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Research Article

Unbounded Perturbations of Nonlinear Second-Order Difference Equations at Resonance

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We study the existence of solutions of nonlinear discrete boundary value problems $\Delta^2 u(t-1)+\mu_1 u(t)+g(t,u(t))=h(t)$, $t\in\mathbb{T}$, $u(a)=u(b+2)=0$, where $\mathbb{T}:=\{a+1,\dots,b+1\}$, $h:\mathbb{T}\to\mathbb{R}$, μ_1 is the first eigenvalue of the linear problem $\Delta^2 u(t-1)+\mu u(t)=0$, $t\in\mathbb{T}$, $u(a)=u(b+2)=0$, $g:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ satisfies some "asymptotic nonuniform" resonance conditions, and $g(t,u)u\ge 0$ for $u\in\mathbb{R}$.

1. Introduction

Let $a,b\in\mathbb{N}$ be two integers with $b-a>2$. Let $\mathbb{T}:=\{a+1,\dots,b+1\}$ and $\hat{\mathbb{T}}:=\{a,a+1,\dots,b+1,b+2\}$.

Definition 1.1. Suppose that a function $y:\hat{\mathbb{T}}\to\mathbb{R}$. If $y(t)=0$, then t is a *zero* of y. If $y(t)=0$ and $\Delta y(t)\neq 0$, then t is a *simple zero* of y. If $y(t)y(t+1)<0$, then y has a *node* at the point $s=(ty(t+1)-(t+1)y(t))/(y(t+1)-y(t))\in(t,t+1)$. The nodes and simple zeros of y are called the *simple generalized zeros* of y.

Let μ is a real parameter. It is well known that the linear eigenvalue problem

$$\begin{gathered}\Delta^2 y(t-1)+\mu y(t)=0,\quad t\in\mathbb{T},\\ u(a)=u(b+2)=0\end{gathered}\tag{1.1}$$

has exactly $N=b-a+1$ eigenvalues

$$\mu_1<\mu_2<\cdots<\mu_N,\tag{1.2}$$

which are real and the eigenspace corresponding to any such eigenvalue is one dimensional. The following lemma is crucial to the study of nonlinear perturbations of the linear problem (1.1). The required results are somewhat scattered in [1, Chapters 6-7].

Lemma 1.2 [1]. *Let (μ_i, ψ_i), $i \in \{1,\dots,N\}$, denote eigenvalue pairs of (1.1) with*

$$\sum_{t=a+1}^{b+1} \psi_j(t)\psi_j(t) = 1, \quad j \in \{1,\dots,N\}. \tag{1.3}$$

Then

(1) *ψ_i has $i-1$ simple generalized zeros in $[a+1, b+1]$; also if $j \neq k$, then*

$$\sum_{t=a+1}^{b+1} \psi_j(t)\psi_k(t) = 0; \tag{1.4}$$

(2) *if $h : \{a+1,\dots,b+1\} \to \mathbb{R}$ is given, then the problem*

$$\begin{gathered} \Delta^2 u(t-1) + \lambda_1 u(t) = h(t), \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{gathered} \tag{1.5}$$

has a solution if and only if $\sum_{t=a+1}^{b+1} h(t)\psi_1(t) = 0$.

In this paper, we study the existence of solutions of nonlinear discrete boundary value problems

$$\begin{gathered} \Delta^2 u(t-1) + \mu_1 u(t) + g(t, u(t)) = h(t), \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0, \end{gathered} \tag{1.6}$$

where $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Definition 1.3. By a solution of (1.6) we mean a function $u : \{a, a+1,\dots,b+1,b+2\} \to \mathbb{R}$ which satisfies the difference equation and the boundary value conditions in (1.6).

