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Research Article

Necessary Conditions of Optimality for Second-Order Nonlinear Impulsive Differential Equations

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We discuss the existence of optimal controls for a Lagrange problem of systems governed by the second-order nonlinear impulsive differential equations in infinite dimensional spaces. We apply a direct approach to derive the maximum principle for the problem at hand. An example is also presented to demonstrate the theory.

1. Introduction

It is well known that Pontryagin maximum principle plays a central role in optimal control theory. In 1960, Pontryagin derived the maximum principle for optimal control problems in finite dimensional spaces (see [1]). Since then, the maximum principle for optimal control problems involving first-order nonlinear impulsive differential equations in finite (or infinite) dimensional spaces has been extensively studied (see [2–10]). However, there are a few papers addressing the existence of optimal controls for the systems governed by the second-order nonlinear impulsive differential equations. By reducing wave equation to the customary vector form, Fattorini obtained the maximum principle for time optimal control problem of the semilinear wave equations (see [6, Chapter 6]). Recently, Peng and Xiang [11, 12] applied the semigroup theory to establish the existence of optimal controls for a class of second-order nonlinear differential equations in infinite dimensional spaces.

Let Y be a reflexive Banach space from which the controls u take the values. We denote a class of nonempty closed and convex subsets of Y by $P_f(Y)$. Assume that the multifunction $\omega : \mathbb{I} = [0,T] \to P_f(Y)$ is measurable and $\omega(\cdot) \subset E$ where E is a bounded set of Y, the admissible control set $U_{\text{ad}} = \{u \in L^p([0,T],Y) \mid u(t) \in \omega(t) \text{ a.e}\}$. $U_{\text{ad}} \neq \emptyset$ (see [13, Page 142 Proposition 1.7 and Page 174 Lemma 3.2]). In this paper, we develop a direct technique to derive the maximum principle for a Lagrange problem of systems governed by a class of the second-order nonlinear impulsive differential equation in infinite dimensional spaces. Consider the following second-order nonlinear impulsive differential equations:

$$\begin{aligned} \ddot{x}(t) &= A\dot{x}(t) + f(t,x(t),\dot{x}(t)) + B(t)u(t), \quad t \in (0,T] \setminus \Theta, \\ x(0) &= x_0, \Delta_l x(t_i) = J_i^0(x(t_i)), \quad t_i \in \Theta,\ i = 1,2,\ldots,n, \\ \dot{x}(0) &= x_1, \Delta_l \dot{x}(t_i) = J_i^1(\dot{x}(t_i)), \quad t_i \in \Theta,\ i = 1,2,\ldots,n, \end{aligned} \tag{1.1}$$

where the A is the infinitesimal generator of a C_0-semigroup in a Banach space X, $\Theta = \{t_i \in \mathbb{I} \mid 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T\}$, J_i^0, J_i^1 $(i = 1,2,\ldots,n)$ are nonlinear maps, and $\Delta_l x(t_i) = x(t_i + 0) - x(t_i)$, $\Delta_l \dot{x}(t_i) = \dot{x}(t_i + 0) - \dot{x}(t_i)$. We denote the jump in the state x, $\dot{x}$ at time t_i, respectively, with J_i^0, J_i^1 determining the size of the jump at time t_i.

As a first step, we use the semigroup $\{S(t),\ t \geq 0\}$ generated by A to construct the semigroup generated by the operator matrix $\mathfrak{A}$ (see Lemma 2.2). Then, the existence and uniqueness of PC_l-mild solution for (1.1) are proved. Next, we consider a Lagrange problem of system governed by (1.1) and prove the existence of optimal controls. In order to derive the optimality conditions for the system (1.1), we consider the associated adjoint equation and convert it to a first-order backward impulsive integro-differential equation with unbounded impulsive conditions. We note that the resulting integro-differential equation cannot be turned into the original problem by simple transformation $s = T - t$ (see (4.9)). Subsequently, we introduce a suitable mild solution for adjoint equation and give a generalized backward Gronwall inequality to find a priori estimate on the solution of adjoint equation. Finally, we make use of Yosida approximation to derive the optimality conditions.

The paper is organized as follows. In Section 2, we give associated notations and preliminaries. In Section 3, the mild solution of second-order nonlinear impulsive differential equations is introduced and the existence result is also presented. In addition, the existence of optimal controls for a Lagrange problem (P) is given. In Section 4, we discuss corresponding the adjoint equation and directly derive the necessary conditions by the calculus of variations and the Yosida approximation. At last, an example is given for demonstration.

2. Preliminaries

In this section, we give some basic notations and preliminaries. We present some basic notations and terminologies. Let $\pounds(X)$ be the class of (not necessary bounded) linear operators in Banach space X. $\pounds_b(X)$ stands for the family of bounded linear operators in X. For $A \in \pounds(X)$, let $\rho(A)$ denote the resolvent set and $R(\lambda, A)$ the resolvent corresponding to $\lambda \in \rho(A)$. Define $PC_l(\mathbb{I}, X)$ $(PC_r(\mathbb{I}, X)) = \{x : \mathbb{I} \to X \mid x$ is continuous at $t \in \mathbb{I} \setminus \Theta, x$ is continuous from left (right) and has right- (left-) hand limits at $t_i \in \Theta\}$. $PC_l^1(\mathbb{I}, X) = \{x \in PC_l(\mathbb{I}, X) \mid \dot{x} \in PC_l(\mathbb{I}, X)\}$, $PC_r^1(\mathbb{I}, X) = \{x \in PC_r(\mathbb{I}, X) \mid \dot{x} \in PC_r(\mathbb{I}, X)\}$. Set

$$\|x\|_{PC} = \max\left\{\sup_{t \in \mathbb{I}} \|x(t+0)\|, \sup_{t \in \mathbb{I}} \|x(t-0)\|\right\}, \quad \|x\|_{PC^1} = \|x\|_{PC} + \|\dot{x}\|_{PC}. \tag{2.1}$$

It can be seen that endowed with the norm $\|\cdot\|_{PC}(\|\cdot\|_{PC^1})$ $PC_l(\mathbb{I},X)(PC_l^1(\mathbb{I},X))$ and $PC_r(\mathbb{I},X)(PC_r^1(\mathbb{I},X))$ are Banach spaces.

In order to construct the C_0-semigroup generated by $\mathfrak{A}$, we need the following lemma ([14, Theorem 5.2.2]).

Lemma 2.1. *Let A be a densely defined linear operator in X with $\rho(A) \neq \emptyset$. Then the Cauchy problem*

$$\begin{gathered}\dot{x}(t) = Ax(t), \quad t > 0, \\ x(0) = x_0\end{gathered} \tag{2.2}$$

has a unique classical solution for each $x_0 \in D(A)$ if, and only if, A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ in X.

In the following lemma we construct the C_0-semigroup generated by $\mathfrak{A}$.

