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Review Article

Do All Integrable Equations Satisfy Integrability Criteria?

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At the price of sacrificing all suspense, we can already announce that the answer to the question of the title is "no." It is indeed our belief that one may find counterexamples to all integrability conjectures, *unless* one constrains the definition of integrability to the point that the integrability criterion becomes tautological. This review is devoted to a critical analysis of the situation.

1. Introduction

The study of integrable systems is intimately related to integrability criteria [1]. This is due to the fact that integrable systems are very rare but also very interesting (which explains the intense activity in this domain). The usefulness of integrability criteria is that they allow the proposal of conjectures which serve for integrability prediction. The integrability detectors thus elaborated to find their full usefulness in the nonconstructive approach to integrability. The opposite, constructive, approach consists in deriving integrable systems starting from the solution, or what is admittedly more customary, from an overdetermined linear system (the Lax pair), the nonlinear integrable equation resulting from the compatibility of the linear one. The nonconstructive approach starts from a nonlinear system resulting from some, often physical, usually realistic, model. It is then of the utmost importance to know whether this system is integrable since integrability conditions the long-term behaviour of its solutions. A reliable integrability detector is a most valuable tool.

In the domain of continuous systems, the use of complex analysis has made possible the development of specific and efficient tools for integrability prediction, and actual integration of systems expressed as (ordinary or partial) differential equations. According to Poincaré [2], to integrate a differential equation is to find for the general solution an expression, possibly multivalued, in terms of a finite number of functions. The word "finite" indicates that integrability is related to a *global* rather than a *local* knowledge of the solution. However, this definition is not very useful unless one defines more precisely what is meant by "function." By extending the solution of a given ordinary differential equation (ODE) into the complex domain, one has the possibility, instead of asking for a global solution for an ODE, to look for solutions locally and obtain a more global result by analytic continuation. If we wish to define a function, we must find a way to treat branch points, that is, points around which two (at least) determinations are exchanged. This can be done through various uniformization procedures provided the branch points are fixed. Linear ODEs are such that all the singularities of their solutions are fixed and are thus considered integrable. In the case of nonlinear ODEs, the situation is not so simple due to the fact that the singular points in this case may depend on the initial conditions: they are movable. The genius of Painlevé [3, 4] was to decide to look for those of the nonlinear ODEs the solutions of which were free from movable branch points. The success of this approach is well known: the Painlevé transcendents have been discovered in that way and their importance in mathematical physics is ever growing. The Painlevé property, that is absence of movable branch points, has been since used with great success in the detection of integrability [5–8].

We must stress one important point here. The Painlevé property as introduced by Painlevé is not just a *predictor* of integrability but practically a *definition* of integrability. As such it becomes a tautology rather than a criterion. It is thus crucial to make the distinction between the Painlevé property and the algorithm for its investigation. The latter can only search for the absence of Painlevé property *within certain assumptions*. The search can thus lead to a conclusion the validity of which is questionable: if we find that the system passes what is usually referred to as the Painlevé test (in one of its several variants), this does not necessarily mean that the system possesses the Painlevé property. Kruskal [9] has stressed this important point on various occasions that, at least as far as its usual practical application is concerned, the Painlevé test may not be sufficient for integrability. The situation becomes further complicated if we consider systems that are integrable through quadratures or linearisation. If we extend the notion of integrability in order to include such systems, it turns out that the connection to the Painlevé property breaks down. As we have shown in [10], the integrable character of linearisable systems is not associated to the Painlevé property. (As a matter of fact, no linearisability detector appears to exist to date, to the authors knowledge.)