Theorem 1.4. *Let $h : \mathbb{T} \to \mathbb{R}$ be a given function, and let $g(t,u)$ be continuous in u for each $t \in \mathbb{T}$. Assume that*

$$g(t,u)u \geq 0 \tag{1.7}$$

for all $t \in \mathbb{T}$ and all $u \in \mathbb{R}$. Moreover, suppose that for all $\sigma > 0$, there exist a constant $R = R(\sigma) > 0$ and a function $b : \{a+1,\dots,b+1\} \to \mathbb{R}$ such that

$$|g(t,u)| \leq (\Gamma(t) + \sigma)|u| + b(t) \tag{1.8}$$

for all $t \in \mathbb{T}$ and all $u \in \mathbb{R}$ with $|u| \geq R$, where $\Gamma : \mathbb{T} \to \mathbb{R}$ is a given function satisfying

$$0 \leq \Gamma(t) \leq \mu_2 - \mu_1, \quad t \in \mathbb{T}, \tag{1.9}$$

$$\Gamma(\tau) < \mu_2 - \mu_1, \quad \text{for some } \tau \in \mathbb{T} \setminus \{\hat{t}\}, \tag{1.10}$$

with $\hat{t}$ is the unique simple generalized zero of ψ_2 in $[a+1, b+1]$.

Then (1.6) has a solution provided

$$\sum_{t=a+1}^{b+1} h(t)\psi_1(t) = 0. \tag{1.11}$$

The analogue of Theorem 1.4 was obtained for two-point BVPs of second-order ordinary differential equations by Iannacci and Nkashama [2]. Our paper is motivated by [2]. However, as we will see, there are very big differences between the continuous case and the discrete case. The main tool we use is the Leray-Schauder continuation theorem, see [3].

The existence of solution of discrete equations subjected to Sturm-Liouville boundary conditions was studied by Rodriguez [4], in which the nonlinearity is required to be bounded. For other related results, see Agarwal and O'Regan [5, 6], Bai and Xu [7], Rachunkova and Tisdell [8], and the references therein. However, all of them do not address the problem under the "asymptotic nonuniform resonance" conditions.

2. Preliminaries

Let

$$D := \{(0, u(a+1), \ldots, u(b+1), 0) \mid u(t) \in \mathbb{R},\ t \in \mathbb{T}\}. \tag{2.1}$$

Then D is a Hilbert space under the inner product

$$\langle u, v \rangle = \sum_{t=a+1}^{b+1} u(t)v(t), \tag{2.2}$$

and the corresponding norm is

$$\|u\| := \sqrt{\langle u, u \rangle} = \left(\sum_{t=a+1}^{b+1} u(t)u(t) \right)^{1/2}. \tag{2.3}$$

We note that D is also a Hilbert space under the inner product

$$\langle u, v \rangle_1 = \sum_{t=a}^{b+1} \Delta u(t)\Delta v(t), \tag{2.4}$$

and the corresponding norm is

$$\|u\|_1 := \sqrt{\langle u, u \rangle_1} = \left(\sum_{t=a}^{b+1} \Delta u(t)\Delta u(t) \right)^{1/2}. \tag{2.5}$$

For $u \in D$, let us write

$$u(t) = \overline{u}(t) + \widetilde{u}(t), \tag{2.6}$$

where

$$\overline{u}(t) = \langle u, \psi_1 \rangle \psi_1(t), \quad \langle \widetilde{u}, \psi_1 \rangle = 0. \tag{2.7}$$

Obviously, $D = \overline{D} \oplus \widetilde{D}$ with

$$\overline{D} = \text{span}\,\{\psi_1\}, \qquad \widetilde{D} = \text{span}\,\{\psi_2, \ldots, \psi_N\}. \tag{2.8}$$

LEMMA 2.1. *Let $u, w \in D$. Then*

$$\sum_{k=a+1}^{b+1} w(k)\Delta^2 u(k-1) = -\sum_{k=a}^{b+1} \Delta u(k)\Delta w(k). \tag{2.9}$$

