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Research Article

Multiple Periodic Solutions to Nonlinear Discrete Hamiltonian Systems

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An existence result of multiple periodic solutions to the asymptotically linear discrete Hamiltonian systems is obtained by using the Morse index theory.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of all integers and real numbers, respectively. For $a,b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots\}$ and $\mathbb{Z}(a,b) = \{a, a+1, \ldots, b\}$ when $a \leq b$. Let A be an $n \times m$ matrix. A^τ denotes the transpose of A. When $n = m$, $\sigma(A)$ and $\det(A)$ denote the set of eigenvalues and the determinant of A, respectively.

In this paper, we study the existence of multiple p-periodic solutions to the following discrete Hamiltonian systems:

$$\Delta x(n) = J\nabla H(Lx(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $p > 2$ is a prime integer, $\Delta x(n) = x(n+1) - x(n)$, $x(n) = \binom{x_1(n)}{x_2(n)}$ with $x_i(n) \in \mathbb{R}^d$, $i = 1,2$, L is defined by $Lx(n) = \binom{x_1(n+1)}{x_2(n)}$, $J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$ is the standard symplectic matrix with I_d the identity matrix on $\mathbb{R}^d$, $H \in C^1(\mathbb{R}^{2d}, \mathbb{R})$, and $\nabla H(z)$ denotes the gradient of H in z.

We may think of systems (1.1) as being a discrete analog of the following Hamiltonian systems:

$$\dot{x} = J\nabla H(x(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

which has been studied extensively by many scholars. For example, by using the critical point theory, some significant results for the existence and multiplicity of periodic and subharmonic solutions to (1.2) were obtained in [1–5].

Some authors have also contributed to the study of (1.1) for the disconjugacy, boundary value problems, oscillations, and asymptotic behavior, see, for example, [6–9]. In recent years, existence and multiplicity results of periodic solutions to discrete Hamiltonian systems employing the minimax theory and the geometrical index theory have appeared in the literature. For example, for the case that H is superquadratic both at zero and at infinity, by using the Z_2 geometrical index theory and the linking theorem, some sufficient conditions for the existence of multiple periodic solutions and subharmonic solutions to (1.1) were obtained in [10]. For the case that H is subquadratic at infinity, some sufficient conditions on the existence of periodic solutions to (1.1) were proved in [11] by using the saddle point theorem. Recently, in [12], the authors have obtained some sufficient conditions on the multiplicity results of periodic solutions to a class of second difference equation by using the Z_p geometrical index theory. Our main purpose in this paper is to give a lower bound of the number of p-periodic solutions to (1.1) by using the Morse index theory and a multiplicity result in [12].

The rest of this paper is organized as follows. In Section 2, we present some useful preliminary results. In Section 3, we firstly introduce the Morse index theory for the p-periodic linear Hamiltonian systems:

$$\Delta x(n) = JS(n)Lx(n), \quad n \in \mathbb{Z}, \tag{1.3}$$

where $S(n)$ is a real symmetric positive definite $2d \times 2d$ matrix with $S(n+p) = S(n)$ for every $n \in \mathbb{Z}$, and then, for any real symmetric positive definite matrix S, we define a pair of index functions $(i(S,p),\nu(S,p)) \in \mathbb{Z}(0,2dp) \times \mathbb{Z}(0,2dp)$ and obtain the formulae of the computations of index functions for a diagonal positive definite matrix. In Section 4, by using the Morse index theory and a multiplicity result in [12], we establish a result on the existence of multiple periodic solutions to (1.1) where H satisfies the asymptotically linear conditions.

2. Preliminaries

In order to apply the Morse index theory to study the existence of multiple p-periodic solutions to (1.1), we now state some basic notations and useful lemmas.

Let Ω be the set of sequences $x = \{x(n)\}_{n\in\mathbb{Z}}$, that is,

$$\Omega = \left\{ x = \{x(n)\} \mid x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \in \mathbb{R}^{2d},\ x_j(n) \in \mathbb{R}^d,\ j = 1,2,\ n \in \mathbb{Z} \right\}. \tag{2.1}$$

x can be rewritten as $x = (\ldots, x^\tau(-n), \ldots, x^\tau(-1), x^\tau(0), x^\tau(1), \ldots, x^\tau(n), \ldots)^\tau$. For any

$x, y \in \Omega$, $a, b \in \mathbb{R}$, $ax + by$ is defined by

$$\begin{aligned} ax + by &\triangleq \{ax(n) + by(n)\} \\ &= \big(\dots, ax^{\tau}(-n) + by^{\tau}(-n), \dots, ax^{\tau}(-1) + by^{\tau}(-1), ax^{\tau}(0) + by^{\tau}(0), \\ &\qquad ax^{\tau}(1) + by^{\tau}(1), \dots, ax^{\tau}(n) + by^{\tau}(n), \dots\big)^{\tau}. \end{aligned} \tag{2.2}$$

Then Ω is a vector space.

