^1 [f_x(n,\theta x) - f_x(n,0)]\,d\theta x$.

We show that the associated variational difference system (2.2) inherits the property of asymptotic equilibrium from the original nonlinear difference system (2.1) in the following theorem.

Theorem 5.1. *If (2.1) has asymptotic equilibrium, then (2.2) has also asymptotic equilibrium.*

Proof. We begin by showing that a fundamental matrix $\Phi(n,n_0,0)$ of (2.2) given by $(\partial/\partial x_0)x(n,n_0,0)$ is convergent as $n \to \infty$. Let x_0 be a vector of length δ in the jth coordinate direction for each $j = 1,\dots,m$. Then the hypothesis implies that $\lim_{n\to\infty} x(n,n_0,x_0) = x_\infty$ exists for fixed nonzero δ. For any given $\varepsilon > 0$, there exists a positive integer N such that $|x(n,n_0,x_0) - x(m,n_0,x_0)| < |\delta|^2$ for any $n,m \geq N$ and $j = 1,\dots,m$, since $x(n,n_0,x_0)$ is Cauchy for each $j = 1,\dots,m$. Then we obtain for each $j = 1,\dots,m$,

$$\begin{aligned} &\left| \frac{\partial}{\partial x_{0j}} x(n,n_0,0) - \frac{\partial}{\partial x_{0j}} x(m,n_0,0) \right| \\ &\quad = \left| \lim_{\delta\to 0} \frac{x(n,n_0,x_0) - x(n,n_0,0)}{\delta} - \lim_{\delta\to 0} \frac{x(m,n_0,x_0) - x(m,n_0,0)}{\delta} \right| \\ &\quad = \left| \lim_{\delta\to 0} \frac{x(n,n_0,x_0) - x(m,n_0,x_0)}{\delta} \right| < \lim_{\delta\to 0} \frac{|\delta^2|}{|\delta|} < \varepsilon, \quad \text{for } n,m \geq N. \end{aligned} \tag{5.2}$$

This implies that $\lim_{n\to\infty} \Phi(n,n_0,0) = \Phi_\infty$ exists.

Now, by using Lemma 4.1, it suffices to prove that limit Φ_∞ is invertible. Given linearly independent vectors $\hat{x}_{0j} \in \mathbb{R}^m$ in the j-coordinate direction for each $j = 1,\dots,m$, it follows from the asymptotic equilibrium of (2.1) that there exist the solutions $x_j(n,n_0,x_{0j})$

of (2.1) which are convergent to $\delta\hat{x}_{0j}$ for each $j = 1,\dots,m$ and fixed $\delta \neq 0$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty}\Phi(n,n_0,0)\\
&\quad= \lim_{n\to\infty}\left[\frac{\partial}{\partial x_{01}}x_1(n,n_0,x_{01}),\dots,\frac{\partial}{\partial x_{0m}}x_m(n,n_0,x_{0m})\right]\\
&\quad= \lim_{n\to\infty}\left[\lim_{\delta\to 0}\frac{x_1(n,n_0,x_{01})-x_1(n,n_0,0)}{\delta},\dots,\lim_{\delta\to 0}\frac{x_m(n,n_0,x_{0m})-x_m(n,n_0,0)}{\delta}\right]\\
&\quad= \left[\lim_{\delta\to 0}\frac{\lim_{n\to\infty}x_1(n,n_0,x_{01})}{\delta},\dots,\lim_{\delta\to 0}\frac{\lim_{n\to\infty}x_m(n,n_0,x_{0m})}{\delta}\right]\\
&\quad= [\hat{x}_{01},\dots,\hat{x}_{0m}] = \Phi_\infty.
\end{aligned} \tag{5.3}$$

Since the vectors $\hat{x}_{01},\dots,\hat{x}_{0m}$ are linearly independent, Φ_∞ is invertible. This completes the proof. □

Note that the converse of Theorem 5.1 does not hold in general. We give the following example.

Example 5.2. We consider the nonlinear difference equation

$$x(n+1) = f(n,x(n)) = x(n)+x^2(n), x(n_0) = x_0 = 1 \tag{5.4}$$

and its variational difference equation

$$v(n+1) = f_x(n,0)v(n) = v(n), \quad v(n_0) = v_0 \neq 0, \tag{5.5}$$

where $f_x(n,x) = 1+2x$.