Lemma 2.2 [12, Lemma 1]. *Suppose A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ on X. Then $\mathfrak{A} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is the infinitesimal generator of a C_0-semigroup $\{\overline{S}(t), t \geq 0\}$ on $X \times X$, given by*

$$\overline{S}(t) = \begin{pmatrix} I & \int_0^t S(\tau)d\tau \\ 0 & S(t) \end{pmatrix}. \tag{2.3}$$

Proof. Obviously, $\mathfrak{A}$ is a densely defined linear operator in $X \times X$ with $\rho(\mathfrak{A}) \neq \emptyset$ according to assumption.

Consider the following initial value problem:

$$\ddot{x}(t) = A\dot{x}(t), \quad t \in (0,T], \quad x(0) = x_0, \quad \dot{x}(0) = x_1 \in D(A). \tag{2.4}$$

It is to see that the classical solution of (2.4) can be given by

$$x(t) = x_0 + \int_0^t S(\tau)x_1 d\tau, \qquad \dot{x}(t) = S(t)x_1. \tag{2.5}$$

Setting $v_0(t) = x(t)$, $v_1(t) = \dot{x}(t)$, $v(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}$, $v_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in D(\mathfrak{A}) = X \times D(A)$, (2.4) can be rewritten as

$$\dot{v}(t) = \mathfrak{A}v(t), \quad t \in (0,T], \quad v(0) = v_0 \in D(\mathfrak{A}), \tag{2.6}$$

and (2.6) has a unique classical solution v given by

$$v(t) = \begin{pmatrix} I & \int_0^t S(\tau)d\tau \\ 0 & S(t) \end{pmatrix} v_0. \tag{2.7}$$

Using Lemma 2.1, $\mathfrak{A}$ generates a C_0-semigroup $\{\overline{S}(t), t \geq 0\}$. □

In order to study the existence of optimal control and necessary conditions of optimality, we also need some important lemmas. For reader's convenience, we state the following results.

Lemma 2.3 [7, Lemma 3.2]. *Suppose A is the infinitesimal generator of a compact semigroup $\{S(t), t \geq 0\}$ in X. Then the operator $Q : L_p([0,T],X) \to C([0,T],X)$ with $p > 1$ given by*

$$(Qf)(t) = \int_0^t S(t-\tau)f(\tau)d\tau \tag{2.8}$$

is strongly continuous.

Lemma 2.4 [15, Lemma 1.1]. *Let $\varphi \in C([0,T],X)$ satisfy the following inequality:*

$$||\varphi(t)|| \leq a + b\int_0^t ||\varphi(s)||ds + c\int_0^t ||\varphi_s||_B ds \quad \forall t \in [0,t], \tag{2.9}$$

where $a,b,c \geq 0$ are constants, and $\|\varphi_s\|_B = \sup_{0\leq\tau\leq s}\|\varphi(\tau)\|$. Then

$$||\varphi(t)|| \leq ae^{(b+c)t}. \tag{2.10}$$

3. Existence of optimal controls

In this section, we not only present the existence of PC_l-mild solution of the controlled system (1.1) but also give the existence of optimal controls of systems governed by (1.1).

We consider the following controlled system:

$$\begin{gathered} \ddot{x}(t) = A\dot{x}(t) + f(t,x(t),\dot{x}(t)) + B(t)u(t), \quad t \in (0,T] \setminus \Theta, \\ \Delta_l x(t_i) = J_i^0(x(t_i)), \quad \Delta_l \dot{x}(t_i) = J_i^1(\dot{x}(t_i)), \quad t_i \in \Theta, \\ x(0) = x_0, \quad \dot{x}(0) = x_1, \quad u \in U_{\text{ad}}, \end{gathered} \tag{3.1}$$

and naturally introduce its mild solution.

Definition 3.1. A function $x \in PC_l^1(\mathbb{I},X)$ is said to be a PC_l-mild solution of the system (3.1) if x satisfies the following integral equation:

$$\begin{aligned} x(t) = x_0 + \int_0^t S(s)x_1 ds + \int_0^t\int_\tau^t S(s-\tau)[f(\tau,x(\tau),\dot{x}(\tau)) + B(\tau)u(\tau)]ds\,d\tau \\ + \sum_{0<t_i<t}\left[J_i^0(x(t_i)) + \int_{t_i}^t S(s-t_i)J_i^1(\dot{x}(t_i))ds\right]. \end{aligned} \tag{3.2}$$

For the forthcoming analysis, we need the following assumptions:

[B]: $B \in L_\infty(\mathbb{I},\pounds(Y,X))$;

[F]: (1) $f : \mathbb{I} \times X \times X \to X$ is measurable in $t \in \mathbb{I}$ and locally Lipschitz continuous with respect to last two variables, that is, for all $x_1,x_2,y_1,y_2 \in X$, satisfying $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\| \leq \rho$, we have

$$||f(t,x_1,y_1) - f(t,x_2,y_2)|| \leq L(\rho)(||x_1 - x_2|| + ||y_1 - y_2||); \tag{3.3}$$

(2) there exists a constant $a > 0$ such that

$$||f(t,x,y)|| \leq a(1 + \|x\| + \|y\|) \quad \forall x,y \in X; \tag{3.4}$$

[J]: (1) $J_i^0(J_i^1): X \to X$ $(i = 1,2,\dots,n)$ map bounded set of X to bounded set of X;
(2) There exist constants $e_i^0, e_i^1 \geq 0$ such that maps $J_i^0, J_i^1 : X \to X$ satisfy

$$||J_i^0(x) - J_i^0(y)|| \leq e_i^0 \|x - y\|, \quad ||J_i^1(x) - J_i^1(y)|| \leq e_i^1 \|x - y\| \quad \forall x, y \in X \ (i = 1,2,\dots,n). \tag{3.5}$$

Similar to the proof of existence of mild solution for the first-order impulsive evolution equation (see [16]), one can verify the basic existence result. Here, we have to deal with space $PC_l^1(\mathbb{I}, X)$ instead.

Theorem 3.2. *Suppose that A is the infinitesimal generator of a C_0-semigroup. Under assumptions [B], [F], and [J](1), the system (3.1) has a unique PC_l-mild solution for every $u \in U_{\text{ad}}$.*

Proof. Consider the map H given by

$$(Hx)(t) = x_0 + \int_0^t S(s)x_1 ds + \int_0^t \int_\tau^t S(s-\tau)[f(\tau, x(\tau), \dot{x}(\tau)) + B(\tau)u(\tau)] ds d\tau \tag{3.6}$$

on

$$B(x_0, x_1, 1) = \left\{x \in C^1([0, T_1], X) \mid ||\dot{x}(t) - x_1|| + ||x(t) - x_0|| \leq 1, \ 0 \leq t \leq T_1\right\}, \tag{3.7}$$

where T_1 would be chosen. Using assumptions and properties of semigroup, we can show that H is a contraction map and obtain local existence of mild solution for the following differential equation without impulse:

$$\begin{gathered} \ddot{x}(t) = A\dot{x}(t) + f(t, x(t), \dot{x}(t)) + B(t)u(t), \quad t \in (0, T], \\ x(0) = x_0, \quad \dot{x}(0) = x_1, \quad u \in U_{\text{ad}}. \end{gathered} \tag{3.8}$$

The global existence comes from a priori estimate of mild solution in space $C^1(\mathbb{I}, X)$ which can be proved by Gronwall lemma.