For any given prime integer $p > 2$, E_p is defined as a subspace of Ω by

$$E_p = \{x = \{x(n)\} \in \Omega \mid x(n+p) = x(n),\ n \in \mathbb{Z}\}. \tag{2.3}$$

E_p can be equipped with the norm $\|\cdot\|_{E_p}$ and the inner product $\langle \cdot, \cdot \rangle_{E_p}$ as follows:

$$\|x\|_{E_p} = \left(\sum_{n=1}^{p} |x(n)|^2 \right)^{1/2}, \quad \langle x, y \rangle_{E_p} = \sum_{n=1}^{p} (x(n), y(n)), \tag{2.4}$$

where $|\cdot|$ denotes the usual Euclidean norm and $(\cdot, \cdot)$ denotes the usual scalar product in $\mathbb{R}^{2d}$.

Define a linear map $\Gamma : E_p \to \mathbb{R}^{2dp}$ by

$$\begin{aligned} \Gamma x = \big(&x_1^1(1), \dots, x_1^d(1), x_1^1(2), \dots, x_1^d(2), \dots, x_1^1(p), \dots, x_1^d(p), \\ &x_2^1(1), \dots, x_2^d(1), x_2^1(2), \dots, x_2^d(2), \dots, x_2^1(p), \dots, x_2^d(p)\big)^{\tau}, \end{aligned} \tag{2.5}$$

where $x = \{x(n)\}$ and $x(i) = (x_1^1(i), \dots, x_1^d(i), x_2^1(i), \dots, x_2^d(i))^{\tau}$ for $i \in \mathbb{Z}(1, p)$. It is easy to see that the map Γ is a linear homeomorphism with $\|x\|_{E_p} = |\Gamma x|$ and $(E_p, \langle \cdot, \cdot \rangle_{E_p})$ is a Hilbert space which can be identified with $\mathbb{R}^{2dp}$.

To get a decomposition of the Hilbert space E_p, in the following we discuss the eigenvalue problem:

$$\Delta x(n) = \lambda J L x(n), \quad n \in \mathbb{Z}, \qquad x(n+p) = x(n), \tag{2.6}$$

where $\lambda \in \mathbb{R}$.

It is obvious that $\lambda = 0$ is an eigenvalue of (2.6) whose eigenfunction can be given by

$$\eta_0(n) = (a_1, a_2, \dots, a_{2d})^{\tau}, \quad a_i \in \mathbb{R},\ i = 1, 2, \dots, 2d,\ n = 1, 2, \dots, p. \tag{2.7}$$

By a simple computation, (2.6) is equivalent to

$$\begin{aligned} \Delta x_1(n) &= -\lambda x_2(n), \quad x_1(n+p) = x_1(n), \\ \Delta x_2(n-1) &= \lambda x_1(n), \quad x_2(n+p) = x_2(n). \end{aligned} \tag{2.8}$$

If $\lambda \neq 0$, then (2.8) is equivalent to

$$\begin{aligned} \Delta^2 x_1(n-1) + \lambda^2 x_1(n) &= 0, \quad x_1(n+p) = x_1(n), \\ \Delta^2 x_2(n-1) + \lambda^2 x_2(n) &= 0, \quad x_2(n+p) = x_2(n). \end{aligned} \tag{2.9}$$

It is known that (2.9) has a nontrivial solution if and only if $\lambda^2 = \lambda_k^2 = 4\sin^2(k\pi/p)$ with $k \in \mathbb{Z}(1,(p-1)/2)$, see, for example, [13, 14]. So in this case (2.6) has a nontrivial solution if and only if $\lambda = \lambda_k = 2\sin(k\pi/p)$ with $k \in \mathbb{Z}(-(p-1)/2,(p-1)/2)\setminus\{0\}$. It is easy to see that the multiplicities of λ_k for each $k \in \mathbb{Z}(-(p-1)/2,(p-1)/2)$ are of the same number $2d$.

To get an explicit decomposition of the Hilbert space E_p, in the following, we also need to compute eigenfunctions of (2.6) corresponding to each λ_k, $k \neq 0$.

