Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2007, Article ID 87818, 15 pages
doi:10.1155/2007/87818

Research Article

Positive Solutions of Two-Point Right Focal Eigenvalue Problems on Time Scales

Yuguo Lin and Minghe Pei

Received 26 May 2007; Revised 20 July 2007; Accepted 21 September 2007

Recommended by Patricia J. Y. Wong

We offer criteria for the existence of positive solutions for two-point right focal eigenvalue problems $(-1)^{n-p}y^{\Delta^n}(t) = \lambda f(t, y(\sigma^{n-1}(t)), y^{\Delta}(\sigma^{n-2}(t)),\dots, y^{\Delta^{p-1}}(\sigma^{n-p}(t)))$, $t \in [0,1]\cap\mathbb{T}$, $y^{\Delta^i}(0) = 0$, $0 \le i \le p-1$, $y^{\Delta^i}(\sigma(1)) = 0$, $p \le i \le n-1$, where $\lambda > 0$, $n \ge 2$, $1 \le p \le n-1$ are fixed and $\mathbb{T}$ is a time scale.

1. Introduction

In this paper, we present results governing the existence of positive solutions to the differential equation on time scales of the form

$$(-1)^{n-p}y^{\Delta^n}(t) = \lambda f\left(t, y(\sigma^{n-1}(t)),\dots, y^{\Delta^{p-1}}(\sigma^{n-p}(t))\right),\quad t \in [0,1]\cap\mathbb{T} \tag{1.1}$$

subject to the two-point right focal boundary conditions

$$\begin{aligned} y^{\Delta^i}(0) &= 0, \quad 0 \le i \le p-1,\\ y^{\Delta^i}(\sigma(1)) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{1.2}$$

where $\lambda > 0$, p, n are fixed integers satisfying $n \ge 2$, $1 \le p \le n-1$, $0,1 \in \mathbb{T}$, with $0 < \sigma(1)$ and $\rho(\sigma(1)) = 1$ and $f : [0,1]\times\mathbb{R}^p \to \mathbb{R}$ is continuous.

We say that $y(t)$ is a positive solution of BVP (1.1), (1.2) if $y(t) \in C^n_{\mathrm{rd}}[0,1]$ is a solution of BVP (1.1), (1.2) and $y^{\Delta^i}(t) > 0$, $t \in (0,\sigma^{n-i}(1))$, $i = 0,1,\dots,p-1$. If, for a particular λ, BVP (1.1), (1.2) has a positive solution y, then λ is called an eigenvalue and y a corresponding eigenfunction of BVP (1.1), (1.2). We let

$$E = \{\lambda > 0 : \text{BVP (1.1), (1.2) has at least one positive solution}\} \tag{1.3}$$

be the set of eigenvalues of BVP (1.1), (1.2).

To understand the notations used in BVP (1.1), (1.2), we recall some standard definitions as follows. The reader may refer to [1] for an introduction to the subject.

(a) Let T be a time scale, that is, $\mathbb{T}$ is a closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Throughout, for any $a, b\,(> a)$, the interval $[a,b]$ is defined as $[a,b] = \{t \in \mathbb{T} \mid a \le t \le b\}$. Analogous notations for open and half-open intervals will also be used in the paper. We also use the notation $\mathbb{R}[c,d]$ to denote the real interval $\{t \in \mathbb{R} \mid c \le t \le d\}$.

(b) For $t < \sup \mathbb{T}$ and $s > \inf \mathbb{T}$, the forward jump operator σ and the backward jump operator ρ are, respectively, defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T}, \qquad \rho(s) = \sup\{\tau \in \mathbb{T} \mid \tau < s\} \in \mathbb{T}. \tag{1.4}$$

We define $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ with $\sigma^0(t) = t$. Similar definition is used for $\rho^n(s)$.

(c) Fix $t \in \mathbb{T}$. Let $y : \mathbb{T} \to \mathbb{R}$. We define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]\right| < \varepsilon\left|\sigma(t) - s\right|. \tag{1.5}$$

We call $y^\Delta(t)$ the delta derivative of $y(t)$. Define $y^{\Delta^n}(t)$ to be the delta derivative of $y^{\Delta^{n-1}}(t)$, that is, $y^{\Delta^n}(t) = (y^{\Delta^{n-1}}(t))^\Delta$.

(d) If $F^\Delta(t) = f(t)$, then we define the integral

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a). \tag{1.6}$$

(e) If $\sigma(t) > t$, then call the point t right-scattered; while if $\rho(t) < t$, then say t is left-scattered. If $\sigma(t) = t$, then call the point t right-dense; while if $\rho(t) = t$, then say t is left-dense.

Focal boundary value problems have attracted a lot of attention in the recent literature, see [2–7]. Recently, many papers have discussed the existence of nonnegative solution of right focal boundary value problem on time scales, see [8–12]. Motivated by the works mentioned above, the purpose of this article is to present results which guarantee the existence of one or more positive solutions to BVP (1.1), (1.2).

The paper is outlined as follows. In Section 2, we will present some lemmas and definitions which will be used later. In Section 3, by using Krasnoselskii's fixed-point theorem in a cone, we offer criteria for the existence of positive solution of BVP (1.1), (1.2).