Step by step, the existence of PC_l-mild solution of (3.1) can be derived. □

Let x^u denote the PC_l-mild solution of system (3.1) corresponding to the control $u \in U_{\text{ad}}$, then we consider the Lagrange problem (P):

find $u^0 \in U_{\text{ad}}$ such that

$$J(u^0) \leq J(u), \quad \forall u \in U_{\text{ad}}, \tag{3.9}$$

where

$$J(u) = \int_0^T l(t, x^u(t), \dot{x}^u(t), u(t)) dt. \tag{3.10}$$

Suppose that

[L]: (1) the functional $l : \mathbb{I} \times X \times X \times Y \to R \cup \{\infty\}$ is Borel measurable;

(2) $l(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for almost all $t \in \mathbb{I}$;

(3) $l(t, x, y, \cdot)$ is convex on Y for each $(x, y) \in X \times X$ and almost all $t \in \mathbb{I}$;

(4) there exist constants $b \geq 0$, $c > 0$ and $\varphi \in L_1(\mathbb{I}, R)$ such that

$$l(t,x,y,u) \geq \varphi(t) + b(\|x\| + \|y\|) + c\|u\|_Y^p \quad \forall x,y \in X,\ u \in Y. \tag{3.11}$$

Now we can give the following result on existence of the optimal controls for problem (P).

Theorem 3.3. *Suppose that A is the infinitesimal generator of a compact semigroup. Under assumptions [F], [L], and [J](2), the problem* (P) *has a solution.*

Proof. If $\inf\{J(u) \mid u \in U_{\mathrm{ad}}\} = +\infty$, there is nothing to prove.

We assume that $\inf\{J(u) \mid u \in U_{\mathrm{ad}}\} = m < +\infty$. By assumption [L], we have $m > -\infty$.

By definition of infimum, there exists a sequence $\{u^n\} \subset U_{\mathrm{ad}}$ such that $J(u^n) \to m$. Since $\{u_n\}$ is bounded in $L_p(\mathbb{I}, Y)$, there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L_p(\mathbb{I}, Y)$ such that

$$u^n \xrightarrow{w} u^0 \quad \text{in } L_p(\mathbb{I}, Y). \tag{3.12}$$

Since U_{ad} is closed and convex, from the Mazur lemma, we have $u^0 \in U_{\mathrm{ad}}$.

Suppose x^n is the PC_l-mild solution of (3.1) corresponding to u^n $(n = 0,1,2,\dots)$. Then x^n satisfies the following integral equation

$$\begin{aligned} x^n(t) = x_0 &+ \int_0^t S(s)x_1\,ds + \int_0^t\int_\tau^t S(s-\tau)\left[f(\tau, x^n(\tau), \dot{x}^n(\tau)) + B(\tau)u^n(\tau)\right]ds\,d\tau \\ &+ \sum_{0<t_i<t} J_i^0(x^n(t_i)) + \sum_{0<t_i<t}\int_{t_i}^t S(s-t_i)J_i^1(\dot{x}^n(t_i))\,ds. \end{aligned} \tag{3.13}$$

Using the boundedness of $\{u^n\}$ and Theorem 3.2, there exists a number $\rho > 0$ such that $\|x^n\|_{PC_l^1(\mathbb{I},X)} \leq \rho$.

Define

$$\eta_n(t) = \int_0^t\int_\tau^t S(s-\tau)B(\tau)u^n(\tau)\,ds\,d\tau - \int_0^t\int_\tau^t S(s-\tau)B(\tau)u^0(\tau)\,ds\,d\tau. \tag{3.14}$$

According to Lemma 2.3, we have

$$\eta_n \longrightarrow 0 \quad \text{in } C(\mathbb{I}, X) \text{ as } u^n \xrightarrow{w} u^0. \tag{3.15}$$

By assumptions [F], [J](2), Theorem 3.2, and Gronwall lemma with impulse (see [17, Lemma 1.7.1]), there exists a constant $M > 0$ such that

$$\|x^n(t) - x^0(t)\| + \|\dot{x}^n(t) - \dot{x}^0(t)\| \leq M\|\eta_n\|_{C^1(\mathbb{I},X)}, \tag{3.16}$$

that is,

$$x^n \longrightarrow x^0 \quad \text{in } PC_l^1(\mathbb{I}, X) \text{ as } n \longrightarrow \infty. \tag{3.17}$$

Since $PC_l^1(\mathbb{I},X) \hookrightarrow L_1(\mathbb{I},X)$, using the assumption [L] and Balder's theorem (see [18]), we can obtain

$$m = \lim_{n\to\infty} \int_0^T l(t,x^n(t),u^n(t))\,dt \geq \int_0^T l(t,x^0(t),u^0(t))\,dt = J(u^0) \geq m. \tag{3.18}$$

This means that J attains its minimum at $u^0 \in U_{\mathrm{ad}}$. □

4. Necessary conditions of optimality

In this section, we present necessary conditions of optimality for Lagrange problem (P). Let (x^0,u^0) be an optimal pair.

[F*] f satisfies the assumptions [F], f is continuously Frechet differentiable at x^0 and $\dot{x}^0$, respectively, $f_x^0 \in L_1(\mathbb{I},\pounds(X))$, $f_{\dot{x}}^0 \in L_\infty(\mathbb{I},\pounds(X))$, $f_x^0(t_i \pm 0) = f_x^0(t_i)$, $f_{\dot{x}}^0(t_i \pm 0) = f_{\dot{x}}^0(t_i)$ for $t_i \in \Theta$, where $f_x^0(t) = f_x(t,x^0(t),\dot{x}^0(t))$, $f_{\dot{x}}^0(t) = f_{\dot{x}}(t,x^0(t),\dot{x}^0(t))$.

[L*] l is continuously Frechet differentiable on x, $\dot{x}$ and u, respectively, $l_x^0(\cdot) \in L_1(\mathbb{I}, X^*)$, $l_{\dot{x}}^0(\cdot) \in W^{1,1}(\mathbb{I},X^*)$, $l_u^0(\cdot) \in L_1(\mathbb{I},Y^*)$, $l_{\dot{x}}^0(T) \in X^*$, $l_{\dot{x}}^0(t_i \pm 0) = l_{\dot{x}}^0(t_i)$ for $t_i \in \Theta$, where $l_x^0(\cdot) = l_x(\cdot,x^0(\cdot),\dot{x}^0(\cdot),u^0(\cdot))$, $l_{\dot{x}}^0(\cdot) = l_{\dot{x}}(\cdot,x^0(\cdot),\dot{x}^0(\cdot),u^0(\cdot))$, $l_u^0(\cdot) = l_u(\cdot,x^0(\cdot),\dot{x}^0(\cdot), u^0(\cdot))$.

[J*] $J_i^0(J_i^1)$ is continuously Frechet differentiable on $x^0(\dot{x}^0)$, and $J_{i\dot{x}}^{10*}(t_i)D(A^*) \subseteq D(A^*)$, where $J_{ix}^{00}(t_i) = J_{ix}^0(x^0(t_i))$, $J_{i\dot{x}}^{10}(t_i) = J_{i\dot{x}}^1(\dot{x}^0(t_i))$ $(i = 1,2,\ldots,n)$.