Discrete systems pose a greater challenge. A first integrability detector was proposed based on the observation that mappings integrable through spectral methods have confined singularities [11], that is, any singularity spontaneously appearing due to the choice of initial conditions disappears after a few iteration steps. What is crucial is that a mapping may at some point lose a degree of freedom. In the mapping of the form

$$x_{n+1} = f(x_n, x_{n-1}), \tag{1.1}$$

this simply means that $\partial x_{n+1}/\partial x_{n-1} = 0$ and the memory of the initial condition x_{n+1} disappears from the iteration. What does the confinement mean in this case? Clearly, the mapping must recover the lost degree of freedom and the only way to do this is by the appearance of an indeterminate form $0/0$, $\infty - \infty$ in the subsequent iterations. As Kruskal points out, the way to treat this difficulty is to use an argument of continuity with respect to the initial conditions and to introduce a small parameter ϵ. For an explicit example, refer to [12]. While singularity confinement has been instrumental in discovering a host of integrable discrete systems [13, 14], it turned out that the confinement property was not sufficient in order to guarantee integrability [15]. We will not go into detailed explanations here. It suffices to say that for discrete systems to be integrable, a proper local singularity structure is not enough. The growth properties of the solutions at infinity enter into play and the best way to qualify this is through the Nevanlinna approach [16]. To put it in a nutshell, for a discrete system to be integrable the requirement is that the Nevanlinna order of the solution be finite (which guarantees not too fast a growth). Just as in the continuous case, the algorithm for the calculation of the Nevanlinna order is not precise enough. The conditions obtained are sufficient but usually largely nonnecessary [17]. In this case, implementing the singularity confinement requirement on top of the "Nevanlinna" constraints allows one to reduce substantially the available parameter space. Linearisable discrete systems are a class of their own. As we have shown in [10], linearisability does not require confined singularities although the solutions must still have finite Nevanlinna order. The finiteness of the Nevanlinna order is to be understood as a condition for mappings with constant coefficients. The case of nonautonomous systems presents additional difficulties. An algorithm which calculates the growth was proposed by Hietarinta and Viallet [15] and is commonly referred to as the algebraic entropy technique.

Finally, there exist systems which are generalised cellular automata and are obtained from discrete systems following the ultradiscretisation procedure [18]. We remind here that the latter consists into introducing an ansatz $x = e^{X/\delta}$ (where x is the solution of the discrete system, which should obviously be positive definite) and obtain for X an equation by going to the limit $\delta \to 0$. The essential identity that allows to derive easily the ultradiscrete forms is $\lim_{\delta\to 0}\delta \log(e^{A/\delta} + e^{B/\delta}) = \max(A, B)$. The ultradiscretisation procedure preserves any integrable character of the initial system. The question of the existence of the ultradiscrete analogue of integrability-related properties, like the singularity confinement, has been already addressed by Joshi and Lafortune [19]. They proposed a singularity analysis approach which is perceived as the ultradiscrete equivalent of singularity confinement. In [20], we have critically examined this approach and have shown that, just as in the discrete case, there exist integrable ultradiscrete systems with unconfined singularities but also nonintegrable systems with confined singularities.

In what follows, we will present a review of these findings, illustrating all situations through concrete examples.

2. Continuous systems

A first instance of integrability without the Painlevé property was the derivation of the integrable system described by the Hamiltonian [21]:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + y^5 + y^3x^2 + \frac{3}{16}yx^4 \tag{2.1}$$

which has the second (besides the energy) constant of motion

$$C = -yp_x^2 + xp_xp_y + \frac{1}{2}y^4x^2 + \frac{3}{8}y^2x^4 + \frac{1}{32}x^6. \tag{2.2}$$