2. Preliminary

Definition 2.1 [9]. (1) Define the function $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in 0, 1, \ldots$, recursively as

$$\begin{aligned} h_0(t,s) &= 1 \quad \forall s,t \in \mathbb{T}, \\ h_{k+1}(t,s) &= \int_s^t h_k(\tau,s)\Delta\tau \quad \forall s,t \in \mathbb{T},\ k = 0,1,\ldots. \end{aligned} \tag{2.1}$$

(2) Define the function $g_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in 0, 1, \ldots$, recursively as

$$\begin{aligned} g_0(t,s) &= 1 \quad \forall s,t \in \mathbb{T}, \\ g_{k+1}(t,s) &= \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \forall s,t \in \mathbb{T},\ k = 0,1,\ldots. \end{aligned} \tag{2.2}$$

(3) Let t_i, $1 \le i \le n$, such that $0 = t_1 = \cdots = t_p < t_{p+1} = \cdots = t_n = \sigma(1)$. Define $T_i : [0,1] \to \mathbb{R}$, $0 \le i \le n-1$ as

$$\begin{gathered} T_0(t) \equiv 1, \\ T_i(t) = T_i(t : t_1, \ldots, t_i) = \int_{t_1}^{t} \int_{t_2}^{\tau_1} \cdots \int_{t_i}^{\tau_{i-1}} \Delta\tau_i \cdots \Delta\tau_2 \Delta\tau_1, \quad 1 \le i \le n-1. \end{gathered} \tag{2.3}$$

Lemma 2.2 [1]. *For nonnegative integer n,*

$$h_n(t,s) = (-1)^n g_n(s,t), \quad t \in T,\ s \in T^{k^n}, \tag{2.4}$$

where

$$T^k = \begin{cases} T, & \text{if } T \text{ is unbounded above}, \\ T \setminus (\rho(\max T), \max T], & \text{otherwise}, \end{cases} \tag{2.5}$$

and $T^{k^n} = (T^{k^{n-1}})^k$. *Further, the functions satisfy the inequalities*

$$h_n(t,s) \ge 0, \quad g_n(t,s) \ge 0 \quad \forall t \ge s. \tag{2.6}$$

Lemma 2.3 [9]. *Green's function of the boundary value problem*

$$\begin{aligned} (-1)^{n-p} y^{\Delta^n}(t) &= 0, \quad t \in [0,1], \\ y^{\Delta^i}(0) &= 0, \quad 0 \le i \le p-1, \\ y^{\Delta^i}(\sigma(1)) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{2.7}$$

may be expressed as

$$K(t,s) = \begin{cases} (-1)^{n-p} \displaystyle\sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(0,\sigma(s)) + (-1)^{n-p+1} h_{n-1}(t,\sigma(s)), & t \le \sigma(s), \\ (-1)^{n-p} \displaystyle\sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(0,\sigma(s)), & t \ge \sigma(s), \end{cases} \tag{2.8}$$

where $t \in [0, \sigma^n(1)]$ *and* $s \in [0,1]$.

Lemma 2.4. *Let $k(t,s)$ be Green's function of the equation*

$$(-1)^{n-p}y^{\Delta^{n-p+1}}(t)=0,\quad t\in[0,\sigma^{n-p+1}(1)] \tag{2.9}$$

subject to the boundary conditions

$$\begin{aligned} y^{\Delta^i}(0)&=0, \quad 0\le i\le p-1,\\ y^{\Delta^i}(\sigma(1))&=0, \quad p\le i\le n-1. \end{aligned} \tag{2.10}$$

Then

$$L(t)\cdot g_{n-p}(\sigma(s),0)\le k(t,s)\le g_{n-p}(\sigma(s),0),\quad (t,s)\in[0,\sigma^{n-p+1}(1)]\times[0,1], \tag{2.11}$$

where

$$L(t)=\frac{t}{\sigma^{n-p+1}(1)}\le 1,\quad t\in[0,\sigma^{n-p+1}(1)]. \tag{2.12}$$

Proof. It is clear that

$$\begin{aligned} k(t,s)&=K^{\Delta_t^{p-1}}(t,s)\\ &=\begin{cases}(-1)^{n-p}\left(h_{n-p}(0,\sigma(s))-h_{n-p}(t,\sigma(s))\right), & t\le\sigma(s),\\ (-1)^{n-p}h_{n-p}(0,\sigma(s)), & t\ge\sigma(s),\end{cases}\\ &=\begin{cases}g_{n-p}(\sigma(s),0)-g_{n-p}(\sigma(s),t)), & t\le\sigma(s),\\ g_{n-p}(\sigma(s),0), & t\ge\sigma(s),\end{cases} \end{aligned} \tag{2.13}$$

where $t\in[0,\sigma^{n-p+1}(1)]$ and $s\in[0,1]$.

Obviously,

$$L(t)g_{n-p}(\sigma(s),0)\le g_{n-p}(\sigma(s),0),\quad t\ge\sigma(s). \tag{2.14}$$

Next, we will prove by induction that for $k=1,2,\ldots,$ and $t\le\sigma(s)$,

$$L(t)g_k(\sigma(s),0)\le g_k(\sigma(s),0)-g_k(\sigma(s),t)\le g_k(\sigma(s),0). \tag{2.15}$$

For $k=1$, we have

$$\begin{aligned} g_1(\sigma(s),0)-g_1(\sigma(s),t)&=\sigma(s)-(\sigma(s)-t)\\ &=t\ge\frac{t}{\sigma^{n-p+1}(1)}\cdot\sigma(s)=L(t)g_1(\sigma(s),0). \end{aligned} \tag{2.16}$$

We now assume that (2.15) holds for some $n\ge 1$.