Fix a $k \in \mathbb{Z}(-(p-1)/2,-1) \cup \mathbb{Z}(1,(p-1)/2)$, any solutions to (2.9) can be written as

$$x_1(n) = c_1\cos(kwn) + c_2\sin(kwn), \qquad x_2(n) = d_1\cos(kwn) + d_2\sin(kwn), \tag{2.10}$$

where $w = 2\pi/p$ and c_1, c_2, d_1, d_2 are constant vectors in $\mathbb{R}^d$. Using the relation between x_1, x_2, that is, (2.8) with $\lambda = \lambda_k$, we have

$$\begin{aligned} c_1\sin\left(\frac{kw}{2}\right) - c_2\cos\left(\frac{kw}{2}\right) &= d_1, \\ c_2\sin\left(\frac{kw}{2}\right) + c_1\cos\left(\frac{kw}{2}\right) &= d_2. \end{aligned} \tag{2.11}$$

If we choose $c_1 = e_j$, $c_2 = 0$, then $d_1 = \sin(kw/2)e_j$, $d_2 = \cos(kw/2)e_j$; if we choose $c_1 = 0$, $c_2 = e_j$, then $d_1 = -\cos(kw/2)e_j$, $d_2 = \sin(kw/2)e_j$, where e_j, $j = 1,2,\ldots,d$ denotes the canonical basis of $\mathbb{R}^d$. So, eigenfunctions of (2.6) corresponding to each $\lambda_k(k \neq 0)$ can be given as

$$\begin{aligned} \eta_{k,j}^{(1)}(n) &= \begin{pmatrix} \cos(kwn)e_j \\ \sin\left(kw\left(n+\dfrac{1}{2}\right)\right)e_j \end{pmatrix}, \quad n = 1,2,\ldots,p, \\ \eta_{k,j}^{(2)}(n) &= \begin{pmatrix} \sin(kwn)e_j \\ -\cos\left(kw\left(n+\dfrac{1}{2}\right)\right)e_j \end{pmatrix}, \quad n = 1,2,\ldots,p. \end{aligned} \tag{2.12}$$

Hereto, E_p can be decomposed as $E_p = X \oplus X_1 \oplus X_2$ with

$$\begin{aligned} X &= \{x = \{x(n)\} \mid x(n) = c_1e_1 + c_2e_2 + \cdots + c_{2d}e_{2d},\ c_i \in \mathbb{R},\ i = 1,2,\ldots,2d,\ n = 1,2,\ldots,p\}, \\ X_1 &= \left\{x = \{x(n)\} \mid x(n) = \sum_{j=1}^{d}\sum_{k=1}^{(p-1)/2}\alpha_{k,j}\eta_{k,j}^{(1)}(n) + \sum_{j=1}^{d}\sum_{k=-(p-1)/2}^{-1}\alpha_{k,j}\eta_{k,j}^{(1)}(n),\ \alpha_{k,j} \in \mathbb{R}\right\}, \\ X_2 &= \left\{x = \{x(n)\} \mid x(n) = \sum_{j=1}^{d}\sum_{k=1}^{(p-1)/2}\beta_{k,j}\eta_{k,j}^{(2)}(n) + \sum_{j=1}^{d}\sum_{k=-(p-1)/2}^{-1}\beta_{k,j}\eta_{k,j}^{(2)}(n),\ \beta_{k,j} \in \mathbb{R}\right\}. \end{aligned} \tag{2.13}$$

Finally, we briefly introduce the Z_p geometrical index theory which can be found in [12].

Define a linear operator $\mu : E_p \to E_p$ as follows. For any $x \in E_p$,

$$\mu x(n) = x(n+1), \quad \forall n \in \mathbb{Z}. \tag{2.14}$$

Clearly, for any $x \in E_p$, $\mu^p x = x$ and $\|\mu x\|_{E_p} = \|x\|_{E_p}$. So μ is an isometric action of group Z_p on E_p. It is easy to see that $\mathrm{Fix}_\mu := \{x \in E_p \mid \mu x = x\} = X$.

Note that if x is a periodic solution to (1.1) with period p, then μx is also a periodic solution to (1.1) with period p. We call $\lceil x \rceil = \{\mu x, \mu^2 x, \dots, \mu^p x\}$ a Z_p-orbit of period solution x to (1.1) with period p.

Let E be a Banach space and let μ be a linear isometric action of Z_p on E. Namely, μ is a linear operator on E satisfying $\|\mu x\| = \|x\|$ for any $x \in E$ and $\mu^p = id_E$, where Z_p is the cyclic group with order p and id_E is the identity map on E.

A subset $A \subset E$ is called μ-invariant if $\mu(A) \subset A$. A continuous map $f : A \to E$ is called μ-equivariant if $f(\mu x) = \mu f(x)$ for any $x \in A$. A continuous functional $F : E \to \mathbb{R}$ is said to be μ-invariant if for any $x \in E$, $F(\mu x) = F(x)$.