There are movable singularities where near some singular point t_0, one has $y \approx \alpha(t-t_0)^{-2/3}$, $x \approx \beta(t-t_0)^{-1/3}$ with $\alpha^3 = -2/9$, β arbitrary. Taking the cube of the variables is not sufficient to regularise them, however. Indeed, a detailed analysis of complex-time singularities shows that their expansions contain *all* powers of $(t-t_0)^{1/3}$. The fact that some multivaluedness was compatible with integrability led to the introduction of the notion of "weak Painlevé" property [21]. However, it was soon realised [22] that (2.1) was a member of a vaster family of integrable Hamiltonian systems associated to the potential $V = (F(\rho+y)+G(\rho-y))/\rho$ where $\rho = \sqrt{x^2+y^2}$. Since the two functions F and G are free, one can easily show that the singularities of the solutions of the equations of motion can be arbitrary. The Hamiltonians of this family are integrable through quadratures and, in fact, the associated Hamilton-Jacobi equations are separable. This leads to the conclusion that this type of integrability is not necessarily related to the Painlevé property. (As a matter of fact, the same conclusion could have been reached if we had simply considered one-dimensional Hamiltonian systems). One may justifiably argue that in the case of Hamiltonian systems the term integrability is to be understood as Liouville integrability which is not the one we refer to in relation to the Painlevé property. Still, Liouville integrability, and the dynamical symmetries to which it is associated, may be of utmost importance for physical applications and a systematic method for the detection would have been most welcome.

We turn now to a second case of integrability where the necessary character of the Painlevé property can be critically examined: that of linearisable systems. The term linearisable is used here to denote systems that can be reduced to linear equations through a local variable transformation. The first family of such systems are the projective ones [23]. As can be easily shown, the projective systems possess the Painlevé property by construction. However, there exists another kind of linearisability for which the Painlevé property need not be satisfied. Let us discuss the best-known second-order case. One of the equations of the Painlevé/Gambier classification [24], bearing the number XXVII, is the equation proposed by Gambier. The Gambier equation is given as a system of two Riccati equations in cascade. This means that we start with a first Riccati for some variable y:

$$y' = -y^2 + c \tag{2.3}$$

and then couple its solution to a second Riccati by making the coefficients of the latter depend explicitly on y:

$$x' = ax^2 + nxy + \sigma. \tag{2.4}$$

(In [25], we have shown that this form is the term linear in y in the right-hand side of (2.3) can be put to zero without loss of generality). The precise form of the coupling introduced in (2.4) is due to integrability requirements. In fact, the application of singularity analysis shows that the Gambier system cannot be integrable unless the coefficient of the xy term in (2.4) is an integer n. This is not the only integrability requirement. Depending on the value of n, one can find constraints on the a, c, σ (where the latter is traditionally taken to be 1 or 0, without loss of generality) which are necessary for integrability. On the other hand, the integration of the two Riccati equations in cascade can always be performed through reduction to linear second-order equations, even when the Painlevé constraints are not satisfied.

Once the Painlevé property is deemed unnecessary for the linearisation of the Gambier system, it is straightforward to extend the latter to the form

$$y' = \alpha y^2 + \beta y + \gamma, \tag{2.5a}$$

$$x' = a(y,t)x^2 + b(y,t)x + c(y,t), \tag{2.5b}$$

where α, β, and γ are arbitrary functions of t while a, b, and c are arbitrary functions of y and t. The integration in cascade of (2.5a) and (2.5b) can be obtained as previously. As a matter of fact, an extension like (2.5a) and (2.5b) gives the handle to the $(N+1)$-variables generalisation of the Gambier system

$$\begin{aligned} x_0' &= a_0(t)x_0^2 + b_0(t)x_0 + c_0(t), \\ x_\mu' &= a_\mu(x_0,\ldots,x_{\mu-1},t)x_\mu^2 + b_\mu(x_0,\ldots,x_{\mu-1},t)x_\mu + c_\mu(x_0,\ldots,x_{\mu-1},t), \quad \mu = 1,\ldots,N, \end{aligned} \tag{2.6}$$

where a_μ, b_μ, and c_μ are arbitrary functions of their arguments. Again, system (2.6) does not possess, generically, the Painlevé property while it can be linearised and integrated in cascade.