Let $k = n+1$. We can obtain that for $\sigma(s) \geq t$,

$$
\begin{aligned}
g_{n+1}(\sigma(s),0) &\geq g_{n+1}(\sigma(s),0) - g_{n+1}(\sigma(s),t) \\
&= \int_0^{\sigma(s)} g_n(\sigma(\tau),0)\Delta\tau - \int_t^{\sigma(s)} g_n(\sigma(\tau),t)\Delta\tau \\
&= \int_0^{t} g_n(\sigma(\tau),0)\Delta\tau + \int_t^{\sigma(s)} [g_n(\sigma(\tau),0) - g_n(\sigma(\tau),t)]\Delta\tau \\
&\geq \int_0^{t} L(t)g_n(\sigma(\tau),0)\Delta\tau + \int_t^{\sigma(s)} L(t)g_n(\sigma(\tau),0)\Delta\tau \\
&= L(t)\int_0^{\sigma(s)} g_n(\sigma(\tau),0)\Delta\tau = L(t)g_{n+1}(\sigma(s),0).
\end{aligned}
\tag{2.17}
$$

Thus, (2.15) holds by induction. Therefore, from (2.14) and (2.15), we get

$$
L(t)g_{n-p}(\sigma(s),0) \leq k(t,s) \leq g_{n-p}(\sigma(s),0) \tag{2.18}
$$

on $[0,\sigma^{n-p+1}(1)] \times [0,1]$. □

Lemma 2.5. *Let $w(t)$ be the solution of BVP:*

$$
\begin{aligned}
(-1)^{(n-p)}u^{\Delta^n}(t) &= 1, \quad t \in [0,1], \\
u^{\Delta^i}(0) &= 0, \quad 0 \leq i \leq p-1, \\
u^{\Delta^i}(\sigma(1)) &= 0, \quad p \leq i \leq n-1.
\end{aligned}
\tag{2.19}
$$

Then

$$
0 \leq w^{\Delta^i}(t) \leq g_{n-p}(\sigma(1),0)h_{p-i}(t,0), \quad t \in [0,\sigma^{n-i}(1)],\ 0 \leq i \leq p-1. \tag{2.20}
$$

Proof. For $\sigma(s) \leq t$,

$$
\begin{aligned}
g_{n-p}(\sigma(s),0) &= (-1)^{n-p}h_{n-p}(0,\sigma(s)) = -\int_{\sigma(s)}^{0} (-1)^{n-p-1}h_{n-p-1}(\tau,\sigma(s))\Delta\tau \\
&= \int_0^{\sigma(s)} (-1)^{n-p-1}h_{n-p-1}(\tau,\sigma(s))\Delta\tau \\
&= \int_0^{\sigma(s)} g_{n-p-1}(\sigma(s),\tau)\Delta\tau \quad \text{(by Lemma 2.2)} \\
&\leq g_{n-p-1}(\sigma(s),0)\int_0^{t}\Delta\tau = g_{n-p-1}(\sigma(s),0)h_1(t,0).
\end{aligned}
\tag{2.21}
$$

For $t \le \sigma(s)$,

$$\begin{aligned}
&g_{n-p}(\sigma(s),0) - g_{n-p}(\sigma(s),t) \\
&\quad = (-1)^{n-p}h_{n-p}(0,\sigma(s)) - (-1)^{n-p}h_{n-p}(t,\sigma(s)) \\
&\quad = \int_0^{\sigma(s)} (-1)^{n-p-1}h_{n-p-1}(\tau,\sigma(s))\Delta\tau - \int_t^{\sigma(s)} (-1)^{n-p-1}h_{n-p-1}(\tau,\sigma(s))\Delta\tau \\
&\quad = \int_0^t (-1)^{n-p-1}h_{n-p-1}(\tau,\sigma(s))\Delta\tau = \int_0^t g_{n-p-1}(\sigma(s),\tau)\Delta\tau \quad (\text{by Lemma 2.2}) \\
&\quad \le g_{n-p-1}(\sigma(s),0)\int_0^t \Delta\tau = g_{n-p-1}(\sigma(s),0)h_1(t,0).
\end{aligned} \tag{2.22}$$

Hence,

$$0 \le k(t,s) \le g_{n-p-1}(\sigma(s),0)h_1(t,0), \quad (t,s) \in [0,\sigma^{n-p+1}(1)] \times [0,1]. \tag{2.23}$$

By defining $w(t)$ as $w(t) = \int_0^{\sigma(1)} K(t,s)\Delta s$, $t \in [0,\sigma^n(1)]$, it is clear that

$$w^{\Delta^{p-1}}(t) = \int_0^{\sigma(1)} k(t,s)\Delta s, \quad t \in [0,\sigma^{n-p+1}(1)]. \tag{2.24}$$

Then

$$\begin{aligned}
0 \le w^{\Delta^{p-1}}(t) = \int_0^{\sigma(1)} k(t,s)\Delta s &\le \int_0^{\sigma(1)} [g_{n-p-1}(\sigma(s),0)h_1(t,0)]\Delta s \\
&= g_{n-p}(\sigma(1),0)h_1(t,0).
\end{aligned} \tag{2.25}$$

Further, since $w^{\Delta^i}(0) = 0$, $0 \le i \le p-1$, we get

$$0 \le w^{\Delta^i}(t) \le g_{n-p}(\sigma(1),0)h_{p-i}(t,0), \quad t \in [0,\sigma^{n-i}(1)],\ 0 \le i \le p-1. \tag{2.26}$$

□

Lemma 2.6 [13]. *Let E be a Banach space, and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_1$; $\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$; $\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.