Proof. Since $w(a) = w(b+2) = 0$, we have

$$\begin{aligned}
\sum_{k=a+1}^{b+1} w(k)\Delta^2 u(k-1) &= \sum_{j=a}^{b} w(j+1)\Delta^2 u(j) \quad \text{(by setting } j = k-1\text{)} \\
&= \sum_{j=a}^{b} w(j+1)\big(\Delta u(j+1) - \Delta u(j)\big) \\
&= \sum_{j=a}^{b} \Delta u(j+1) w(j+1) - \sum_{j=a}^{b} \Delta u(j) w(j+1) \\
&= \sum_{l=a+1}^{b+1} \Delta u(l) w(l) - \sum_{j=a}^{b} \Delta u(j) w(j+1) \quad \text{(by setting } l = j+1\text{)} \\
&= \left[\Delta u(b+1) w(b+1) + \sum_{l=a+1}^{b} \Delta u(l) w(l)\right] \\
&\quad - \left[\Delta u(a) w(a+1) + \sum_{j=a+1}^{b} \Delta u(j) w(j+1)\right] \\
&= \Delta u(b+1)\big[w(b+1) - w(b+2)\big] - \sum_{l=a+1}^{b} \Delta u(l)\Delta w(l) \\
&\quad - \Delta u(a)\big[w(a+1) - w(a)\big] = -\sum_{l=a}^{b+1} \Delta u(l)\Delta(l).
\end{aligned} \tag{2.10}$$

□

Lemma 2.2. *Let $\Gamma : \mathbb{T} \to \mathbb{R}$ be a given function satisfying*

$$0 \le \Gamma(t) \le \mu_2 - \mu_1, \quad t \in \mathbb{T},$$
$$\Gamma(\tau) < \mu_2 - \mu_1 \quad \text{for some } \tau \in \mathbb{T} \setminus \{\hat{t}\}, \tag{2.11}$$

with $\hat{t}$ is the unique simple generalized zero of ψ_2 in $[a+1, b+1]$.

Then there exists a constant $\delta = \delta(\Gamma) > 0$ such that for all $u \in D$, one has

$$\sum_{t=a+1}^{b+1} \left[\Delta^2 u(t-1) + \mu_1 u(t) + \Gamma(t) u(t)\right]\left(\overline{u}(t) - \widetilde{u}(t)\right) \ge \delta \|\widetilde{u}\|_1^2. \tag{2.12}$$

Proof. Let

$$u(t) = \sum_{i=1}^{N} c_i \psi_i(t), \quad t \in \mathbb{T}. \tag{2.13}$$

Then

$$\Delta^2 u(t-1) = -\sum_{i=1}^{N} c_i \mu_i \psi_i(t), \tag{2.14}$$

$$\overline{u}(t) = c_1 \psi_1(t), \qquad \widetilde{u}(t) = \sum_{i=2}^{N} c_i \psi_i(t). \tag{2.15}$$

Taking into account the orthogonality of $\overline{u}$ and $\widetilde{u}$ in D, we have

$$\begin{aligned}
&\sum_{t=a+1}^{b+1} \left[\Delta^2 u(t-1) + \mu_1 u(t) + \Gamma(t) u(t)\right]\left(\overline{u}(t) - \widetilde{u}(t)\right) \\
&\quad = \sum_{t=a+1}^{b+1} \left(-\sum_{i=1}^{N} c_i \mu_i \psi_i(t) + \sum_{i=1}^{N} c_i \mu_1 \psi_i(t) + \Gamma(t) u(t)\right)\left(c_1 \psi_1(t) - \sum_{i=2}^{N} c_i \psi_i(t)\right) \\
&\quad = \sum_{t=a+1}^{b+1} \left(\Gamma(t) c_1^2 \psi_1^2(t) + \sum_{i=2}^{N} c_i^2 \mu_i \psi_i^2(t) - \sum_{i=2}^{N} c_i^2 \mu_1 \psi_i^2(t) - \Gamma(t) \sum_{i=2}^{N} c_i^2 \psi_i^2(t)\right) \\
&\quad = \sum_{t=a+1}^{b+1} \left[\sum_{i=2}^{N} c_i^2 \mu_i \psi_i^2(t) - \left(\mu_1 + \Gamma(t)\right) \sum_{i=2}^{N} c_i^2 \psi_i^2(t)\right] + \sum_{t=a+1}^{b+1} \Gamma(t) c_1^2 \psi_1^2(t) \\
&\quad \ge \sum_{t=a+1}^{b+1} \left[\sum_{i=2}^{N} c_i^2 \mu_i \psi_i^2(t) - \left(\mu_1 + \Gamma(t)\right) \sum_{i=2}^{N} c_i^2 \psi_i^2(t)\right] \\
&\quad = \sum_{t=a+1}^{b+1} \left[\sum_{i=2}^{N} c_i^2 \psi_i(t)\left(-\Delta^2 \psi_i(t-1)\right) - \left(\mu_1 + \Gamma(t)\right) \sum_{i=2}^{N} c_i^2 \psi_i^2(t)\right] \\
&\quad = \sum_{i=2}^{N} \sum_{t=a+1}^{b+1} c_i^2 \psi_i(t)\left(-\Delta^2 \psi_i(t-1)\right) + \sum_{t=a+1}^{b+1} \left[-\left(\mu_1 + \Gamma(t)\right) \sum_{i=2}^{N} c_i^2 \psi_i^2(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{N} \sum_{t=a}^{b+1} c_i^2 [\Delta\psi_i(t)]^2 + \sum_{t=a+1}^{b+1} \left[-(\mu_1 + \Gamma(t)) \sum_{i=2}^{N} c_i^2 \psi_i^2(t) \right] \\
&= \sum_{t=a}^{b+1} \left[(\Delta\widetilde{u}(t))^2 - (\mu_1 + \Gamma(t)) (\widetilde{u}(t))^2 \right].
\end{aligned}
\tag{2.16}
$$