Since the fundamental solution $\phi(n) = 1$ of (5.5) is nonzero, (5.5) has asymptotic equilibrium. But (5.4) does not have asymptotic equilibrium because of the unboundedness of the solution $x(n,n_0,x_0)$ of (5.4). In fact, there exists a solution $x(n,0,1)$ of (5.4) such that

$$x(n,0,1) = x(n) > n, \tag{5.6}$$

for each $n \geq 1$.

Now, for the converse of Theorem 5.1, we examine the asymptotic equilibrium for the perturbed system of linear difference system (2.2) by using the comparison principle. To do this we need the following comparison principle in [5] which is a slight modification of [10].

Lemma 5.3 [5, Lemma 9]. *Let $\iota(n,r)$ be a nondecreasing function in r for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for $n \geq n_0$,*

$$v(n) - \sum_{l=n_0}^{n-1}\iota(l,v(l)) < u(n) - \sum_{l=n_0}^{n-1}\iota(l,u(l)). \tag{5.7}$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$, for all $n \geq n_0$.

Theorem 5.4. *Assume that*

(i) *equation (2.2) has asymptotic equilibrium,*

(ii) *for each $n \geq s \geq n_0$ and $x \in \mathbb{R}^m$,*

$$|G(n,x)| \leq \omega(n,|x|), \tag{5.8}$$

where $\omega : \mathbb{N}(n_0) \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega(n,u)$ is continuous and nondecreasing in u for $n \geq n_0$.

Also, we consider the scalar difference equation

$$u(n+1) = u(n) + M^2\omega(n,u(n)), \quad u(n_0) = u_0 > 0, \tag{5.9}$$

where M is a positive constant, and suppose that

(iii) *all solutions of (5.9) are bounded on $\mathbb{N}(n_0)$.*

Then (2.1) has asymptotic equilibrium provided $d < u_0$ with $d = M|x_0|$.

Proof. Let $x(n,n_0,x_0)$ be any solution of (2.1). From the variation of constants formula in [14] and conditions (i) and (ii), we obtain

$$\begin{aligned} |x(n)| &= \left|\Psi(n,n_0)\left[x_0 + \sum_{s=n_0}^{n-1} \Psi^{-1}(s+1,n_0)G(s,x(s))\right]\right| \\ &\leq |\Psi(n,n_0)|\,|x_0| + |\Psi(n,n_0)| \sum_{s=n_0}^{n-1} |\Psi^{-1}(s+1,n_0)|\,|G(s,x(s))| \\ &\leq M|x_0| + M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|) \\ &= d + M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|), \end{aligned} \tag{5.10}$$

where M is a bounded constant of $\Psi(n,m)$ for each $n,m \geq n_0$ and $d = M|x_0|$. Then we have the following summable inequality:

$$|x(n)| - M^2 \sum_{s=n_0}^{n-1} \omega(s,|x(s)|) = d < u_0 = u(n) - M^2 \sum_{s=n_0}^{n-1} \omega(s,u(s)). \tag{5.11}$$

By letting $\iota(n,u) = M^2\omega(s,u)$ and using Lemma 5.3, we obtain

$$|x(n)| \leq u(n), \quad \text{for each } n \geq n_0, \tag{5.12}$$

provided $d < u_0$.

Now, we prove that the solution $x(n)$ of (2.1) which can be written as (5.1) is convergent. Consider the sequence

$$v(n) = \sum_{s=n_0}^{n-1} \Psi^{-1}(s+1,n_0)G(s,x(s)). \tag{5.13}$$

By using the monotonicity of the function ω and asymptotic equilibrium of (2.2), we obtain

$$\begin{aligned} |v(n)-v(m)| &\le \sum_{s=m}^{n-1} |\Psi^{-1}(s+1,n_0)|\,|G(s,x(s))| \\ &\le M\sum_{s=m}^{n-1}\omega(s,|x(s)|) \le M\sum_{s=m}^{n-1}\omega(s,u(s)) \\ &= M(u(n)-u(m)), \end{aligned} \tag{5.14}$$