Then, T has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

In this section, by using Lemma 2.6, we offer criteria for the existence of positive solution of BVP (1.1), (1.2).

To begin, we will list the conditions that are needed later as follows. In these conditions, $f(t,u_1,u_2,\dots,u_p)$ is a continuous function such that $f : [0,1] \times \mathbb{R}[0,\infty)^p \to \mathbb{R}[0,\infty)$.

(A_1) There exists constant $\varepsilon \in (0,1)$ such that

$$\lim_{u_1,u_2,\dots,u_p \to \infty} \min_{t \in [\varepsilon,1]} \frac{f(t,u_1,u_2,\dots,u_p)}{u_p} = \infty. \tag{3.1}$$

(A_2) There exists constant $a > 0$ such that

$$\lim_{u_p\to 0^+} \min_{(t,u_1,u_2,\dots,u_{p-1})\in[0,1]\times\mathbb{R}[0,a]^{p-1}} \frac{f(t,u_1,u_2,\dots,u_p)}{u_p} = \infty. \tag{3.2}$$

(A_3) $f(t,u_1,u_2,\dots,u_p)$ is nondecreasing in u_j for each fixed $(t,u_1,u_2,\dots,u_{j-1},u_{j+1},\dots,u_p)$

Definition 3.1. Define $f \in C_{\mathrm{rd}}(\mathbb{T}:\mathbb{R})$ to be right-dense continuous if for all $t \in \mathbb{T}$, $\lim_{s\to t^+} f(s) = f(t)$ at every right-dense point $t \in \mathbb{T}$, $\lim_{s\to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

Let $C^n_{\mathrm{rd}}([0,1])$ denote the space of functions:

$$C^n_{\mathrm{rd}}([0,1]) = \{y : y \in C([0,\sigma^n(1)]),\dots,y^{\Delta^{n-1}} \in C([0,\sigma(1)]),\ y^{\Delta^n} \in C_{\mathrm{rd}}([0,1])\}. \tag{3.3}$$

Let $B = \{y \in C^n_{\mathrm{rd}}([0,1]) : y^{\Delta^i}(0) = 0,\ 0 \le i \le p-2\}$ be a Banach space with the norm $\|y\| = \sup_{t\in[0,\sigma^{n-p+1}(1)]} |y^{\Delta^{p-1}}(t)|$, and let

$$C = \{y \in B : y^{\Delta^{p-1}}(t) \ge L(t)\|y\|,\ t \in [0,\sigma^{n-p+1}(1)]\}, \tag{3.4}$$

where $L(t)$ is given in Lemma 2.4.

It is obvious that C is a cone in B. From $y^{\Delta^i}(0) = 0$, $0 \le i \le p-2$, it follows that for all $y \in C$,

$$\frac{h_{p-i}(t,0)}{\sigma^{n-p+1}(1)} \cdot \|y\| \le y^{\Delta^i}(t) \le \delta\|y\|, \quad i = 1,2,\dots,p-1, \tag{3.5}$$

where

$$\delta := [\sigma^n(1)]^{p-1}. \tag{3.6}$$

Remark 3.2. If $u,v \in C$ and $u^{\Delta^{p-1}}(t) \ge v^{\Delta^{p-1}}(t)$, $t \in [0,\sigma^{n-p+1}(1)]$, it follows from $u^{\Delta^i}(0) = v^{\Delta^i}(0) = 0$, $0 \le i \le p-2$ that $u^{\Delta^i}(t) \ge v^{\Delta^i}(t)$, $t \in [0,\sigma^{n-i}(1)]$, $0 \le i \le p-1$.

Let the operator $S : C \to B$ be defined by

$$\begin{aligned} (Sy)(t) &= \lambda \int_0^{\sigma(1)} K(t,s) f(s, y(\sigma^{n-1}(s)),\dots,y^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s, \quad t \in [0,\sigma^n(1)], \\ (Sy)^{\Delta^{p-1}}(t) &= \lambda \int_0^{\sigma(1)} k(t,s) f(s, y(\sigma^{n-1}(s)),\dots,y^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s, \quad t \in [0,\sigma^{n-p+1}(1)]. \end{aligned} \tag{3.7}$$

To obtain a positive solution of BVP (1.1), (1.2), we seek a fixed point of the operator S in the cone C.