The Gambier systems above are rather straightforward generalisations of integrable systems which violate the Painlevé property while preserving their linearisability. However, there exist other methods of linearisation which again lead to integrable systems not possessing the Painlevé property [26]. The idea is the following: we start from a linear second-order equation in the form

$$\frac{\alpha x'' + \beta x' + \gamma x + \delta}{\epsilon x'' + \zeta x' + \eta x + \theta} = K, \tag{2.7}$$

where $\alpha, \beta, \ldots, \theta$ are functions of t with K a constant, and a *nonlinear* second-order equation of the form

$$f(x'', x', x) = M, \tag{2.8}$$

where f is a (possibly inhomogeneous) polynomial of degree two in x together with its derivatives, but linear in x'', and with M a constant. We then ask that the derivatives of both equations with respect to the independent variable, that is the resulting third-order equations, be identical up to an overall factor. This is a novel linearisation approach. The explicit integration procedure is the following. We start from (2.8) with *given* M and initial conditions x_0, x_0' for some value t_0 of the independent variable t. We use (2.8) to compute x_0'' at t_0. Having these values, we can use (2.7) to compute the value of K. Since the latter is assumed to be a constant, we can integrate the linear equation (2.7) for all values of t. Since this solution will satisfy the third-order equation mentioned above, it will also be a solution of (2.8).

In order to illustrate this approach, we derive one equation that can be integrated through this linearisation. Our starting assumption is that (2.8) contains a term $x''x'$. The more general term $x''(x' + cx + d)$ can always be reduced to this form, that is $c = d = 0$ through a rescaling and translation of x. It is then straightforward to obtain the full expression in the homogeneous subcase $\delta = \theta = 0$. We thus find

$$\frac{tx'' + (at - 1/2)x' + btx}{x'' + ax' + bx} = K \tag{2.9}$$

for the linear equation, and

$$x''x' + 2ax'^2 + 3bx'x + \left(2ab - b'\right)x^2 = M \tag{2.10}$$

for the nonlinear one, with $b = a^2 - a'/2$ and a satisfying the equation

$$a''' = 6a''a + 7a'^2 - 16a'a^2 + 4a^4 \tag{2.11}$$

which is equation XII in the Chazy classification [27]. Given a and the corresponding b, (2.10) is integrable by linearisation through (2.9). On the other hand, (2.10) violates the Painlevé property. Solving it for x'', we find a term proportional to x^2/x' (or, for that matter, to $1/x'$) which is incompatible with it.

It is thus natural, given the results presented above on linearisable ODEs, to wonder whether linearisable PDEs without the Painlevé property may exist. Calogero [28] has stressed the importance of the existence of PDEs integrable by methods different from the spectral ones. He has dubbed the members of this class C-integrable systems. One large class of the C-integrable systems of Calogero comprises equations which are obtained from some other integrable (sometimes linear) equations through hodograph transformations. The prototypical equation of this class is the Dym equation [29]

$$u_t = u^3 u_{xxx} \tag{2.12}$$

which is related to the KdV equation. Equations of this class quite often possess the "weak" Painlevé property. This is the case for the Dym equation. The expansion around a singularity manifold $\phi(x,t)$ is $u_0\phi(x,t)^{2/3} + \sum_{p=1}^{\infty} u_p \phi(x,t)^{(p+2)/3}$. Moreover, there exist equations in the Calogero list of C-integrable PDEs belonging to the class of solvable through hodograph transformations which do not satisfy the Painlevé property at all. An example of such an equation is $u_t = f(u_x)/u_{xx} + g(u_x) + uh(u_x)$, where f, g, and h are arbitrary functions.

We turn now to the question whether linearisable PDEs without the Painlevé property do exist. The answer to this question is an unqualified "yes" [30]. Let us construct a specific example. We will adopt the construction we follow for the derivation of the Burgers' equation. For the latter, we start from a linear equation $v_t + v_{xx} = 0$ and obtain a nonlinear one through a Cole-Hopf relation $v_x + uv = 0$. In order to derive the equation we are seeking, we start from a nonlinear, linearisable (Riccati) equation in one variable $v_t + v^2 = 0$ and couple it through a Cole-Hopf-like relation to another variable in a new direction $u_x + uv = 0$. Eliminating v, we obtain a nonlinear equation for u:

$$uu_{xt} - u_x u_t - u_x^2 = 0. \tag{2.13}$$

This is obviously a linearisable equation since its solution proceeds through the solution of a linearisable equation and a linear one, in cascade. The solution of this equation does *not* possess the Painlevé property. Instead of performing a standard Painlevé analysis, let us profit from the fact that the solution of (2.13) can be explicitly constructed. Solving the equation for v, we find $v = (t - \phi(x))^{-1}$. Next we integrate for u and obtain $\log u = -\int (t - \phi(x))^{-1} dx$. A singularity will appear in the expansion of u whenever we have $x = \xi$ such that $\phi(\xi) = t$. We solve for ξ and find $\xi = \psi(t)$ (where ψ is the inverse function of ϕ). Expanding $\phi(x)$ around ξ, we have $\phi(x) = \phi(\xi) + (x - \xi)\phi'(\xi) + \cdots$ and the integration for u can be performed order by order. We find $u \propto (x - \psi(t))^{\psi'(t)} + \cdots$. Thus, since the exponent $\psi'(t) \equiv 1/\phi'(\psi(t))$ is arbitrary, the solution does *not* possess the Painlevé property.

Equation (2.13) may be easily generalised. The principle remains the same. One starts from a linearisable equation in one independent and one dependent variable, say $v(t)$. If, for instance, we take for v a higher-order projective equation, we are guaranteed that the solution for v will satisfy the Painlevé property. Next we couple this equation to a linear PDE of the form $f(v)u_x + g(v)u_t + h(v)u = 0$, where f, g, and h can be taken as inhomogeneous linear functions of v. Eliminating v, one obtains an equation for u which is linearisable and can be shown to violate the Painlevé property, the exponent of the leading singular term being again an arbitrary function of t.

3. Discrete systems

In the case of discrete systems, a difficulty appears from the outset in the sense that the discrete analogue of the Painlevé property, namely singularity confinement, does not guarantee integrability. There exist mappings which have only confined singularities and which are *not* integrable [15]. In [31], we have presented such an example. The mapping

$$\frac{x_{n+1}}{x_{n-1}} = x_n + \frac{1}{x_n} \tag{3.1}$$

has a confined singularity pattern $\{1, 0, \infty, 1\}$ but it is not integrable. This can be established by studying the growth properties of the iterates of some initial condition following the approach of [15]. Indeed starting from initial data x_0, x_1, we introduce homogeneous variables through $x_0 = p$, $x_1 = q/r$ and compute the homogeneity degree of the iterates of the mapping in q, r, to which we assign the same degree 1, while p is assigned the degree 0. For a generic, nonintegrable mapping, the degree growth of the iterates is exponential. In the case of (3.1) we obtain the following sequence: 0, 1, 2, 4, 8, 14, 24, 40, 66, 108, 176, 286, ... which, for n large enough, obeys the recursion relation $\delta_{n+1} - 2\delta_n + \delta_{n-2} = 0$. Thus we have indeed an exponential growth of the degree, the asymptotic ratio of two consecutive x being $(1 + \sqrt{5})/2$, that is, the same as that of the Fibonacci sequence.

For integrable mappings, the growth is just polynomial. Moreover, a detailed analysis of discrete Painlevé equations [32] and linearisable mappings [33] has shown that the latter have even slower growth properties (which can be used not only as a detector of integrability but as an indicator of the integration method). In what follows, we will examine the results of the application of the two methods to integrable discrete systems. We will not present in any detail the case of projective mappings [23]. It suffices to say that any singularity appearing in projective systems is confined in one step. Moreover, the study of the degree of the iterates [34] shows that there is no growth at all: the degree is constant. Thus both criteria are satisfied in this case.