In order to derive a priori estimate on solution of adjoint equation, we need the following generalized backward Gronwall lemma.

Lemma 4.1. *Let $\varphi \in C(\mathbb{I},X^*)$ satisfy the following inequality:*

$$||\varphi(t)||_{X^*} \leq a + b\int_t^T ||\varphi(s)||_{X^*}\,ds + c\int_t^T ||\varphi_s||_{B_0}\,ds \quad \forall t \in \mathbb{I}, \tag{4.1}$$

where $a,b,c \geq 0$ are constants, and $\|\varphi_s\|_{B_0} = \sup_{s\leq\tau\leq T}\|\varphi(\tau)\|_{X^*}$. Then

$$||\varphi(t)||_{X^*} \leq a\exp[(b+c)(T-t)]. \tag{4.2}$$

Proof. Setting $\varphi(T-t) = \psi(t)$ for $t \in \mathbb{I}$, $\|\psi_t\|_B = \sup_{0\leq\tau\leq t}\|\varphi(\tau)\|_{X^*}$, we have

$$||\psi(t)||_{X^*} \leq a + b\int_0^t ||\psi(s)||_{X^*}\,ds + c\int_0^t ||\psi_s||_B\,ds. \tag{4.3}$$

Using Lemma 2.4, we obtain

$$||\psi(t)||_{X^*} \leq a\exp[(b+c)t]; \tag{4.4}$$

further,

$$||\varphi(t)||_{X^*} \leq a\exp[(b+c)(T-t)]. \tag{4.5}$$

The proof is completed. □

Let X be a reflexive Banach space, let A^* be the adjoint operator of A, and let $\{S^*(t), t \geq 0\}$ be the adjoint semigroup of $\{S(t), t \geq 0\}$. It is a C_0-semigroup and its generator is just A^* (see [14, Theorem 2.4.4]).

We consider the following adjoint equation:

$$\begin{gathered}\varphi''(t) = -(A^*\varphi(t))' - (f_x^{0*}(t)\varphi(t))' + f_x^{0*}(t)\varphi(t) + l_x^0(t) - l_{\dot{x}}^{0'}(t), \quad t \in [0,T) \setminus \Theta, \\ \varphi(T) = 0, \quad \Delta_r \varphi(t_i) = J_{i\dot{x}}^{10*}(t_i)\varphi(t_i), \quad t_i \in \Theta, \\ \varphi'(T) = -l_{\dot{x}}^0(T), \quad \Delta_r \varphi'(t_i) = G_i(\varphi(t_i), \varphi'(t_i)), \quad t_i \in \Theta,\end{gathered} \tag{4.6}$$

where

$$\begin{aligned}&G_i(\varphi(t_i), \varphi'(t_i)) \\ &\quad = [J_{ix}^{00*}(t_i)(A^* + f_{\dot{x}}^{0*}(t_i)) - (A^* + f_{\dot{x}}^{0*}(t_i))J_{i\dot{x}}^{10*}(t_i)]\varphi(t_i) + J_{ix}^{00*}(t_i)\varphi'(t_i) + J_{ix}^{00*}(t_i)l_{\dot{x}}^0(t_i).\end{aligned} \tag{4.7}$$

A function $\varphi \in PC_r^1(\mathbb{I}, X^*) \bigcap PC_r(\mathbb{I}, D(A^*))$ is said to be a PC_r-mild solution of (4.6) if φ is given by

$$\begin{aligned}\varphi(t) = &\int_t^T S^*(\tau - t)\left[\int_\tau^T (f_x^{0*}(s)\varphi(s) - l_x^0(s) + l_{\dot{x}}^{0'}(s))ds + f_{\dot{x}}^{0*}(\tau)\varphi(\tau) + l_{\dot{x}}^0(T)\right]d\tau \\ &+ \sum_{t_i > t} S^*(t_i - t)J_{i\dot{x}}^{10*}(t_i)\varphi(t_i) + \sum_{t_i > t}\int_t^{t_i} S^*(\tau - t)G_i(\varphi(t_i), \varphi'(t_i))d\tau.\end{aligned} \tag{4.8}$$

Lemma 4.2. *Assume that X is a reflexive Banach space. Under the assumptions [F*], [L*], [J*], the evolution (4.6) has a unique PC_r-mild solution $\varphi \in PC_r^1(\mathbb{I}, X^*)$.*

Proof. Consider the following equation:

$$\begin{aligned}&\varphi'(t) + (A^* + f_{\dot{x}}^{0*}(t))\varphi(t) + \int_t^T [f_x^{0*}(s)\varphi(s) + l_x^0(s) - l_{\dot{x}}^{0'}(s)]ds \\ &\qquad = \sum_{t_i > t} G_i(\varphi(t_i), \varphi'(t_i)) - l_{\dot{x}}^0(T), \quad t \in \mathbb{I} \setminus \Theta, \\ &\varphi(T) = 0, \quad \Delta_r \varphi(t_i) = J_{i\dot{x}}^{10*}(t_i)\varphi(t_i), \quad t_i \in \Theta.\end{aligned} \tag{4.9}$$

Equation (4.9) is a linear impulsive integro-differential equation. Setting $t = T - s$, $\psi(s) = \varphi(T - s)$, (4.9) can be rewritten as

$$\begin{gathered}\psi'(s) = (A^* + f_{\dot{x}}^{0*}(T - s))\psi(s) + F(s) + \sum_{s_i < s} g_i(\psi(s_i), \psi'(s_i)), \quad s \in [0,T) \setminus \Lambda, \\ \psi(0) = 0, \quad \Delta_l \psi(s_i) = J_{i\dot{x}}^{10*}(t_i)\psi(s_i), \quad s_i \in \Lambda = \{s_i = T - t_i \mid t_i \in \Theta\},\end{gathered} \tag{4.10}$$

where

$$g_i(\psi(s_i),\psi'(s_i)) = [(A^* + f_x^{0*}(t_i))J_{ix}^{10*}(t_i) - J_{ix}^{00*}(t_i)(A^* + f_x^{0*}(t_i))]\psi(s_i) + J_{ix}^{00*}(t_i)\psi'(t_i) - J_{ix}^{00*}(t_i)l_x^0(t_i),$$
$$F(s) = \int_{T-s}^{T} [f_x^{0*}(\theta)\psi(T-\theta) + l_x^0(\theta) - l_x^{0'}(\theta)]d\theta + l_x^0(T). \tag{4.11}$$

Obviously, if φ is the classical solution of (4.9), then it must be the PC_r-mild solution of (4.6). Now we show that (4.9) has a unique classical solution $\varphi \in PC^1(\mathbb{I}, X^*)\bigcap PC(\mathbb{I}, D(A^*))$.