Lemma 3.3. *The operator S maps C into C.*

Proof. From Lemma 2.4, we know that for $t \in [0, \sigma^{n-p+1}(1)]$,

$$
\begin{aligned}
(Sy)^{\Delta^{p-1}}(t) &= \lambda \int_0^{\sigma(1)} k(t,s) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s \\
&\le \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s.
\end{aligned}
\tag{3.8}
$$

So

$$
\|Sy\| \le \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s, \quad t \in [0, \sigma^{n-p+1}(1)].
\tag{3.9}
$$

From Lemma 2.4 again, it follows that for $t \in [0, \sigma^{n-p+1}(1)]$,

$$
\begin{aligned}
(Sy)^{\Delta^{p-1}}(t) &= \lambda \int_0^{\sigma(1)} k(t,s) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s \\
&\ge \lambda \int_0^{\sigma(1)} L(t) g_{n-p}(\sigma(s),0) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s \\
&\ge L(t)\|Sy\|.
\end{aligned}
\tag{3.10}
$$

Hence, S maps C into C. □

LEMMA 3.4. *The operator $S : C \to C$ is completely continuous.*

Proof. First we will prove that the operator S is continuous. Let $\{y_m\}$, $y \in C$ be such that $\lim_{m\to\infty}\|y_m - y\| = 0$. From $y^{\Delta^i}(0) = 0$, $i = 0, 1, \dots, p-2$, we have

$$
\sup_{t \in [0, \sigma^{n-i}(1)]} \left| y_m^{\Delta^i} - y^{\Delta^i} \right| \longrightarrow 0, \quad i = 0, 1, \dots, p-1.
\tag{3.11}
$$

Then, it is easy to see that

$$
\begin{aligned}
\rho_m = \sup_{s \in [0,1]} &\left| f\left(s, y_m(\sigma^{n-1}(s)), \dots, y_m^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \right. \\
&\left. - f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \right| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.
\end{aligned}
\tag{3.12}
$$

Hence, we get from Lemma 2.4 that for $t \in [0, \sigma^{n-p+1}(1)]$,

$$
\begin{aligned}
&\left| (Sy_m)^{\Delta^{p-1}}(t) - (Sy)^{\Delta^{p-1}}(t) \right| \\
&\quad = \left| \lambda \int_0^{\sigma(1)} k(t,s) \left[f\left(s, y_m(\sigma^{n-1}(s)), \dots, y_m^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \right. \right. \\
&\qquad \left. \left. - f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \right] \Delta s \right| \\
&\quad \le \lambda \rho_m \int_0^{\sigma(1)} k(t,s) \Delta s \le \lambda \rho_m \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0) \Delta s \longrightarrow 0
\end{aligned}
\tag{3.13}
$$

as $m \to \infty$. This shows that $S : C \to C$ is continuous.

Next, to show complete continuity, we will apply Arzela-Ascoli theorem. Let Ω be a bounded subset of C and let $y \in \Omega$. Now there exists $L > 0$ such that for all $y \in \Omega$,

$$\sup \left| y^{\Delta^{p-1}} \right| \leq L, \quad \sup \left| y^{\Delta^{i}} \right| \leq \delta L, \quad i = 0, 1, \dots, p-2, \tag{3.14}$$

where δ is given in (3.6). Let

$$M = \sup_{(s,u_1,u_2,\dots,u_p)\in[0,1]\times\mathbb{R}[0,\delta L]^{p-1}\times\mathbb{R}[0,L]} \left| f(s,u_1,u_2,\dots,u_p) \right|. \tag{3.15}$$

Clearly, we have for $t \in [0, \sigma^n(1)]$,

$$\left|(Sy)(t)\right| \leq \lambda M \int_0^{\sigma(1)} K(t,s)\Delta s \leq \lambda M \sup_{t\in[0,\sigma^n(1)]} \int_0^{\sigma(1)} K(t,s)\Delta s, \tag{3.16}$$

and for $t, t' \in [0, \sigma^n(1)]$,

$$\left|(Sy)(t) - (Sy)(t')\right| \leq \lambda M \int_0^{\sigma(1)} \left|K(t,s) - K(t',s)\right| \Delta s. \tag{3.17}$$

The Arzela-Ascoli theorem guarantees that $S\Omega$ is relatively compact, so $S : C \to C$ is completely continuous. □

For any $L > 0$, define

$$r_L = \frac{L}{M_L}\left[g_{n-p+1}(\sigma(1),0)\right]^{-1}, \tag{3.18}$$

where

$$M_L = \sup_{(t,u_1,u_2,\dots,u_p)\in[0,1]\times\mathbb{R}[0,\delta L]^{p-1}\times\mathbb{R}[0,L]} f(t,u_1,u_2,\dots,u_p), \tag{3.19}$$

and δ is given in (3.6).

Theorem 3.5. *Let (A_1) hold. For any $\lambda \in \mathbb{R}(0, r_L]$, BVP (1.1), (1.2) has at least one positive solution y such that $\|y\| \geq L$.*

Proof. Let $L > 0$ be given and let $\lambda \in \mathbb{R}(0, r_L]$ be fixed. We separate the proof into the following two steps.

Step 1. Let

$$\Omega_1 = \{y \in B : \|y\| < L\}. \tag{3.20}$$

It follows from Lemma 2.4 that for all $y \in \partial\Omega_1 \cap C$,

$$\begin{aligned}(Sy)^{\Delta^{p-1}}(t) &= \lambda \int_0^{\sigma(1)} k(t,s) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right)\Delta s \\ &\leq \lambda M_L \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0)\Delta s \\ &= \lambda M_L \cdot g_{n-p+1}(\sigma(1),0) \leq L, \quad t \in [0, \sigma^{n-p+1}(1)].\end{aligned} \tag{3.21}$$