Set

$$
\Lambda_\Gamma(\widetilde{u}) := \|\widetilde{u}\|_1^2 - \sum_{t=a}^{b+1} (\mu_1 + \Gamma(t)) (\widetilde{u}(t))^2. \tag{2.17}
$$

We claim that $\Lambda_\Gamma(\widetilde{u}) \geq 0$ with the equality only if $\widetilde{u} = A\psi_2$ for some $A \in \mathbb{R}$.

In fact, we have from (1.9), (1.3), (1.4), and Lemma 2.1 that

$$
\begin{aligned}
\Lambda_\Gamma(\widetilde{u}) &= \sum_{t=a}^{b+1} [\Delta\widetilde{u}(t)]^2 - \sum_{t=a+1}^{b+1} (\mu_1 + \Gamma(t)) [\widetilde{u}(t)]^2 \\
&= -\sum_{t=a+1}^{b+1} \widetilde{u}(t) \Delta^2 \widetilde{u}(t-1) - \sum_{t=a+1}^{b+1} (\mu_1 + \Gamma(t)) (\widetilde{u}(t))^2 \\
&= \sum_{t=a+1}^{b+1} \sum_{i=2}^{N} c_i \psi_i(t) \sum_{i=2}^{N} c_i \mu_i \psi_i(t) - \sum_{t=a+1}^{b+1} (\mu_1 + \Gamma(t)) \left(\sum_{i=2}^{N} c_i \psi_i(t) \right)^2 \\
&\geq \sum_{t=a+1}^{b+1} \sum_{i=2}^{N} c_i \psi_i(t) \sum_{j=2}^{N} c_j \mu_j \psi_j(t) - \sum_{t=a+1}^{b+1} \mu_2 \left(\sum_{i=2}^{N} c_i \psi_i(t) \right) \left(\sum_{j=2}^{N} c_j \psi_j(t) \right) \\
&= \sum_{i=2}^{N} \sum_{j=2}^{N} c_i c_j \mu_j \sum_{t=a+1}^{b+1} \psi_i(t) \psi_j(t) - \sum_{i=2}^{N} \sum_{j=2}^{N} c_i c_j \mu_2 \sum_{t=a+1}^{b+1} \psi_i(t) \psi_j(t) \\
&= \sum_{j=2}^{N} c_j^2 (\mu_j - \mu_2) \geq 0.
\end{aligned}
\tag{2.18}
$$

Obviously, $\Lambda_\Gamma(\widetilde{u}) = 0$ implies that $c_3 = \cdots = c_N = 0$, and accordingly $\widetilde{u} = A\psi_2$ for some $A \in \mathbb{R}$. But then we get

$$
0 = \Lambda_\Gamma(\widetilde{u}) = A^2 \sum_{t=a}^{b+1} (\mu_2 - \mu_1 - \Gamma(t)) \psi_2^2(t) = A^2 \sum_{t=a+1}^{b+1} (\mu_2 - \mu_1 - \Gamma(t)) \psi_2^2(t) \tag{2.19}
$$

so that by our assumption, $A = 0$ and hence $\widetilde{u} = 0$.