Let us recall the definition of the Palais-Smale condition.

Let E be a real Banach space and $F \in C^1(E,\mathbb{R})$. F is said to satisfy the Palais-Smale condition ((PS) condition) if any sequence $\{x^{(m)}\} \subset E$ for which $\{F(x^{(m)})\}$ is bounded and $F'(x^{(m)}) \to 0 (m \to \infty)$ possesses a convergent subsequence in E.

Our result is based on the following theorem (see [12, Theorem 2.1]).

Theorem 2.1. *Let $F \in C^1(E,\mathbb{R})$ be a μ-invariant functional satisfying the "PS" condition. Let Y and Z be closed μ-invariant subspaces of E with $co\dim Y$ and $\dim Z$ finite and*

$$co\dim Y < \dim Z. \tag{2.15}$$

Assume that the following conditions are satisfied.

(F1) $\mathrm{Fix}_\mu \subset Y$, $Z \cap \mathrm{Fix}_\mu = \{0\}$;

(F2) $\inf_{x \in Y} F(x) > -\infty$;

(F3) *there exist $r > 0$ and $c < 0$ such that $F(x) \le c$ whenever $x \in Z$ and $\|x\| = r$;*

(F4) *if $x \in \mathrm{Fix}_\mu$ and $F'(x) = 0$, then $F(x) \ge 0$.*

Then there exist at least $\dim Z - co\dim Y$ distinct Z_p-orbits of critical points of F outside of Fix_μ with critical value less or equal to c.

The following estimate will be useful in the subsequent sections.

Proposition 2.2. *For any $x \in E_p$, the following inequality holds:*

$$\sum_{n=1}^{p} |\Delta x(n)|^2 \le 2\left(1 + \cos\frac{\pi}{p}\right) \sum_{n=1}^{p} |x(n)|^2. \tag{2.16}$$

Proof. We note that

$$\sum_{n=1}^{p} |\Delta x(n)|^2 = 2\sum_{n=1}^{p} \left[\left(x(n),x(n)\right) - \left(x(n+1),x(n)\right) \right] = (A\Gamma x, \Gamma x), \tag{2.17}$$

where

$$A = \begin{pmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{pmatrix}_{2dp\times 2dp} \quad \text{with } B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{p\times p}. \tag{2.18}$$

It follows from [15] that p distinct eigenvalues of matrix B are $\overline{\lambda}_k = 4\sin^2(k\pi/p)$ with $k \in \mathbb{Z}(0, p-1)$ and $\overline{\lambda}_{\max} = \max\{\overline{\lambda}_k \mid k \in \mathbb{Z}(0, p-1)\} = 2(1+\cos(\pi/p))$. Since $|\Gamma x|^2 = \|x\|^2_{E_p} = \sum_{n=1}^{T} |x(n)|^2$, inequality (2.16) now follows from (2.17). □

Remark 2.3. Noticing that the set of eigenvalues $\{\lambda_k \mid k \in \mathbb{Z}(-(p-1)/2, (p-1)/2)\}$ is bounded from below by -2 and bounded from above by 2 which are different from the differential case. So, we can avoid the fussy process of finding the dual action which is necessary for the differential case (see [4, Chapter 7]).

3. The Morse index of a linear positive definite Hamiltonian systems

In this section, we define a pair of index functions $(i(S,p), \nu(S,p)) \in \mathbb{Z}(0, 2dp) \times \mathbb{Z}(0, 2dp)$ for any real symmetric positive definite matrix S and obtain the formulae of the computations of index functions for a diagonal positive definite matrix.

As stated in [10, 11], the corresponding action functional of (1.3) is defined on E_p by

$$F_S(x) = \frac{1}{2}\sum_{n=1}^{p}\left[(J\Delta x(n), Lx(n)) + (S(n)Lx(n), Lx(n))\right]. \tag{3.1}$$

Definition 3.1. The index $i(S,p)$ is the Morse index of F_S, that is, the supremum of the dimensions of the subspaces of E_p on which F_S is negative definite.