For $s \in [0, s_n]$, prove that the following equation:

$$\psi'(s) = A^*\psi(s) + f_x^{0*}(T-s)\psi(s) + F(s),\quad \psi(0) = 0, \tag{4.12}$$

has a unique classical solution $\psi \in C^1([0,s_n],X^*)\bigcap C([0,s_n],D(A^*))$ given by

$$\psi(s) = \int_0^s S^*(s-\tau)(f_x^{0*}(T-\tau)\psi(\tau) + F(\tau))d\tau. \tag{4.13}$$

By following the same procedure as in [16, Theorem 4.A], one can verify that (4.12) has a unique mild solution $\psi \in C([0,s_n],X^*)$ given by expression (4.13).

By the definition of F, it is easy to see that $F \in L_1([0,s_n],X^*)\bigcap C((0,s_n),X^*)$. Using (4.13) and the basic properties of C_0-semigroup, we obtain $\psi(s) \in D(A^*)$ for $s \in [0,s_n]$ and

$$\psi'(s) = f_x^{0*}(T-s)\psi(s) + F(s) + A^*\int_0^s S^*(s-\tau)(f_x^{0*}(T-\tau)\psi(\tau) + F(\tau))d\tau. \tag{4.14}$$

This implies $\psi \in C^1((0,s_n),X^*)$ and $\psi'(s_n-) = \psi'(s_n)$. Using [14, Theorem 5.2.13], (4.12) has a unique classical solution $\psi \in C^1((0,s_n),X^*)\bigcap C([0,s_n],D(X^*))$ given by the expression (4.13). In addition, the expressions (4.13) and (4.12) imply $\psi(0) = 0$, $\psi'(0) = l_x^0(T)$, and $\psi(s_n - 0)$, $\psi'(s_n - 0)$ exist. Furthermore, $\psi \in C^1([0,s_n],X^*)\bigcap C([0,s_n],D(A^*))$.

By assumption [J*], we have

$$\psi_n^0 = \psi(s_n) + J_{nx}^{10*}(t_n)\psi(s_n) \in D(A^*),\qquad \psi_n^1 = \psi'(s_n) + g_n(\psi(s_n),\psi'(s_n)) \in X^*. \tag{4.15}$$

For $s \in (s_n, s_{n-1}]$, consider the following equation:

$$\psi'(s) = (A^* + f_x^{0*}(T-s))\psi(s) + \int_{T-s}^{T-s_n} [f_x^{0*}(\theta)\psi(T-\theta) + l_x^0(\theta) - l_x^{0'}(\theta)]d\theta + \psi_n^1,\quad \psi(s_n+) = \psi_n^0, \tag{4.16}$$

that is, study the following equation:

$$\psi'(s) = (A^* + f_{\dot{x}}^{0*}(T-s))\psi(s) + F(s) + g_n(\psi(s_n), \psi'(s_n)),$$
$$\psi(s_n+) = \psi_n^0. \tag{4.17}$$

By following the same procedure as on time interval $[0, s_n]$, it has a unique classical solution given by

$$\psi(s) = S^*(s-s_n)\psi_n^0 + \int_{s_n}^{s} S^*(s-\tau)[f_{\dot{x}}^{0*}(T-\tau)\psi(\tau) + F(\tau) + g_n(\psi(s_n), \psi'(s_n))]d\tau. \tag{4.18}$$

In general, for $s \in (s_i, s_{i-1}]$ $(i = 0, 1, \ldots, n)$, consider the following equation:

$$\psi'(s) = (A^* + f_{\dot{x}}^{0*}(T-s))\psi(s) + F(s) + g_i(\psi(s_i), \psi'(s_i)),$$
$$\psi(s_i) = \psi(s_i) + J_{i\dot{x}}^{10*}(t_i)\psi(s_i) \in D(A^*). \tag{4.19}$$

It has a unique classical solution given by

$$\psi(s) = S^*(s-s_i)\psi_i^0 + \int_{s_i}^{s} S^*(s-\tau)[f_{\dot{x}}^{0*}(T-\tau)\psi(\tau) + F(\tau) + g_i(\psi(s_i), \psi'(s_i))]d\tau. \tag{4.20}$$

Repeating the procedure till the time interval which is expanded, and combining all of the solutions on $[t_i, t_{i+1}]$ $(i = 0, 1, \ldots, n)$, we obtain classical solution of (4.10) given by

$$\begin{aligned}\psi(s) = &\int_0^s S^*(s-\tau)[f_{\dot{x}}^{0*}(T-\tau)\psi(\tau) + F(\tau)]d\tau \\ &+ \sum_{0<s_i<s}\left[S^*(s-s_i)J_{i\dot{x}}^{10*}(t_i)\psi(s_i) + \int_{s_i}^{s} S^*(s-\tau)g_i(\psi(s_i), \psi'(s_i))d\tau\right].\end{aligned} \tag{4.21}$$

Further, (4.9) has a unique classical solution $\varphi \in PC^1(\mathbb{I}, X^*) \bigcap PC(\mathbb{I}, D(A^*))$ given by (4.8). □

Using the assumption [F*], [3, Corollary 3.2], and [2, Theorem 2], $\{A^*(t) = A^* + f_{\dot{x}}^{0*}(t) \mid t \in \mathbb{I}\}$ generates a strongly continuous evolution operator $U^*(t,s)$, $0 \le s \le t \le T$. For simplicity, we have the following result.

Remark 4.3. The PC-mild solution φ of (4.6) can be rewritten as

$$\begin{aligned}\varphi(t) = &\int_t^T U^*(\tau,t)\left[\int_\tau^T (f_{\dot{x}}^{0*}(s)\varphi(s) + l_x^0(s) - l_{\dot{x}}^{0'}(s))ds + l_{\dot{x}}^0(T)\right]d\tau \\ &+ \sum_{t_i>t} U^*(t_i,t)J_{i\dot{x}}^{10*}(t_i)\varphi(t_i) + \sum_{t_i>t}\int_t^{t_i} U^*(\tau,t)G_i(\varphi(t_i), \varphi'(t_i))d\tau.\end{aligned} \tag{4.22}$$

Now we can give the necessary conditions of optimality for Lagrange problem (P).

Theorem 4.4. *Suppose both X and Y be reflexive Banach spaces. Under the assumption of Theorem 3.2 and assumptions [B], [F^*], [L^*], and [J^*], then, in order that the pair $\{x^0, u^0\}$ be optimal, it is necessary that there exists a function $\varphi \in PC_r^1(\mathbb{I}, X^*) \bigcap PC_r(\mathbb{I}, D(A^*))$ such that the following evolution equations and inequality hold:*

$$\begin{aligned}
&\ddot{x}^0(t) = A\dot{x}^0(t) + f(t, x^0(t), \dot{x}^0(t)) + B(t)u^0(t), \quad t \in (0,T] \setminus \Theta, \\
&x^0(0) = x_0, \Delta_l x^0(t_i) = J_i^0(x^0(t_i)), \quad t_i \in \Theta, \\
&\dot{x}^0(0) = x_1, \Delta_l \dot{x}^0(t_i) = J_i^1(\dot{x}^0(t_i)), \quad t_i \in \Theta;
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
&\varphi''(t) = -(A^*\varphi(t))' + f_x^{0*}(t)\varphi(t) - (f_{\dot{x}}^{0*}(t)\varphi(t))' + l_x^0(t) - l_{\dot{x}}^{0'}(t), \quad t \in [0,T) \setminus \Theta, \\
&\varphi(T) = 0, \Delta_r \varphi(t_i) = J_{i\dot{x}}^{10}(t_i)\varphi(t_i), \quad t_i \in \Theta, \\
&\varphi'(T) = l_{\dot{x}}^0(T), \Delta_r \varphi'(t_i) = G_i(\varphi(t_i), \varphi'(t_i)), \quad t_i \in \Theta;
\end{aligned} \tag{4.24}$$

$$\int_0^T \langle l_u^0(t) + B^*(t)\varphi(t), u(t) - u^0(t) \rangle_{Y^*,Y} dt \geq 0, \quad \forall u \in U_{\text{ad}}. \tag{4.25}$$

Proof. Since $(x^0, u^0) \in PC_l^1(\mathbb{I}, X) \times U_{\text{ad}}$ is an optimal pair, it must satisfy (4.23).