We claim that there is a constant $\delta = \delta(\Gamma) > 0$ such that

$$
\Lambda_\Gamma(\widetilde{u}) \geq \delta \|\widetilde{u}\|_1^2. \tag{2.20}
$$

Assume that the claim is not true. Then we can find a sequence $\{\widetilde{u}_n\} \subset D$ and $\widetilde{u} \in D$, such that, by passing to a subsequence if necessary,

$$0 \leq \Lambda_\Gamma(\widetilde{u}_n) \leq \frac{1}{n}, \qquad \|\widetilde{u}_n\|_1 = 1, \tag{2.21}$$

$$\|\widetilde{u}_n - \widetilde{u}\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.22}$$

From (2.22) and the fact that $\widetilde{u}_n(a) = \widetilde{u}(a) = 0 = \widetilde{u}_n(b+2) = \widetilde{u}(b+2)$, it follows that

$$\begin{aligned}
\left| \sum_{t=a}^{b+1} [\Delta \widetilde{u}_n(t)]^2 - \sum_{t=a}^{b+1} [\Delta \widetilde{u}(t)]^2 \right| &= \left| \sum_{t=a}^{b+1} [\widetilde{u}_n(t+1) - \widetilde{u}_n(t)]^2 - \sum_{t=a}^{b+1} [\widetilde{u}(t+1) - \widetilde{u}(t)]^2 \right| \\
&\leq \sum_{t=a}^{b+1} \left| \widetilde{u}_n^2(t+1) - \widetilde{u}^2(t+1) \right| + \sum_{t=a}^{b+1} \left| \widetilde{u}_n^2(t) - \widetilde{u}^2(t) \right| \\
&\quad + 2\sum_{t=a}^{b+1} \big(\left| \widetilde{u}_n(t) \right| \left| \widetilde{u}_n(t+1) - \widetilde{u}(t+1) \right| \\
&\quad + \left| \widetilde{u}(t+1) \right| \left| \widetilde{u}_n(t) - \widetilde{u}(t) \right| \big) \longrightarrow 0.
\end{aligned} \tag{2.23}$$

By (2.17), (2.21), and (2.22), we obtain, for $n \to \infty$,

$$\sum_{t=a}^{b+1} [\Delta \widetilde{u}_n(t)]^2 \longrightarrow \sum_{t=a}^{b+1} (\mu_1 + \Gamma(t)) [\widetilde{u}(t)]^2, \tag{2.24}$$

and hence

$$\sum_{t=a}^{b+1} [\Delta \widetilde{u}(t)]^2 \leq \sum_{t=a}^{b+1} (\mu_1 + \Gamma(t)) [\widetilde{u}(t)]^2, \tag{2.25}$$

that is,

$$\Lambda_\Gamma(\widetilde{u}) \leq 0. \tag{2.26}$$

By the first part of the proof, $\widetilde{u} = 0$, so that, by (2.24), $\sum_{t=a}^{b+1} [\Delta \widetilde{u}_n(t)]^2 \to 0$, a contradiction with the second equality in (2.21), and the proof is complete. □

Lemma 2.3. *Let Γ be like in Lemma 2.2 and let $\delta > 0$ be associated with Γ by that lemma. Let $\sigma > 0$. Then, for all function $p : \mathbb{T} \to \mathbb{R}$ satisfying*

$$0 \leq p(x) \leq \Gamma(x) + \sigma \tag{2.27}$$

and all $u \in D$,

$$\sum_{t=a+1}^{b+1} [\Delta^2 u(t-1) + \mu_1 u(t) + p(t) u(t)] (\overline{u}(t) - \widetilde{u}(t)) \geq \left(\delta - \frac{\sigma}{\mu_2} \right) \|\widetilde{u}\|_1^2. \tag{2.28}$$