Our assumption follows the existence of $\delta_p > 0$ such that $(S(n)x, x) \geq \delta_p |x|^2$ for every $n \in \mathbb{Z}$ and $x \in \mathbb{R}^{2d}$. The symmetric bilinear form given by $(x,y)_S = \sum_{n=1}^{p}(S(n)Lx(n), Ly(n))$ defines an inner product on E_p. The corresponding norm $\|\cdot\|_S$ is such that

$$\|x\|_S^2 \geq \delta_p \sum_{n=1}^{p} \left|Lx(n)\right|^2 = \delta_p \sum_{n=1}^{p} \left|x(n)\right|^2. \tag{3.2}$$

For any $x, y \in E_p$, if we define a bilinear function as $a(x,y) = \sum_{n=1}^{p}(Jx(n), \Delta Ly(n-1))$, then by Proposition 2.2 and (3.2) we have

$$\begin{aligned}
|a(x,y)| &\leq \left(\sum_{n=1}^{p}|Jx(n)|^2\right)^{1/2}\left(\sum_{n=1}^{p}|\Delta Ly(n-1)|^2\right)^{1/2} \\
&= \left(\sum_{n=1}^{p}|x(n)|^2\right)^{1/2}\left(\sum_{n=1}^{p}|\Delta y(n)|^2\right)^{1/2} \\
&\leq \sqrt{2\left(1+\cos\left(\frac{\pi}{p}\right)\right)}\left(\sum_{n=1}^{p}|x(n)|^2\right)^{1/2}\left(\sum_{n=1}^{p}|y(n)|^2\right)^{1/2} \\
&\leq \frac{\sqrt{2(1+\cos(\pi/p))}}{\delta_p}\|x\|_S\|y\|_S.
\end{aligned} \tag{3.3}$$

So, by [16, Theorem 2.2.2], we can define the unique continuous linear operator K on E_p by $(Kx,y)_S = \sum_{n=1}^{p}(Jx(n), \Delta Ly(n-1))$. Since

$$\sum_{n=1}^{p}(Jx(n), \Delta Ly(n-1)) = -\sum_{n=1}^{p}(J\Delta x(n), Ly(n)), \tag{3.4}$$

we have

$$2F_S(x) = (x - Kx, x)_S. \tag{3.5}$$

It is obvious that K is self-adjoint. So, it follows from (3.5) that E_p will be the orthogonal sum of $\ker(I-K) = H^0(S)$, $H^-(S)$ and $H^+(S)$ with $I-K$ positive definite (resp., negative definite) on $H^+(S)$ (resp., $H^-(S)$). Clearly, $i(S,p) = \dim H^-(S) \in \mathbb{Z}(0, 2dp)$. On the other hand, there exists $\overline{\delta} > 0$ such that

$$\begin{aligned}
(x - Kx, x)_S &\geq \overline{\delta}\|x\|_S^2, \quad x \in H^+(S), \\
(x - Kx, x)_S &\leq -\overline{\delta}\|x\|_S^2, \quad x \in H^-(S).
\end{aligned} \tag{3.6}$$

Setting $\delta = \overline{\delta}\delta_p > 0$, we deduce from (3.2) and (3.5) the estimates

$$F_S(x) \geq \frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2, \quad x \in H^+(S), \tag{3.7}$$

$$F_S(x) \leq -\frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2, \quad x \in H^-(S). \tag{3.8}$$

Definition 3.2. The nullity $\nu(S,p)$ is the dimension of $\ker(I-K)$.

We now state and prove a result which offers another interpretation of the nullity $\nu(S,p)$.

Proposition 3.3. $\ker(I-K)$ *is isomorphic to the space of solutions to (1.3).*

Proof. By the fact that $J\Delta x(n) = \Delta Jx(n)$ we have

$$\begin{aligned}
x \in \ker(I-K) &\Longleftrightarrow ((I-K)x, y)_S = 0, \quad \forall y \in E_p, \\
&\Longleftrightarrow \sum_{n=1}^{p} \left[(S(n)Lx(n), Ly(n)) - (Jx(n), \Delta Ly(n-1)) \right] = 0, \quad \forall y \in E_p, \\
&\Longleftrightarrow \sum_{n=1}^{p} (\Delta Jx(n) + S(n)Lx(n), Ly(n)) = 0, \quad \forall y \in E_p, \\
&\Longleftrightarrow J\Delta x(n) + S(n)Lx(n) = 0, \quad n \in \mathbb{Z}(1,p),
\end{aligned} \tag{3.9}$$

which implies that $\ker(I-K)$ is isomorphic to the space of solutions to (1.3). □

To get more information on the index functions, in the following we will compute the index and the nullity of the diagonal positive definite matrix. By direct computation, it is easy to get the following.

Proposition 3.4. *Let* $A = \operatorname{diag}\{a_1, a_2, \ldots, a_{2d}\}$ *with* $a_i > 0$, $i \in \mathbb{Z}(1,2d)$. *Then, all the eigenvalues of* JA *must be pure imaginary and*

$$\sigma(JA) = \{ \pm i\alpha_j \mid \alpha_j > 0,\ j = 1,2,\ldots,d \} \tag{3.10}$$

with $\alpha_j = \sqrt{a_j a_{j+d}}$.