Since U_{ad} is convex, it is clear that $u^\varepsilon = u^0 + \varepsilon(u - u^0) \in U_{\text{ad}}$ for $\varepsilon \in [0,1]$, $u \in U_{\text{ad}}$. Let x^ε denote the PC_l-mild solution of (3.1) corresponding to the control u^ε. Using assumption [J*], J is Gateaux differentiable, and the G-derivative of J at u^0 in the direction $u - u^0$ can be given by

$$\begin{aligned}
\lim_{\varepsilon \to 0} &\frac{J(u^\varepsilon) - J(u^0)}{\varepsilon} \\
&= \int_0^T \langle l_x^0(t), y(t) \rangle_{X^*,X} dt + \int_0^T \langle l_{\dot{x}}^0(t), \dot{y}(t) \rangle_{X^*,X} dt + \int_0^T \langle l_u^0(t), u(t) - u^0(t) \rangle_{Y^*,Y} dt \\
&= \int_0^T \langle l_x^0(t) - l_{\dot{x}}^{0'}(t), y(t) \rangle_{X^*,X} dt + \int_0^T \langle l_u^0(t), u(t) - u^0(t) \rangle_{Y^*,Y} dt \\
&\quad + \langle l_{\dot{x}}^0(T), y(T) \rangle_{X^*,X} - \langle l_{\dot{x}}^0(0), y(0) \rangle_{X^*,X} - \sum_{i=1}^n \langle l_{\dot{x}}^0(t_i), \Delta_l y(t_i) \rangle_{X^*,X},
\end{aligned} \tag{4.26}$$

where the process $y \in PC_l^1(\mathbb{I}, X)$ is the Gateaux derivative of solution x at u^0 in the direction $u - u^0$ which satisfies the following equation:

$$\begin{aligned}
&\ddot{y}(t) = (A + f_{\dot{x}}^0(t))\dot{y}(t) + f_x^0(t)y(t) + B(t)[u(t) - u^0(t)], \quad t \in (0,T] \setminus \Theta, \\
&y(0) = 0, \qquad \Delta_l y(t_i) = J_{ix}^{00}(t_i)y(t_i), \quad t_i \in \Theta, \\
&\dot{y}(0) = 0, \qquad \Delta_l \dot{y}(t_i) = J_{i\dot{x}}^{10}(t_i)\dot{y}(t_i), \quad t_i \in \Theta.
\end{aligned} \tag{4.27}$$

This is usually known as the variational equation. By following the same procedure as in Theorem 3.2, one can easily establish that (4.27) has a unique PC_l-mild solution y given by

$$y(t)=\int_0^t\int_0^{t-\tau}S(\nu)[f_x^0(\tau)y(\tau)+f_{\dot{x}}^0(\tau)\dot{y}(\tau)+B(\tau)(u(\tau)-u^0(\tau))]d\nu d\tau \\ +\sum_{0<t_i<t}\left[J_{ix}^{00}(t_i)y(t_i)+\int_{t_i}^t S(\nu-t_i)J_{i\dot{x}}^{10}(t_i)\dot{y}(t_i)d\nu\right]. \tag{4.28}$$

Since u^0 is the optimal control, we have the following inequality:

$$\int_0^T\langle l_x^0(t)-l_{\dot{x}}^{0'}(t),y(t)\rangle_{X^*,X}dt+\int_0^T\langle l_u^0(t),u(t)-u^0(t)\rangle_{Y^*,Y}dt \\ +\langle l_{\dot{x}}^0(T),y(T)\rangle_{X^*,X}-\sum_{i=1}^n\langle l_{\dot{x}}^0(t_i),\Delta_l y(t_i)\rangle_{X^*,X}\geq 0. \tag{4.29}$$

Due to the reflexivity of Banach space X, we have the Yosida approximation $\lambda_k R(\lambda_k, A^*)\to I^*$ as $\lambda_k\to\infty$, where $R(\lambda_k,A^*)$ is the resolvent of A^* for $\lambda_k\in\rho(A^*)$ and I^* stands for the identity operator in X^*. Consider the Yosida approximation of f_x^{0*}, $f_{\dot{x}}^{0*}$, l_x^0, $l_{\dot{x}}^{0'}$, $l_{\dot{x}}^0(T)$, $J_{ix}^{00*}(t_i)$, $J_{ix}^{10*}(t_i)$ given by

$$f_x^{k*}(\cdot)=\lambda_kR(\lambda_k,A^*)f_x^{0*}(\cdot),\quad l_x^k(\cdot)=\lambda_kR(\lambda_k,A^*)l_x^0(\cdot),\quad l_{\dot{x}}^{\prime k}(\cdot)=\lambda_kR(\lambda_k,A^*)l_{\dot{x}}^{0'}(\cdot), \\ J_{ix}^{k*}(t_i)=\lambda_kR(\lambda_k,A^*)J_{ix}^{00*}(t_i),\quad J_{i\dot{x}}^{k*}(t_i)=\lambda_kR(\lambda_k,A^*)J_{i\dot{x}}^{10*}(t_i),\quad l_{\dot{x}}^k(T)=\lambda_kR(\lambda_k,A^*)l_{\dot{x}}^0(T), \tag{4.30}$$

which take values in $D(A^*)$.