Proof. Using the computations of Lemma 2.2, we obtain

$$\begin{aligned}\sum_{t=a+1}^{b+1} & \left[\Delta^2 u(t-1)+\mu_1 u(t)+p(t)u(t)\right](\overline{u}(t)-\widetilde{u}(t)) \\ & \geq \sum_{t=a}^{b+1}\left[(\Delta\widetilde{u}(t))^2-(\mu_1+p(t))(\widetilde{u}(t))^2\right]=:\Lambda_p(\widetilde{u}).\end{aligned} \tag{2.29}$$

Therefore, by the second inequality in (2.27), we get

$$\Lambda_p(\widetilde{u}) \geq \Lambda_\Gamma(\widetilde{u})-\sigma\sum_{t=a}^{b+1}(\widetilde{u}(t))^2. \tag{2.30}$$

So that, using (2.13)-(2.14), the relation $\widetilde{u}(t)=\sum_{i=2}^{N}c_i\psi_i(t)$, and Lemma 2.2, it follows that

$$\Lambda_p(\widetilde{u}) \geq \left(\delta-\frac{\sigma}{\mu_2}\right)\|\widetilde{u}\|_1^2, \tag{2.31}$$

and the proof is complete. □

3. Proof of the main result

Let $\delta>0$ be associated to the function Γ by Lemma 2.2. Then, by assumption (1.8), there exist $R(\delta)>0$ and $b:\mathbb{T}\to\mathbb{R}$, such that

$$|g(t,u)| \leq \left(\Gamma(t)+\frac{\mu_2\delta}{4}\right)|u|+b(t) \tag{3.1}$$

for all $t\in\mathbb{T}$ and all $u\in\mathbb{R}$ with $|u|\geq R$. Without loss of generality, we can choose R so that $b(t)/|u|<(\mu_2\delta)/4$ and all $u\in\mathbb{R}$ with $u\geq R$.

Proof of Theorem 1.4. Let us define $\gamma:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ by

$$\gamma(t,u)=\begin{cases} u^{-1}g(t,u), & |u|\geq R,\\ R^{-1}g(t,R)\left(\dfrac{u}{R}\right)+\left(1-\dfrac{u}{R}\right)\Gamma(t), & 0\leq u\leq R,\\ R^{-1}g(t,-R)\left(\dfrac{u}{R}\right)+\left(1+\dfrac{u}{R}\right)\Gamma(t), & -R\leq u\leq 0.\end{cases} \tag{3.2}$$

Then by assumption (1.7) and the relations (3.1), we have that

$$0\leq\gamma(t,u)\leq\Gamma(t)+\frac{\mu_2\delta}{2}, \quad t\in\mathbb{T},\ u\in\mathbb{R}. \tag{3.3}$$

Define $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ by

$$f(t,u)=g(t,u)-\gamma(t,u)u. \tag{3.4}$$

Then

$$\left| f(t,u) \right| \le \nu(t), \quad t \in \mathbb{T}, \tag{3.5}$$

for some function $\nu : \mathbb{T} \to \mathbb{R}$.

To prove that (1.6) has at least one solution, it suffices, according to the Leray-Schauder continuation method [3], to show that the possible solutions of the family of equations

$$\begin{gathered} \Delta^2 u(t-1) + \mu_1 u(t) + (1-\eta) q u(t) + \eta \gamma(t,u(t)) u(t) + \eta f(t,u(t)) = \eta h(t), \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{gathered} \tag{3.6}$$

(in which $\eta \in (0,1)$, $q \in (0, \mu_2 - \mu_1)$ with $q < (\mu_2 \delta)/2$, q fixed) are a priori bounded in D, independent of $\eta \in [0,1)$. Notice that, by (3.3), we have

$$0 \le (1-\eta) q + \eta \gamma(t,u) \le \Gamma(t) + \frac{\mu_2 \delta}{2}, \quad t \in \mathbb{T},\ u \in \mathbb{R}. \tag{3.7}$$