for any $n \ge m \ge n_0$. Since $u(n)$ is convergent, $v(n)$ is Cauchy. Thus $v(n)$ is also convergent. Hence there exist a vector $\xi \in \mathbb{R}^m$ and $r > 0$ such that any solution $x(n,n_0,x_0)$ of (5.1) with $|x_0| < r$ satisfies the following asymptotic relationship:

$$x(n) = \xi + o(1) \quad \text{as } n \longrightarrow \infty. \tag{5.15}$$

Conversely, let $\xi \in \mathbb{R}^m$ be any vector. Setting $x_0 = \Psi_\infty^{-1}\xi - v_\infty$ with $v_\infty = \lim_{n\to\infty} v(n)$, any solution $x(n)$ of (5.1) equivalent to (2.1) satisfies

$$\begin{aligned} x(n,n_0,x_0) &= \Psi(n,n_0)\left[x_0 + \sum_{s=n_0}^{\infty}\Psi^{-1}(s+1,n_0)G(s,x(s))\right] \\ &\quad - \Psi(n,n_0)\sum_{s=n}^{\infty}\Psi^{-1}(s+1,n_0)G(s,x(s)) \\ &= \xi + o(1) \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{5.16}$$

since

$$\Psi(n,n_0)\sum_{s=n}^{\infty}\Psi^{-1}(s+1,n_0)G(s,x(s)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.17}$$

This completes the proof. □

As a consequence of Theorem 5.4 we obtain the following corollary.

Corollary 5.5. *Instead of the condition (i) of Theorem 5.4, one assumes that* $\lim_{n\to\infty}\Phi(n) = \Psi_\infty$ *exists and* $|\det(\Phi(n))| > \alpha > 0$, *for each* $n \ge n_0$ *and some positive constant* α. *Then (2.1) has asymptotic equilibrium.*

Proof. Since $\det\Psi_\infty = \lim_{n\to\infty}\det\Phi(n) \ge \alpha$, Ψ_∞ is invertible. It follows from Lemma 4.1 that (2.2) has asymptotic equilibrium. Hence (2.1) has also asymptotic equilibrium by Theorem 5.4. □

Theorem 5.6. *Let the assumptions be the same as in Theorem 5.4. Then (2.1) and (2.2) are asymptotically equivalent.*

Proof. We can prove this by the same method as in Theorem 4.9. □

Corollary 5.7. *Assume that (2.1) has asymptotic equilibrium and for some $\delta > 0$, one has*

$$\sum_{n=n_0}^{\infty} |f_x(n,0) - f_x(n,x(n,n_0,x_0))| < \infty, \tag{5.18}$$

for some $|x_0| \leq \delta$. Then (2.1) has asymptotic equilibrium in variation.

Proof. It follows from Theorem 5.1 that (2.2) has asymptotic equilibrium. Letting $F(n) = |f_x(n,n_0,0) - f_x(n,x(n,n_0,x_0))|$ with $S(n) = I$ for each $n \geq n_0 \geq 0$, we obtain that $F(n)$ is absolutely summable. Thus $f_x(n,x(n,n_0))$ and $f_x(n,0)$ are n_∞-summably similar. This implies that (2.3) has also asymptotic equilibrium by Theorem 4.6. □

Corollary 5.8. *Let the assumptions except the condition (i) be the same as in Theorem 5.4. Suppose that (2.1) has asymptotic equilibrium in variation and for some $\delta > 0$*

$$\sum_{n=n_0}^{\infty} |f_x(n,0) - f_x(n,x(n,n_0,x_0))| < \infty, \tag{5.19}$$

for some $|x_0| \leq \delta$. Then (2.1) has also asymptotic equilibrium. Furthermore (2.1) and (2.2) are asymptotically equivalent.

Remark 5.9. We see that two concepts of asymptotic equilibrium and asymptotic equilibrium in variation for nonlinear difference system (2.1) are equivalent by means of n_∞-similarity of the associated variational difference systems and the results of Theorem 5.4 and Corollary 5.7.