However, there exist linearisable mappings which are not projective. One such example is

$$x_{n+1} = a x_{n-1} \frac{x_n - a}{x_n - 1}. \tag{3.2}$$

As shown in [31], two singularities exist when either $x = 1$ or $x = a$. The first singularity is confined leading to a finite singularity pattern $\{1, \infty, a\}$. The second singularity never confines unless $a = 1$ (in which case the mapping is trivial) or a is a cubic root of unity (with the resulting mapping being periodic with period six). On the other hand, (3.2) is linearisable. Indeed, introducing $y_n = x_n x_{n-1} - x_n - a x_{n-1}$ we reduce (3.2) to the linear mapping

$$y_{n+1} = a y_n. \tag{3.3}$$

It goes without saying that the degree growth of the iterates of (3.2) is linear as expected from algebraic entropy arguments.

Next we turn now to the case of the Gambier mapping [35]. The latter is, in perfect analogy to the continuous case, a system of two (discrete) Riccati equations in cascade:

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}, \tag{3.4a}$$

$$x_{n+1} = \frac{a y_n x_n + b x_n + c y_n + d}{f y_n x_n + g x_n + h y_n + k}, \tag{3.4b}$$

where $\alpha, \ldots, \delta$ and $a, \ldots, k$ are all functions of the independent discrete variable n. In [35], it was shown that system (3.4a) and (3.4b) is not confining unless the coefficients entering in the equation satisfy certain conditions. On the other hand, the same argument presented in the continuous case can be transposed here: the integration of the two Riccati equations in cascade can always be performed through reduction to linear second-order mappings. The study of the degree growth of the iterates of (3.4a) and (3.4b) was performed in [33] where it was found that the growth is always linear, *independently* of the conditions we referred to above.

This result leads naturally to the following generalisation of the discrete Gambier system, the singularities of which are, in general, not confined:

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}, \tag{3.5a}$$

$$x_{n+1} = \frac{a(y_n) x_n + b(y_n)}{c(y_n) x_n + d(y_n)}, \tag{3.5b}$$

where $a, \ldots, d$ are polynomials in y the coefficients of which may depend on the independent variable n. The study of the degree growth of the iterates of (3.5a) and (3.5b) is straightforward. We find that the degree growth of x is linear. Again, system (3.5a) and (3.5b) can be integrated in cascade. On the other hand, (3.5a) and (3.5b) cannot be written as a three-point mapping for x. Indeed, if we eliminate y_n, y_{n+1} between (3.5a), (3.5b) and the upshift of the latter, we obtain an equation relating x_n, x_{n+1} and x_{n+2} which is polynomial in all three variables, generically *not* linear in x_{n+2}. This does not define a mapping but rather a correspondence which in general leads to exponential proliferation of the number of images and preimages. This correspondence is not integrable but this is not in contradiction with the integrability of (3.5a) and (3.5b). The two systems are not equivalent.

Another point concerns the discrete analogues of the linearisable systems we have presented at the end of Section 2. The procedure can be transposed to a discrete setting in a pretty straightforward way [26]. We have a linear equation

$$\frac{\alpha x_{n+1} + \beta x_n + \gamma x_{n-1} + \delta}{\epsilon x_{n+1} + \zeta x_n + \eta x_{n-1} + \theta} = K, \tag{3.6}$$

where $\alpha, \dots, \theta$ are all functions of n with K a constant, and a nonlinear mapping

$$f\left(x_{n-1}, x_n, x_{n+1}; n\right) = M, \tag{3.7}$$

where f is globally polynomial of degree two in all the x's but not more than linear separately in each of x_{n-1} and x_{n+1}. Writing that the left-hand side of (3.6) is the same as that of its upshift, we get an equation relating x_{n-1}, x_n, x_{n+1}, and x_{n+2}. For appropriate choices of $\alpha, \dots, \theta$, this four-point equation can be identical (up to unimportant factors) to the four-point equation obtained from (3.7) by writing $f(x_{n-1}, x_n, x_{n+1}; n) = f(x_n, x_{n+1}, x_{n+2}; n+1)$. The integration method is quite similar to that described in the continuous case. Given M, and starting with x_{n-1}, x at some n, one gets x_{n+1} from (3.7). Implementing (3.6), this fixes the value of K. From now on, one integrates the linear equation (3.6) for all n. Since the four-point equation is always satisfied, this means that f computed at any n has a constant value, which is just M, so (3.7) is satisfied.