It is clear that for $\eta = 0$, (3.6) has only the trivial solution. Now if $u \in D$ is a solution of (3.6) for some $\eta \in (0,1)$, using Lemma 2.3 and Cauchy inequality, we get

$$\begin{aligned} 0 &= \sum_{t=a}^{b+1} (\overline{u}(t) - \widetilde{u}(t)) \left(\Delta^2 u(t-1) + \mu_1 u(t) + \left[(1-\eta) q + \eta \gamma(t,u(t)) \right] u(t) \right) \\ &\quad + \sum_{t=a+1}^{b+1} (\overline{u}(t) - \widetilde{u}(t)) \left(\eta f(t,u(t)) - \eta h(t) \right) \\ &\ge (\delta/2) \sum_{t=a}^{b+1} \left[\widetilde{\Delta} u(t) \right]^2 - \left(\|\overline{u}\| + \|\widetilde{u}\| \right) (b-a+1)^{1/2} \left(\|\nu\| + \|h\| \right), \end{aligned} \tag{3.8}$$

so that by the relation $\sum_{t=a}^{b+1} \Delta [w(t)]^2 \ge \mu_1 \|w\|^2$, $w \in D$, we deduce

$$0 \ge \left(\frac{\delta}{2} \right) \|\widetilde{u}\|_1^2 - \beta \left(\|\widetilde{u}\|_1 + \|\overline{u}\|_1 \right) \tag{3.9}$$

for some constant $\beta > 0$, dependent only on γ and h (but not on u or μ). Taking $\alpha = \beta \delta^{-1}$, we get

$$\|\widetilde{u}\|_1 \le \alpha + \left(\alpha^2 + 2\alpha \|\overline{u}\|_1 \right)^{1/2}. \tag{3.10}$$

We claim that there exists $\rho > 0$, independent of u and μ, such that for all possible solutions of (3.6),

$$\|u\|_1 < \rho. \tag{3.11}$$

Suppose on the contrary that the claim is false, then there exists $\{(\eta_n, u_n)\} \subset (0,1) \times D$ with $\|u_n\|_1 \geq n$ and for all $n \in \mathbb{N}$,

$$\Delta^2 u_n(t-1) + \mu_1 u_n(t) + (1-\eta_n) q u_n(t) + \eta_n g(t, u_n(t)) = \eta_n h(t), \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0. \tag{3.12}$$

Set $v_n = (u_n/\|u_n\|_1)$, we have

$$\begin{aligned} &\Delta^2 v_n(t-1) + \mu_1 v_n(t) + q v_n(t) \\ &\quad = \eta_n\left(\frac{h}{\|u_n\|_1}\right) + \eta_n q v_n(t) - \eta_n\left(\left(g\left(t, \frac{u_n(t)}{\|u_n\|_1}\right)\right)\right), \quad t \in \mathbb{T}, \\ &v_n(a) = v_n(b+2) = 0. \end{aligned} \tag{3.13}$$

Define an operator $L : D \to D$ by

$$(Lw)(t) := \Delta^2 w(t-1) + \mu_1 w(t) + q w(t), \quad t \in \mathbb{T}, \\ (Lw)(a) := 0, \quad (Lw)(b+2) := 0. \tag{3.14}$$

Then $L^{-1} : D \to D$ is completely continuous since D is finite-dimensional. Now, (3.13) is equivalent to

$$v_n(t) = L^{-1}\left[\eta_n\left(\frac{h(\cdot)}{\|u_n\|_1}\right) + \eta_n q v_n(\cdot) - \eta_n\left(g\left(\cdot, \frac{u_n(\cdot)}{\|u_n\|_1}\right)\right)\right](t), \quad t \in \mathbb{T}. \tag{3.15}$$

By (3.1) and (3.15), it follows that $\{(g(\cdot, u_n(\cdot))/\|u_n\|_1\}$ is bounded. Using (3.15) again, we may assume that (taking a subsequence and relabelling if necessary) $v_n \to v$ in $(D, \|\cdot\|_1)$, $\|v\| = 1$, and $v(a) = v(b+2) = 0$.

On the other hand, using (3.10), we deduce immediately that

$$\|\tilde{v}_n\|_1 \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.16}$$

Therefore, $v \in \overline{D}$, that is,

$$v(t) = B\psi_1(t), \quad t \in \hat{T}. \tag{3.17}$$

Since $\|v\|_1 = 1$, we follows that $B = \pm{\mu_1}^{1/2}$ and

$$v(t) = \pm{\mu_1}^{1/2}\psi_1(t), \quad t \in \hat{T}. \tag{3.18}$$

In what follows, we will suppose that

$$v(t) = {\mu_1}^{1/2}\psi_1(t), \quad t \in \hat{T}. \tag{3.19}$$

The case $v(t) = -{\mu_1}^{1/2}\psi_1(t)$ can be treated in a similar way.