Consider the following evolution equation:

$$\varphi_k''(t)=-(A^*(t)\varphi_k(t))'+f_x^{k*}(t)\varphi_k(t)+l_x^k(t)-l_{\dot{x}}^{k'}(t),\quad t\in[0,T)\setminus\Theta, \\ \varphi_k(T)=0,\quad \Delta_r\varphi_k(t_i)=J_{ix}^{k*}(t_i)\varphi_k(t_i),\quad t_i\in\Theta, \\ \varphi_k'(T)=l_{\dot{x}}^k(T),\quad \Delta_r\varphi_k'(t_i)=G_i^k(\varphi_k(t_i),\varphi_k'(t_i)),\quad t_i\in\Theta, \tag{4.31}$$

where

$$G_i^k(\varphi_k(t_i),\varphi_k'(t_i))=[J_x^{k*}(t_i)A^*(t_i)-A^*(t_i)J_{i\dot{x}}^{1k*}(t_i)]\varphi_k(t_i)+J_{ix}^{k*}(t_i)\varphi_k'(t_i)+J_{ix}^{k*}(t_i)l_{\dot{x}}^k(t_i). \tag{4.32}$$

Similar to the proof of Lemma 4.2, one can show that (4.31) has a unique class solution φ_k given by

$$\varphi_k(t)=\int_t^T U^*(\tau,t)\left[\int_\tau^T(f_x^{k*}(s)\varphi_k(s)+l_x^k(s)-l_{\dot{x}}^{k'}(s))ds+l_{\dot{x}}^k(T)\right]d\tau \\ +\sum_{t_i>t}\left[U^*(t_i,t)J_{ix}^{k*}(t_i)\varphi_k(t_i)+\int_t^{t_i}U^*(\tau,t)G_i^k(\varphi_k(t_i),\varphi_k'(t_i))d\tau\right]. \tag{4.33}$$

Next, show that

$$\varphi_k \longrightarrow \varphi \quad \text{in } PC_r^1(\mathbb{I}, X^*) \text{ as } \lambda_k \longrightarrow \infty. \tag{4.34}$$

Employing the method of proof for Lemma 4.2, there exists a number $M_0 > 0$ such that

$$\|\varphi\|_{PC^1(\mathbb{I},X^*)}, \|\varphi_k\|_{PC^1(\mathbb{I},X^*)} \leq M_0 \quad (k = 1, 2, \dots). \tag{4.35}$$

Setting

$$\begin{aligned} F_k(t) &= \int_t^T \left(f_x^{k*}(s)\varphi_k(s) + l_x^k(s) - l_{\dot{x}}^{k'}(s)\right) ds + l_{\dot{x}}^k(T) \quad (k = 0, 1, \cdots), \\ a_k &= \|l_x^k - l_{\dot{x}}^{\prime k} - l_x^0 + l_{\dot{x}}^{0'}\|_{L_1(\mathbb{I},X^*)} + \|l_{\dot{x}}^k(T) - l_{\dot{x}}^0(T)\|_{X^*} + M_0 \|f_x^{k*} - f_x^{0*}\|_{L_1(\mathbb{I},\pounds(X^*))}, \end{aligned} \tag{4.36}$$

it follows that

$$\begin{aligned} &\|F_k(t) - F_0(t)\|_{X^*} \\ &\quad \leq a_k + \|f_x^{0*}\|_{L_1(\mathbb{I},\pounds(X^*))} \|(\varphi_k)_t - \varphi_t\|_{B_0} + \|f_x^{0*}\|_{L_1(\mathbb{I},\pounds(X^*))} \|\varphi_k(t) - \varphi(t)\|_{X^*}. \end{aligned} \tag{4.37}$$

For $t \in [t_n, T]$, we have

$$\|\varphi_k(t) - \varphi(t)\|_{X^*} \leq \alpha T a_k + \alpha\theta \int_t^T \|\varphi_k(\tau) - \varphi(\tau)\|_{X^*} d\tau + \alpha\theta \int_t^T \|(\varphi_k)_\tau - \varphi_\tau\|_{B_0} d\tau, \tag{4.38}$$

where $\theta = \|f_x^{0*}\|_{L_1([0,T],\pounds(X^*))}$, $\alpha = \sup\{\|U^*(t,s)\|_{\pounds(X^*)} \mid 0 \leq s \leq t \leq T\}$. By Lemma 4.1, we obtain

$$\|\varphi_k(t) - \varphi(t)\|_{X^*} \leq a_k \alpha T e^{2\alpha T\theta}. \tag{4.39}$$

Further,

$$\|\varphi_k'(t) - \varphi'(t)\|_{X^*} \leq \lambda a_k e^{2\alpha T\theta}, \tag{4.40}$$

where $\omega = \sup_{0\leq t\leq T} \|A^*(t)\|_{\pounds(D(A^*),X^*)}$, $\lambda = (1+T)(1+\alpha\omega+2\alpha\theta)$. Hence

$$\|\varphi_k(t) - \varphi(t)\|_{X^*} + \|\varphi_k'(t) - \varphi'(t)\|_{X^*} \leq 2\lambda a_k e^{2\alpha T\theta} \quad \text{for } t \in [t_n, T]. \tag{4.41}$$

Using (4.8), (4.33), and (4.41), we have

$$\begin{aligned} &\|\varphi_k(t_n-0)-\varphi(t_n-0)\|_{X^*}\le h_k\equiv b_k+c_k+\lambda(2\delta+1)a_k e^{2\alpha T\theta},\\ &\|\varphi_k'(t_n-0)-\varphi'(t_n-0)\|_{X^*}\le h_k, \end{aligned} \tag{4.42}$$

where

$$\begin{aligned} b_k&=M_0(\omega+1)\sum_{i=1}^{n}\left(2\|J_{ix}^{k*}(t_i)-J_{ix}^{00*}(t_i)\|_{\pounds(X^*)}+\|J_{i\dot{x}}^{k*}(t_i)-J_{i\dot{x}}^{10}(t_i)\|_{\pounds(X^*)}\right),\\ c_k&=\sum_{i=1}^{n}\|J_{ix}^{k*}(t_i)l_{\dot{x}}^{k}(t_n)-J_{ix}^{00*}(t_i)l_{\dot{x}}^{0}(t_n)\|_{X^*},\\ \delta&=(\omega+1)\sum_{i=1}^{n}\left(2\|J_{ix}^{00*}(t_i)\|_{\pounds(X^*)}+\|J_{i\dot{x}}^{10*}(t_i)\|_{\pounds(X^*)}\right). \end{aligned} \tag{4.43}$$

Hence, for $t\in(t_{n-1},t_n)$, we also obtain

$$\|\varphi_k(t)-\varphi(t)\|_{X^*}+\|\varphi_k'(t)-\varphi'(t)\|_{X^*}\le 2\lambda(a_k+h_k)e^{2\alpha T\theta}. \tag{4.44}$$

By the same procedure, there exists $\gamma>0$ such that

$$\|\varphi_k(t)-\varphi(t)\|_{X^*}+\|\varphi_k'(t)-\varphi'(t)\|_{X^*}\le\gamma(a_k+b_k+c_k)\quad\text{for } t\in\mathbb{I}. \tag{4.45}$$

This proves that

$$\varphi_k\longrightarrow\varphi\quad\text{in } PC_r^1(\mathbb{I},X^*)\text{ as }\lambda_k\longrightarrow\infty. \tag{4.46}$$