Hence

$$\|Sy\| \le \|y\|, \quad y \in \partial\Omega_1 \cap C. \tag{3.22}$$

Step 2. From (A_1), we know that there exists $\eta > L$ (η can be chosen arbitrarily large) such that for all $(u_1, u_2, \dots, u_p) \in \mathbb{R}[\sigma_1\eta, \infty) \times \mathbb{R}[\sigma_2\eta, \infty) \times \cdots \times \mathbb{R}[\sigma_p\eta, \infty)$,

$$\min_{t\in[\varepsilon,1]} \frac{f(t, u_1, u_2, \dots, u_p)}{u_p} \ge \frac{\left[\int_\varepsilon^{\sigma(1)} g_{n-p}(\sigma(s), 0)\Delta s\right]^{-1}}{\lambda\sigma_p}, \tag{3.23}$$

where

$$\sigma_i = \frac{h_{p-i+1}(\varepsilon, 0)}{\sigma^{n-p+1}(1)}, \quad i = 1, 2, \dots, p. \tag{3.24}$$

So,

$$f(t, u_1, u_2, \dots, u_p) \ge \frac{\left[\int_\varepsilon^{\sigma(1)} g_{n-p}(\sigma(s), 0)\Delta s\right]^{-1}\eta}{\lambda} \tag{3.25}$$

on $[\varepsilon, 1] \times \mathbb{R}[\sigma_1\eta, \infty) \times \mathbb{R}[\sigma_2\eta, \infty) \times \cdots \times \mathbb{R}[\sigma_p\eta, \infty)$.

Using Lemma 2.4, we know that

$$\begin{aligned} (Sy)^{\Delta^{p-1}}(\sigma^{n-p+1}(1)) &= \lambda\int_0^{\sigma(1)} k(\sigma^{n-p+1}(1), s) f(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s \\ &\ge \lambda\int_\varepsilon^{\sigma(1)} g_{n-p}(\sigma(s), 0)\Delta s \cdot \frac{\left[\int_\varepsilon^{\sigma(1)} g_{n-p}(\sigma(s), 0)\Delta s\right]^{-1}\eta}{\lambda} = \eta. \end{aligned} \tag{3.26}$$

By letting $\Omega_2 = \{y \in B : \|y\| < \eta\}$, we have

$$\|Sy\| \ge \|y\|, \quad y \in \partial\Omega_2 \cap C. \tag{3.27}$$

Therefore, it follows from Lemma 2.6 that BVP (1.1), (1.2) has a solution $y \in C$ such that $\|y\| \ge L$. □

Theorem 3.6. *Let (A_2) hold. For any $\lambda \in \mathbb{R}(0, r_L]$ ($L \in \mathbb{R}(0, a]$), BVP (1.1), (1.2) has at least one positive solution y such that $0 < \|y\| \le L$.*

Proof. Let $L \in \mathbb{R}(0, a]$ be given and let $\lambda \in \mathbb{R}(0, r_L]$ be fixed. Let

$$\Omega_3 = \{y \in B : \|y\| < L\}. \tag{3.28}$$

Then for $y \in C \cap \partial\Omega_3$, we have from Lemma 2.4 that for $t \in [0, \sigma^{n-p+1}(1)]$,

$$\begin{aligned} (Sy)^{\Delta^{p-1}}(t) &= \lambda\int_0^{\sigma(1)} k(t, s) f(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s \\ &\le \lambda M_L \int_0^{\sigma(1)} g_{n-p}(\sigma(s), 0)\Delta s = \lambda M_L \cdot g_{n-p+1}(\sigma(1), 0) \le L. \end{aligned} \tag{3.29}$$

Therefore,

$$\|Sy\| \le \|y\|, \quad y \in C \cap \partial\Omega_3. \tag{3.30}$$

From (A_2), there exists η, r_0 where $\lambda\eta \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0)L(s)\Delta s > 1$ with $r_0 < L$ such that

$$f(t,u_1,u_2,\dots,u_p) \ge \eta u_p, \tag{3.31}$$

on $[0,1] \times \mathbb{R}[0,\delta r_0]^{p-1} \times \mathbb{R}[0,r_0]$, where δ is given in (3.6).

For $y \in C$ and $\|y\| = r_0$, we have from Lemma 2.4 that

$$\begin{aligned} (Sy)^{\Delta^{p-1}}(\sigma^{n-p+1}(1)) &= \lambda \int_0^{\sigma(1)} k(\sigma^{n-p+1}(1),s) f(s,y(\sigma^{n-1}(s)),\dots,y^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s \\ &\ge \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0)\,\eta y^{\Delta^{p-1}}(\sigma^{n-p}(s))\Delta s \\ &\ge \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0)L(s)\eta\|y\|\Delta s > \|y\| = r_0. \end{aligned} \tag{3.32}$$

By letting $\Omega_4 = \{y \in B : \|y\| < r_0\}$, we have

$$\|Sy\| \ge \|y\|, \quad y \in C \cap \partial\Omega_4. \tag{3.33}$$

Therefore, it follows from Lemma 2.6 that BVP (1.1), (1.2) has a solution $y \in C$ such that $0 < r_0 \le \|y\| \le L$. □

Theorem 3.7. *Let (A_2) and (A_3) hold. Suppose that $\lambda_0 \in E$. Then $\mathbb{R}(0,\lambda_0] \subseteq E$.*

Proof. Let y_0 be the eigenfunction corresponding to the eigenvalue λ_0. Then for $t \in [0,\sigma^{n-p+1}(1)]$,

$$y_0^{\Delta^{p-1}}(t) = \lambda_0 \int_0^{\sigma(1)} k(t,s) f(s,y_0(\sigma^{n-1}(s)),\dots,y_0^{\Delta^{p-1}}(\sigma^{n-p}(s)))\Delta s. \tag{3.34}$$

From $y_0 \in C$, we have

$$\frac{t}{\sigma^{n-p+1}(1)} \cdot ||y_0|| \le y_0^{\Delta^{p-1}}(t) \le ||y_0||, \quad t \in [0,\sigma^{n-p+1}(1)]. \tag{3.35}$$

We will consider two cases.