On the formulae of the computations of the index and the nullity, we have the following.

Proposition 3.5. *For the above matrix* A, *one has*

$$\begin{aligned}
i(A,p) &= 2 \sum_{j=1}^{d} \sharp \left\{ k \in \mathbb{Z}\left(1, \frac{p-1}{2}\right) \mid \alpha_j < 2 \sin \frac{k\pi}{p} \right\}, \\
\nu(A,p) &= 2 \sum_{j=1}^{d} \sharp \left\{ k \in \mathbb{Z}\left(1, \frac{p-1}{2}\right) \mid \alpha_j = 2 \sin \frac{k\pi}{p} \right\}.
\end{aligned} \tag{3.11}$$

Proof. If $(I-K)x = \lambda x$ with $x \in E_p$, then for all $y \in E_p$, we have

$$\sum_{n=1}^{p} (J\Delta x(n), Ly(n)) + (ALx(n), Ly(n)) = \sum_{n=1}^{p} (A\lambda Lx(n), Ly(n)) \tag{3.12}$$

which implies that

$$\Delta x(n) = JA(1-\lambda)Lx(n), \quad n \in \mathbb{Z}, \quad x(n) = x(n+p). \tag{3.13}$$

Assume that the general solutions to (3.13) are of the form

$$x(n) = \mu^n \xi = \mu^n \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{3.14}$$

where ξ_1, ξ_2 are vectors in $\mathbb{R}^d$. By $x(0) = x(p)$, we have $\mu^p = 1$, so $\mu = e^{ikw}$, $k = 0, 1, 2, \ldots, p-1$, where $w = 2\pi/p$. Therefore, any nontrivial solution to (3.13) can be expressed as

$$x(n) = e^{ikwn} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{3.15}$$

Substituting (3.15) into (3.13), we have

$$2i \sin \frac{kw}{2} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = JA(1-\lambda) \begin{pmatrix} e^{ikw/2} I_d & 0 \\ 0 & e^{-ikw/2} I_d \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{3.16}$$

Noticing that

$$\sigma\left(JA \begin{pmatrix} e^{ikw/2} I_d & 0 \\ 0 & e^{-ikw/2} I_d \end{pmatrix} \right) = \sigma(JA), \tag{3.17}$$

by Definitions 3.1 and 3.2 and Proposition 3.4, we get the conclusion. □

4. Periodic solutions to convex asymptotically linear autonomous discrete Hamiltonian systems

In this section, we consider the existence of multiple p-periodic solutions to (1.1) where $H \in C^1(\mathbb{R}^{2d}, \mathbb{R})$ is strictly convex and satisfies the following asymptotically linear conditions:

$$\nabla H(x) = A_0 x + o(|x|) \quad \text{as } |x| \longrightarrow 0, \tag{4.1}$$

$$\nabla H(x) = A_\infty x + o(|x|) \quad \text{as } |x| \longrightarrow \infty \tag{4.2}$$

with real symmetric positive definite matrices A_0, A_∞. Our main result is the following.

THEOREM 4.1. *Assume that*

(A1) $\nu(A_\infty, p) = 0$,

(A2) $i(A_0, p) > i(A_\infty, p)$.

Then (1.1) has at least $i(A_0, p) - i(A_\infty, p)$ distinct nonconstant Z_p-periodic orbits.

Remark 4.2. (1) It follows from (A1) and Proposition 3.3 that the linear systems

$$J\Delta x(n) + A_\infty L x(n) = 0, \quad n \in \mathbb{Z} \tag{4.3}$$

do not have any nontrivial p-periodic solutions. Thus (A1) is a nonresonance condition at infinity.

(2) Since H is strictly convex and $\nabla H(0)=0$ by (4.1), 0 is the unique equilibrium point of (1.1). Without loss of generality, we can assume that $H(0)=0$. The action functional of (1.1) defined by

$$F_H(x)=\sum_{n=1}^{p}\left[\frac{1}{2}(J\Delta x(n),Lx(n))+H(Lx(n))\right] \tag{4.4}$$

is continuously differentiable on E_p. Since F_H is a μ-invariant functional, we are in a position to apply Theorem 2.1.

(3) It is convenient in this section to use the inner product $(x,y)_{A_\infty}=\sum_{n=1}^{p}(A_\infty Lx(n), Ly(n))$ and the corresponding norm $\|\cdot\|_{A_\infty}$ in E_p. The norm is equivalent to the standard norm of E_p.

The proof of Theorem 4.1 depends on the following lemmas. The first one implies that F_H satisfies the "PS" condition.