Define

$$\eta_k=\int_0^T\langle\varphi(t)-\varphi_k(t),B(t)(u(t)-u^0(t))\rangle_{X^*,X}dt, \tag{4.47}$$

and observe that $\eta_k\to 0$ as $k\to\infty$. Thus

$$\begin{aligned} &\int_0^T\langle\varphi(t),B(t)(u(t)-u^0(t))\rangle_{X^*,X}dt\\ &\quad=\int_0^T\langle\varphi(t)-\varphi_k(t),B(t)(u(t)-u^0(t))\rangle_{X^*,X}dt+\int_0^T\langle\varphi_k(t),B(t)(u(t)-u^0(t))\rangle_{X^*,X}dt\\ &\quad=\eta_k+\int_0^T\langle l_x^k(t)-\dot{l}_{\dot{x}}^k(t),y(t)\rangle_{X^*,X}dt+\langle l_{\dot{x}}^k(T),y(T)\rangle_{X^*,X}-\sum_{i=1}^{n}\langle l_{\dot{x}}^k(t_i),\Delta_l y(t_i)\rangle_{X^*,X} \end{aligned} \tag{4.48}$$

for $\lambda_k \in \rho(A^*) > 0$. Taking the limit $k \to \infty$, we find that

$$\begin{aligned}
&\int_0^T \langle \varphi(t), B(t)(u(t) - u^0(t))\rangle_{X^*,X} dt \\
&\quad = \int_0^T \langle l_x^0(t) - \dot{l}_{\dot{x}}^0(t), y(t)\rangle_{X^*,X} dt + \langle l_{\dot{x}}^0(T), y(T)\rangle_{X^*,X} - \sum_{i=1}^{n} \langle l_{\dot{x}}^0(t_i), \Delta_l y(t_i)\rangle_{X^*,X}.
\end{aligned} \tag{4.49}$$

Further,

$$\int_0^T \langle l_u(t, x^0(t), u^0(t)) + B^*(t)\varphi(t), u(t) - u^0(t)\rangle_{Y^*,Y} dt \geq 0, \quad \forall u \in U_{\text{ad}}. \tag{4.50}$$

Thus, we have proved all the necessary conditions of optimality given by (4.23)–(4.25). □

At the end of this section, an example is given to illustrate our theory. Consider the following problem:

$$\begin{gathered}
\frac{\partial^2}{\partial t^2} x(t,y) \\
= \Delta \frac{\partial}{\partial t} x(t,y) + \sqrt{x^2(t,y)+1} + \sqrt{\left(\frac{\partial}{\partial t} x(t,y)\right)^2 + 1} + u(t,y), \quad y \in \Omega,\ t \in (0,1] \setminus \left\{\frac{1}{3}, \frac{2}{3}\right\}, \\
x(0,y) = 0, \quad x\left(\frac{i}{3} + 0, y\right) - x\left(\frac{i}{3} - 0, y\right) = x\left(\frac{i}{3}, y\right), \quad i = 1,2,\ y \in \Omega, \\
\frac{\partial}{\partial t} x(t,y)|_{t=0} = 0, \quad \frac{\partial}{\partial t} x(t,y)|_{t=i/3+0} - \frac{\partial}{\partial t} x(t,y)|_{t=i/3-0} = \frac{\partial}{\partial t} x(t,y)|_{t=i/3}, \quad i = 1,2,\ y \in \Omega, \\
x(t,y)|_{[0,1]\times\partial\Omega} = 0, \qquad \frac{\partial}{\partial t} x(t,y)|_{[0,1]\times\partial\Omega} = 0,
\end{gathered} \tag{4.51}$$

with the cost function

$$J(u) = \int_0^1 \int_\Omega |x(t,\xi)|^2 d\xi\, dt + \int_0^1 \int_\Omega \left|\frac{\partial}{\partial t} x(t,\xi)\right|^2 d\xi\, dt + \int_0^1 \int_\Omega |u(t,\xi)|^2 d\xi\, dt, \tag{4.52}$$

where $\Omega \subset R^3$ is bounded domain, $\partial\Omega \in C^3$.

For the problem (4.51), one can show the following theorem.

Theorem 4.5. *In order that the pair* $\{x^0, u^0\} \in PC_l^1([0,1],\ L_2(\Omega)) \times L_2([0,1], L_2(\Omega))$ *be optimal, it is necessary that there exists a* $\varphi \in PC_r^1([0,1], L_2(\Omega))$ *such that the following* *evolution equations and inequality hold:*

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}x^0(t,y)=\Delta\frac{\partial}{\partial t}x^0(t,y)+\sqrt{\left(x^0(t,y)\right)^2+1}+\sqrt{\left(\frac{\partial}{\partial t}x^0(t,y)\right)^2+1}+u^0(t,y),\\
&\qquad\qquad y\in\Omega,\ t\in(0,1]\setminus\left\{\frac{1}{3},\frac{2}{3}\right\},\\
&x^0(0,y)=0,\quad x^0\left(\frac{i}{3}+0,y\right)-x^0\left(\frac{i}{3}-0,y\right)=x^0\left(\frac{i}{3},y\right),\quad i=1,2,\ y\in\Omega,\\
&\frac{\partial}{\partial t}x^0(t,y)_{t=0}=0,\quad \frac{\partial}{\partial t}x^0(t,y)|_{t=i/3+0}-\frac{\partial}{\partial t}x^0(t,y)|_{t=i/3-0}=\frac{\partial}{\partial t}x^0(t,y)|_{t=i/3},\\
&\qquad\qquad i=1,2,\ y\in\Omega,\\
&x^0(t,y)|_{[0,1]\times\partial\Omega}=0,\quad \frac{\partial}{\partial t}x^0(t,y)|_{[0,1]\times\Omega}=0;\\
&\frac{\partial^2}{\partial t^2}\varphi(t,y)=-\frac{\partial}{\partial t}\left(\Delta\varphi(t,y)+\frac{(\partial/\partial t)x^0(t,y)\varphi(t,y)}{\sqrt{\left((\partial/\partial t)x^0(t,y)\right)^2+1}}\right)\\
&\qquad+\frac{x^0(t,y)\varphi(t,y)}{\sqrt{\left(x^0(t,y)\right)^2+1}}+2x^0(t,y)-\frac{\partial^2}{\partial t^2}x^0(t,y),\quad y\in\Omega,\ t\in[0,1)\setminus\left\{\frac{1}{3},\frac{2}{3}\right\},\\
&\varphi\left(\frac{i}{3}-0,y\right)-\varphi\left(\frac{i}{3}+0,y\right)=\varphi(t,y)|_{t=i/3},\quad i=1,2,\ y\in\Omega,\\
&\frac{\partial}{\partial t}\varphi(t,y)|_{t=i/3-0}-\frac{\partial}{\partial t}\varphi(t,y)|_{t=i/3+0}=\frac{\partial}{\partial t}\left[\varphi(t,y)+2x^0(t,y)\right]\big|_{t=i/3},\quad i=1,2,\ y\in\Omega,\\
&\varphi(1,y)=0,\quad \frac{\partial}{\partial t}\varphi(t,y)|_{t=1}=2\frac{\partial}{\partial t}x(t,y)|_{t=1},\quad y\in\Omega,\\
&\varphi(t,y)|_{[0,1]\times\partial\Omega}=0,\quad \frac{\partial}{\partial t}\varphi(t,y)|_{[0,1]\times\partial\Omega}=0;\\
&\int_0^1\int_\Omega\left\langle 2u^0(t,\xi)+\varphi(t,\xi),u(t,\xi)-u^0(t,\xi)\right\rangle_{L_2(\Omega),L_2(\Omega)}d\xi\,dt\geq 0,\quad \forall u\in U_{\mathrm{ad}}.
\end{aligned}
\tag{4.53}
$$

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