Case 1. $f(t,0,0,\dots,0) \not\equiv 0$, $t \in [0,1]$. Define

$$K = \{y \in C : 0 \le y^{\Delta^{p-1}}(t) \le y_0^{\Delta^{p-1}}(t),\ t \in [0,\sigma^{n-p+1}(1)]\}. \tag{3.36}$$

For $y \in K$ and $\lambda \in \mathbb{R}(0,\lambda_0)$, from Lemma 3.3, (A_3) and Remark 3.2, we have that for $t \in [0,\sigma^{n-p+1}(1)]$,

$$\begin{aligned} 0 < \lambda \int_0^{\sigma(1)} k(t,s) f(s,0,\dots,0)\Delta s &\le (Sy)^{\Delta^{p-1}}(t) \\ &= \lambda \int_0^{\sigma(1)} k(t,s) f\left(s, y(\sigma^{n-1}(s)),\dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right)\Delta s \\ &\le \lambda_0 \int_0^{\sigma(1)} k(t,s) f\left(s, y_0(\sigma^{n-1}(s)),\dots, y_0^{\Delta^{p-1}}(\sigma^{n-p}(s))\right)\Delta s = y_0^{\Delta^{p-1}}(t). \end{aligned} \tag{3.37}$$

Hence, S maps K into K. Moreover, S is completely continuous, Schauder's fixed point theorem guarantees that S has a fixed point in K, which is a positive solution of BVP (1.1), (1.2). Thus $\lambda \in E$.

Case 2. $f(t,0,\dots,0) \equiv 0$, $t \in [0,1]$. Let

$$M^* = \frac{1}{2}\left[g_{n-p}(\sigma(1),0)\cdot \sigma^{n-p+1}(1)\right]^{-1}\cdot \|y_0\|. \tag{3.38}$$

From the continuity of f, there exists $b \in \mathbb{R}(0,a]$ such that

$$M^* \ge f(t,u_1,u_2,\dots,u_p) \ge 0, \quad (t,u_1,u_2,\dots,u_p) \in [0,1]\times \mathbb{R}[0,\delta b]^{p-1}\times \mathbb{R}[0,b], \tag{3.39}$$

where δ is given in (3.6). From the proof of Theorem 3.6, let

$$L = b, \quad r_L = \min\left\{1, \frac{L}{M^*}\left[g_{n-p+1}(\sigma(s),0)\right]^{-1}\right\}, \tag{3.40}$$

we know that $\mathbb{R}(0,r_L] \subseteq E$. If $r_L \ge \lambda_0$, then the proof is completed. If $r_L < \lambda_0$, we still need to prove that $\mathbb{R}(r_L,\lambda_0) \subseteq E$.

If $r_L < \lambda_0$, let $\lambda_* \in \mathbb{R}(0,r_L]$ and let y_* be the eigenfunction corresponding to the eigenvalue λ_*. It follows from Lemma 2.5 and (3.5) that for $t \in [0,\sigma^{n-p+1}(1)]$,

$$\begin{aligned} y_*^{\Delta^{p-1}}(t) &= \lambda_* \int_0^{\sigma(1)} k(t,s) f\left(s, y_*(\sigma^{n-1}(s)),\dots, y_*^{\Delta^{p-1}}(\sigma^{n-p}(s))\right)\Delta s \\ &\le \int_0^{\sigma(1)} M^* k(t,s)\Delta s \le M^*\cdot g_{n-p}(\sigma(1),0) h_1(t,0) \\ &= \frac{1}{2}\frac{t}{\sigma^{n-p+1}(1)}\cdot \|y_0\| < y_0^{\Delta^{p-1}}(t). \end{aligned} \tag{3.41}$$

Define

$$K = \left\{y \in C : y_*^{\Delta^{p-1}}(t) \le y^{\Delta^{p-1}}(t) \le y_0^{\Delta^{p-1}}(t),\ t \in [0,\sigma^{n-p+1}(1)]\right\}. \tag{3.42}$$

For $y \in K$ and $\lambda \in \mathbb{R}(\lambda_*, \lambda_0)$, from Remark 3.2 and (A_3), we have that for $t \in [0, \sigma^{n-p+1}(1)]$,

$$\begin{aligned} y_*^{\Delta^{p-1}}(t) &= \lambda_* \int_0^{\sigma(1)} k(t,s) f\left(s, y_*(\sigma^{n-1}(s)), \dots, y_*^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s \\ &\le \lambda \int_0^{\sigma(1)} k(t,s) f\left(s, y(\sigma^{n-1}(s)), \dots, y^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s = (Sy)^{\Delta^{p-1}}(t) \\ &\le \lambda_0 \int_0^{\sigma(1)} k(t,s) f\left(s, y_0(\sigma^{n-1}(s)), \dots, y_0^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s = y_0^{\Delta^{p-1}}(t). \end{aligned} \tag{3.43}$$

Hence, S maps K into K. Schauder's fixed point theorem guarantees that S has a fixed point in K. Thus $\mathbb{R}(r_L, \lambda_0) \subseteq \mathbb{R}(\lambda_*, \lambda_0) \subseteq E$.