Lemma 4.3. *Every sequence $\{x^{(j)}\}$ in E_p such that $F_H'(x^{(j)})\to 0(j\to\infty)$ contains a convergent subsequence.*

Proof. Let us define the operator Q over E_p, using the Riesz theorem, by the formula

$$(Qx,y)_{A_\infty}=\sum_{n=1}^{p}(\nabla H(Lx(n))-A_\infty Lx(n),Ly(n)). \tag{4.5}$$

Since

$$\langle F_H'(x),y\rangle=\sum_{n=1}^{p}(J\Delta x(n),Ly(n))+(\nabla H(Lx(n),Ly(n))), \tag{4.6}$$

we have

$$\langle F_H'(x),y\rangle=(x-Kx+Qx,y)_{A_\infty}. \tag{4.7}$$

Let $f^{(j)}=x^{(j)}-Kx^{(j)}+Qx^{(j)}$. Then by assumption $F_H'(x^{(j)})\to 0(j\to\infty)$, we have $f^{(j)}\to 0$ as $j\to\infty$. In particular, there exists $R>0$ such that $\|f^{(j)}\|\le R$ for every j. Assumption (A1) implies that $P=I-K$ is invertible. Thus, it follows from (4.2) that there exists some $c>0$ such that $\|Qx\|\le 1/2\|P^{-1}\|^{-1}\|x\|+c$ for all $x\in E_p$. Therefore, we have

$$||x^{(j)}||=||P^{-1}Px^{(j)}||\le||P^{-1}||(||f^{(j)}||+||Qx^{(j)}||)\le\frac{1}{2}||x^{(j)}||+||P^{-1}||(c+R) \tag{4.8}$$

and hence $\{x^{(j)}\}$ is bounded. The proof is complete since E_p is a finite dimensional space. □

We now verify the condition (F2) of Theorem 2.1 for F_H.

Lemma 4.4. *The functional F_H is bounded from below on a closed μ-invariant subspace Y of E_p with codimension $i(A_\infty,p)$.*

Proof. By assumption (A1), E_p is the orthogonal direct sum of $H^+(A_\infty)$ and $H^-(A_\infty)$. Hence, $co\dim H^+(A_\infty) = \dim H^-(A_\infty) = i(A_\infty, p)$ and there exists a closed μ-invariant subspace $Y = H^+(A_\infty)$ of E_p with codimension $i(A_\infty, p)$. By (3.7), there exists $\delta > 0$ such that for each $x \in Y$,

$$F_{A_\infty}(x) = \frac{1}{2}\sum_{n=1}^{p}\left[\left(J\Delta Lx(n-1), x(n)\right) + \left(A_\infty Lx(n), Lx(n)\right)\right] \geq \frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2. \tag{4.9}$$

It follows from (4.2) that there exists $c > 0$ such that $|\nabla H(x) - A_\infty x| \leq \delta|x|/2 + c$ for each $x \in \mathbb{R}^{2d}$. Hence, by direct integrating, we have

$$\begin{aligned} \left|H(x) - \frac{1}{2}(A_\infty x, x)\right| &\leq \int_0^1 \left|\left(\nabla H(tx) - A_\infty tx, x\right)\right| dt \\ &\leq \int_0^1 \left(\frac{\delta}{2}t|x|^2 + c|x|\right) dt \\ &= \frac{\delta}{4}|x|^2 + c|x|. \end{aligned} \tag{4.10}$$

Consequently, we have, for $x \in Y$,

$$\begin{aligned} F_H(x) &= F_{A_\infty}(x) + \sum_{n=1}^{p}\left(H\left(Lx(n)\right) - \frac{1}{2}\left(A_\infty Lx(n), Lx(n)\right)\right) \\ &\geq \frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2 - \sum_{n=1}^{p}\left[\frac{\delta}{4}|Lx(n)|^2 + c|Lx(n)|\right] \\ &\geq \frac{\delta}{4}\sum_{n=1}^{p}|x(n)|^2 - cp^{1/2}\left(\sum_{n=1}^{p}|x(n)|^2\right)^{1/2} \end{aligned} \tag{4.11}$$

and hence F_H is bounded from below on Y. □

Now, we show that the condition (F3) of Theorem 2.1 holds for F_H.