Now, using the facts that $v_n(a) = v(b+2) = 0$ and $v_n(t) \to v(t)$ for $t \in \mathbb{T}$ and $v(t) > 0$ for $t \in \mathbb{T}$, we have that there exists $n_0 \in \mathbb{N}$ such that

$$v_n(t) > 0, \quad t \in \mathbb{T},\ n \geq n_0. \tag{3.20}$$

Writing $v_n = \overline{v}_n + \widetilde{v}_n$, we have that $\overline{v}_n(t) = K_n(t)\psi_1(t)$ with $K_n \to 1$ as $n \to \infty$.

Let us come back to (3.12). Taking the inner product in $(D, \|\cdot\|)$ of (3.12) with $\overline{u}_n$, noticing that $\eta_n \in (0,1)$, and considering the assumption (1.11), we deduce that

$$(\eta_n/\|u_n\|_1) \sum_{t=a}^{b+1} g(t, u_n(t))\overline{v}_n(t) < 0 \tag{3.21}$$

for all n sufficiently large, so $\sum_{t=a}^{b+1} g(t,u_n(t))\overline{v}_n(t) < 0$. This is a contradiction, since by (3.21) and (1.7), $g(t,u_n(t))\overline{v}_n(t) \geq 0$ for $t \in \mathbb{T}$ and $n \geq n_0$, and the proof is complete. □

4. An example

From [1, Example 4.1], we know that the linear eigenvalues and the eigenfunctions of the problem

$$\begin{gathered} \Delta^2 y(t-1) + \mu y(t) = 0, \quad t \in \mathbb{T}_1 := \{1,2,3\}, \\ u(0) = u(4) = 0 \end{gathered} \tag{4.1}$$

are as follows:

$$\begin{aligned} &\overline{\mu}_1 = 2 - \sqrt{2}, \qquad \psi_1(t) = \sin\left(\frac{\pi}{4}t\right), \quad t \in \mathbb{T}_1, \\ &\overline{\mu}_2 = 2, \qquad \psi_2(t) = \sin\left(\frac{\pi}{2}t\right), \quad t \in \mathbb{T}_1, \\ &\overline{\mu}_3 = 2 + \sqrt{2}, \qquad \psi_3(t) = \sin\left(\frac{3\pi}{4}t\right), \quad t \in \mathbb{T}_1. \end{aligned} \tag{4.2}$$

Obviously,

$$\{t \in \mathbb{T}_1 \mid \psi_1(t) = 0\} = \varnothing, \qquad \{t \in \mathbb{T}_1 \mid \psi_2(t) = 0\} = \{2\}, \qquad \{t \in \mathbb{T}_1 \mid \psi_3(t) = 0\} = \varnothing. \tag{4.3}$$

Example 4.1. Let us consider the discrete boundary value problem

$$\begin{gathered} \Delta^2 y(t-1) + \overline{\mu}_1 y(t) + g_0(t, y(t)) = h(t), \quad t \in \mathbb{T}_1, \\ u(0) = u(4) = 0, \end{gathered} \tag{4.4}$$

where

$$g_0(t,s) = (\overline{\mu}_2 - \overline{\mu}_1) \sin\left(\frac{\pi}{4}t\right)\left(s + \frac{s}{1+s^2}\right), \quad (t,s) \in \mathbb{T}_1 \times \mathbb{R}. \tag{4.5}$$

It is easy to verify that g_0 satisfies all conditions of Theorem 1.4 with

$$\Gamma(t) = (\overline{\mu}_2 - \overline{\mu}_1)\left|\sin\left(\frac{\pi}{4}t\right)\right|. \tag{4.6}$$

Therefore, (4.4) has at least one solution for every $h : \mathbb{T}_1 \to \mathbb{R}$ with

$$\sum_{t=a+1}^{b+1} h(t)\sin\left(\frac{\pi}{4}\right)t = 0. \tag{4.7}$$

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