Therefore, $\mathbb{R}(0, \lambda_0] \subset E$. □

From Theorems 3.5, 3.6, and 3.7, we can easily get the following results.

Corollary 3.8. *Let (A_2) and (A_3) hold. Then E is an interval.*

Corollary 3.9. *Let (A_1), (A_2), and (A_3) hold. For any $\lambda \in \mathbb{R}(0, r_L]$ ($L \in \mathbb{R}(0, a]$), BVP (1.1), (1.2) has at least two positive solutions.*

Theorem 3.10. *Let*

$$\lim_{u_1, u_2, \dots, u_p \to \infty} \min_{t \in [0,1]} \frac{f(t, u_1, u_2, \dots, u_p)}{u_p} = \infty, \tag{3.44}$$

and (A_2), (A_3) hold. Then there are $\lambda^* > 0$ such that BVP (1.1), (1.2) has no solution for $\lambda > \lambda^*$.

Proof. First, the function $f(t, u_1, u_2, \dots, u_p)/u_p$ has the minimal value on $[0,1] \times \mathbb{R}[0, \infty)^p$, whose existence is guaranteed by the continuity and nondecreasing property of f and by assumption (3.44) and (A_2). Let

$$N = \min_{(t, u_1, \dots, u_p) \in [0,1] \times \mathbb{R}[0,\infty)^p} \frac{f(t, u_1, u_2, \dots, u_p)}{u_p}. \tag{3.45}$$

Let $\lambda \in E$, then there exists y_λ satisfying BVP (1.1), (1.2). By Lemma 2.4 and (3.5),

$$\begin{aligned} &(Sy_\lambda)^{\Delta^{p-1}}(\sigma^{n-p+1}(1)) \\ &\quad = \lambda \int_0^{\sigma(1)} k(\sigma^{n-p+1}(1), s) f\left(s, y_\lambda(\sigma^{n-1}(s)), \dots, y_\lambda^{\Delta^{p-1}}(\sigma^{n-p}(s))\right) \Delta s \\ &\quad = \lambda \int_0^{\sigma(1)} k(\sigma^{n-p+1}(1), s) \frac{f\left(s, y_\lambda(\sigma^{n-1}(s)), \dots, y_\lambda^{\Delta^{p-1}}(\sigma^{n-p}(s))\right)}{y_\lambda^{\Delta^{p-1}}(\sigma^{n-p}(s))} \cdot y_\lambda^{\Delta^{p-1}}(\sigma^{n-p}(s)) \Delta s \\ &\quad \ge \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s), 0) N y_\lambda^{\Delta^{p-1}}(\sigma^{n-p}(s)) \Delta s \\ &\quad \ge \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s), 0) N L(s) \Delta s \cdot \|y_\lambda\|. \end{aligned} \tag{3.46}$$

It follows that

$$1 \geq \lambda \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0) N L(s) \Delta s. \tag{3.47}$$

Let

$$\lambda^* = \left[N \int_0^{\sigma(1)} g_{n-p}(\sigma(s),0) L(s) \Delta s \right]^{-1}. \tag{3.48}$$

Therefore, BVP (1.1), (1.2) has no solution for $\lambda > \lambda^*$. □

Acknowledgment

The authors thank the referee for valuable suggestions which led to improvement of the original manuscript.

References

[1] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.

[2] R. P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Teaneck, NJ, USA, 1986.

[3] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, "Singular $(p, n-p)$ focal and (n, p) higher order boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 2, pp. 215–228, 2000.

[4] X. He and W. Ge, "Positive solutions for semipositone $(p, n-p)$ right focal boundary value problems," *Applicable Analysis*, vol. 81, no. 2, pp. 227–240, 2002.

[5] R. Ma, "Multiple positive solutions for a semipositone fourth-order boundary value problem," *Hiroshima Mathematical Journal*, vol. 33, no. 2, pp. 217–227, 2003.

[6] P. J. Y. Wong and R. P. Agarwal, "Multiple positive solutions of two-point right focal boundary value problems," *Mathematical and Computer Modelling*, vol. 28, no. 3, pp. 41–49, 1998.

[7] P. J. Y. Wong and R. P. Agarwal, "On two-point right focal eigenvalue problems," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 17, no. 3, pp. 691–713, 1998.

[8] R. P. Agarwal and D. O'Regan, "Nonlinear boundary value problems on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 4, pp. 527–535, 2001.

[9] K. L. Boey and P. J. Y. Wong, "Two-point right focal eigenvalue problems on time scales," *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1281–1303, 2005.

[10] K. L. Boey and P. J. Y. Wong, "Positive solutions of two-point right focal boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 52, no. 3-4, pp. 555–576, 2006.

[11] K. L. Boey and P. J. Y. Wong, "Existence of triple positive solutions of two-point right focal boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 50, no. 10–12, pp. 1603–1620, 2005.

[12] J. Henderson and C. C. Tisdell, "Topological transversality and boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 289, no. 1, pp. 110–125, 2004.

[13] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

Yuguo Lin: Department of Mathematics, Bei Hua University, Jilin City 132013, China
Email address: yglinoa@163.com

Minghe Pei: Department of Mathematics, Bei Hua University, Jilin City 132013, China
Email address: pmh@beihua.edu.cn