Lemma 4.5. *There exists a closed μ-invariant subspace Z of E_p with dimension $i(A_0, p)$ and some $r > 0$ such that $F(x) < 0$ whenever $x \in Z$ and $\|x\|_{A_\infty} = r$.*

Proof. By (3.8), there exists a μ-invariant subspace $Z = H^-(A_0)$ of E_p with dimension $i(A_0, p)$ and some $\delta > 0$ such that

$$F_{A_0}(x) = \frac{1}{2}\sum_{n=1}^{p}\left[\left(J\Delta Lx(n-1), x(n)\right) + \left(A_0 Lx(n), Lx(n)\right)\right] \leq -\frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2 \tag{4.12}$$

whenever $x \in Z$. By (4.1), there exists $r > 0$ such that $|\nabla H(x) - A_0 x| \leq \delta|x|/2$ for each $x \in \mathbb{R}^{2d}$ with $\|x\|_{A_\infty} \leq r$. Hence, by direct integrating, we have

$$\left|H(x) - \frac{1}{2}(A_0 x, x)\right| \leq \int_0^1 \left|\left(\nabla H(tx) - A_0 tx, x\right)\right| dt \leq \int_0^1 \frac{\delta}{2}t|x|^2 dt = \frac{\delta}{4}|x|^2 \tag{4.13}$$

whenever $\|x\|_{A_\infty} \le r$. Consequently, if $x \in Z$ and $0 < \|x\|_{A_\infty} \le r$, we get

$$\begin{aligned} F_H(x) &= F_{A_0}(x) + \sum_{n=1}^{p}\left(H(Lx(n)) - \frac{1}{2}(A_0Lx(n), Lx(n))\right) \\ &\le -\frac{\delta}{2}\sum_{n=1}^{p}|x(n)|^2 + \frac{\delta}{4}\sum_{n=1}^{p}|x(n)|^2 = -\frac{\delta}{4}\sum_{n=1}^{p}|x(n)|^2 \end{aligned} \tag{4.14}$$

and the proof is complete. □

Proof of Theorem 4.1. We apply Theorem 2.1 to F_H which is μ-invariant and satisfies the "PS" condition by Lemma 4.3. The spaces Y and Z introduced, respectively, in Lemma 4.4 and Lemma 4.5 satisfy the assumption $i(A_\infty, p) = co\dim Y < \dim Z = i(A_0, p)$. Since $\mathrm{Fix}_\mu = X$ for all $0 \ne x \in X$, we have $F_{A_\infty}(x) = 1/2\sum_{n=1}^{p}(A_\infty Lx(n), Lx(n)) > 0$, so $x \in H^+(A_\infty) = Y$. At the same time, it is easy to verify that $\mathrm{Fix}_\mu \cap Z = \mathrm{Fix}_\mu \cap H^-(A_0) = \{0\}$. So, the condition (F1) of Theorem 2.1 holds for F_H. Finally, if $x \in \mathrm{Fix}_\mu$ and $F_H'(x) = 0$, then $\langle F_H'(x), y\rangle = \sum_{n=1}^{p}(\nabla H(Lx(n)), Ly(n)) = 0$. By (2) of Remark 4.2 we have $x = 0$, so $F_H(0) = 0$ and the condition (F4) of Theorem 2.1 holds for F_H. Thus, all the conditions of Theorem 2.1 are satisfied. Then there exist at least $i(A_0, p) - i(A_\infty, p)$ distinct nonconstant Z_p-orbits of critical points of F_H and the proof is complete. □

Remark 4.6. In addition to the assumptions in Theorem 4.1, if we further assume that the Hamiltonian function is odd on $\mathbb{R}^{2N}$, then for any prime integer $p > 2$, (1.1) possesses at least $2[i(A_0, p) - i(A_\infty, p)]$ distinct Z_p-orbits of solutions with period p(see [12, Corollary 1.1]).

Example 4.7. Let $H \in C^1(\mathbb{R}^{2d}, \mathbb{R})$ be strictly convex such that $H(0) = 0$ and $\nabla H(0) = 0$. Let $p > 0$ be a prime integer. Assume that there exists $\gamma > 2$ such that

$$\nabla H(x) = \gamma x + o(|x|) \quad \text{as } |x| \longrightarrow \infty \tag{4.15}$$

and some $1 \le j \le (p-1)/2$ and $2\sin((j-1)\pi/p) < \beta < 2\sin(j\pi/p)$ such that

$$\nabla H(x) = \beta x + o(|x|) \quad \text{as } |x| \longrightarrow 0. \tag{4.16}$$

By Proposition 3.5, we get $\nu(\gamma I_d, p) = 0$, $i(\gamma I_d, p) = 0$, and $i(\beta I_d, p) = d(p - 2j + 1)$. Then, the problem

$$J\Delta x(n) + \nabla H(Lx(n)) = 0, \quad n \in \mathbb{Z},\ x(n+p) = x(n) \tag{4.17}$$

has at least $d(p - 2j + 1)$ distinct nonconstant Z_p-periodic orbits.

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