Example 5.10. To illustrate Theorem 5.6, we consider the nonlinear difference equation

$$x(n+1) = f(n,x(n)) = x(n) + \frac{a^n x(n)}{\sqrt{1+2x^2(n)}} \tag{5.20}$$

and its associated variational difference equation

$$v(n+1) = f_x(n,0)v(n) = (1+a^n)v(n), \tag{5.21}$$

where $f(n,x) = x + a^n x/\sqrt{1+2x^2}$ and $f_x(n,x) = 1 + a^n/(1+2x^2)^{3/2}$ with $0 < a < 1$. Then (5.20) and (5.21) are asymptotically equivalent. Furthermore, (5.20) has asymptotic equilibrium in variation.

Proof. Setting $f_x(n,0) = A(n)$ and using the mean value theorem, (5.20) can be written as

$$x(n+1) = A(n)x(n) + G(n,x(n)) = (1+a^n)x(n) + a^n\left[\frac{1}{\sqrt{1+2x^2(n)}} - 1\right]x(n), \tag{5.22}$$

where $G(n,x) = \int_0^1 [f_x(n,\theta x) - f_x(n,0)]\,d\theta\, x$.

Then we obtain

$$\begin{aligned} |G(n,x)| &\leq \left| a^n \left[\int_0^1 \frac{d\theta}{\sqrt{1+2(\theta x)^2}} - 1 \right] d\theta x \right| \\ &= \left| a^n \left[\frac{1}{\sqrt{1+2x^2}} - 1 \right] x \right| \\ &\leq a^n |x| = \omega(n,|x|), \end{aligned} \tag{5.23}$$

where $\omega(n,u) = a^n u$ is nondecreasing in $u > 0$. For the scalar difference equation

$$u(n+1) = u(n) + M^2 \omega(n,u(n)) = u(n) + M^2 a^n u(n), \quad u(0) = u_0 > 0, \tag{5.24}$$

we have

$$u(n) = u_0 + M^2 \sum_{s=0}^{n-1} a^s = u_0 + M^2 \left(\frac{1-a^n}{1-a} \right), \quad n \geq n_0 = 0. \tag{5.25}$$

Thus all solutions $u(n)$ of (5.24) are bounded on $\mathbb{N}(0)$. Putting $d = \lim_{n\to\infty} \prod_{s=0}^{n-1}(1+a^s)|x_0|$, we easily see that (5.21) has asymptotic equilibrium.

Also, all conditions of Theorem 5.6 are satisfied. It follows that (5.20) and (5.21) are asymptotically equivalent by Theorem 5.6.

Next, we consider associated variational difference equation

$$z(n+1) = f_x(n, x(n,n_0,x_0)) z(n) = \left[1 + \frac{a^n}{\left[1+2x^2(n)\right]^{3/2}} \right] z(n). \tag{5.26}$$

Then $f_x(n,0)$ and $f_x(n,x(n))$ are n_∞-summably similar with $S(n) = I$ and $F(n) = f_x(n,0) - f_x(n,x(n))$. Note that we have

$$\sum_{n=0}^{\infty} |F(n)| = \sum_{n=0}^{\infty} a^n \left[1 - \frac{1}{\left[1+2x^2(n)\right]^{3/2}} \right] \leq \sum_{n=0}^{\infty} a^n < \infty, \tag{5.27}$$

where $0 < a < 1$. It follows from Theorem 4.7 that (5.26) has asymptotic equilibrium. Hence (5.20) has asymptotic equilibrium in variation. This completes the proof. □

Theorem 5.11. *Assume that (2.1) and (2.3) are asymptotically equivalent. If (2.1) has asymptotic equilibrium, then (2.3) has also asymptotic equilibrium. Also the converse holds.*

Proof. Let $z(n,n_0,v_0)$ be any solution of (2.3). Then there exists a solution $x(n)$ of (2.1) such that the following asymptotic relationship holds:

$$z(n) = x(n) + o(1) \quad \text{as } n \longrightarrow \infty, \tag{5.28}$$

by means of asymptotic equivalence between (2.1) and (2.3). Since (2.1) has asymptotic equilibrium, $x(n)$ is convergent to x_∞. This implies that $\lim_{n\to\infty} z(n) = x_\infty$ by the above asymptotic relationship.

For the converse, let ξ be any vector in $\mathbb{R}^m$. Then there exists a solution $z(n)$ of (2.3) such that

$$z(n) = \xi + o(1) \quad \text{as } n \longrightarrow \infty, \tag{5.29}$$

by the above method. Hence (2.3) has asymptotic equilibrium. This completes the proof. □

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