Several mappings derived in [26] as special limits of discrete Painlevé equations can be linearised in this way. For instance, the nonlinear equation

$$\left(\frac{x_{n+1} + x_n - a}{z_{n+1}} - \frac{x_n}{\zeta_n}\right)\left(\frac{x_{n-1} + x_n - a}{z_n} - \frac{x_n}{\zeta_n}\right) - \frac{x_n^2}{\zeta_n^2} = M \tag{3.8}$$

with a a constant, where z and ζ are defined from a single arbitrary function g of n through $z_n = g_{n+1} + g_{n-1}$, $\zeta_n = g_{n+1} + g_n$, can be solved through the linear equation:

$$\frac{A_n x_{n+1} + B_n\left(x_n - a\right) + A_{n+1} x_{n-1}}{z_n x_{n+1} + \left(z_{n+1} + z_n\right)\left(x_n - a\right) + z_{n+1} x_{n-1}} = K, \tag{3.9}$$

where $A_n = g_n^2(g_{n+1} + g_{n-1})$ and $B_n = -(g_{n+1} + g_n)g_{n+2}g_{n-1} - (g_{n+2} + g_{n-1})g_{n+1}g_n$. Mapping (3.8) is generically nonconfining unless g is a constant.

A question that can be asked at this point is whether there exist integrable systems which violate the algebraic entropy criterion. (We remind that following the approach of Hietarinta and Viallet the algebraic entropy of a rational mapping is defined from the degree d_n of the nth iterate as $\epsilon = \lim_{n\to\infty} \log(d_n)/n$.) Families of mappings with exponential growth of the degree of the iterates, that is with positive algebraic entropy, have been proposed in [36]. It all hinges on what we mean by linearisability. We start from a linear equation:

$$\omega_{n+1} + \omega_{n-1} = k\omega_n. \tag{3.10}$$

The solution of (3.10) is straightforward: $\omega_n = a\rho_1^n + b\rho_2^n$ where $\rho_{1,2} = (k \pm \sqrt{k^2-4})/2$. Putting $x_n = \tan\omega_n$, we find

$$\frac{x_{n+1} + x_{n-1}}{1 - x_{n+1}x_{n-1}} = f_k(x_n), \tag{3.11}$$

where, when k is an integer, f_k is a rational function of x_n. It is in fact the expression of tan $k\omega_n$ in terms of tan $\omega_n \equiv x_n$. For the first few values of k, we have $f_1(x_n) = x_n$, $f_2(x_n) = 2x_n/(1-x_n^2)$, $f_3(x_n) = (3x_n - x_n^3)/(1-3x_n^2)$ and so forth. The cases $k = 1$ and $k = 2$ are trivially integrable. The case $k = 3$ is more interesting. Indeed, the mapping

$$x_{n+1} = \frac{3x_n - x_n^3 - x_{n-1}(1-3x_n^2)}{1 - 3x_n^2 + (3x_n - x_n^3)x_{n-1}} \tag{3.12}$$

is both chaotic and integrable (since it is linearisable). The value of the algebraic entropy is $\log((3+\sqrt{5})/2)$. The same would apply to all cases $k \geq 3$: they all have a positive algebraic entropy and are also linearisable. The main difference of this mapping compared to the previous example is that while (3.12) is rational the transformation leading to the linear equation (3.10) is transcendental.

We conclude this section with the case of lattice linearisable equations [30]. We start with a simple homographic mapping (the index m is dummy at this level):

$$v_{m,n+1} + 1 + \frac{1}{v_{m,n}} = 0 \tag{3.13}$$

and couple it to a linear equation

$$u_{m+1,n} - u_{m,n}v_{m,n} = 0. \tag{3.14}$$

Eliminating v, we find for u the equation

$$u_{m+1,n+1}u_{m+1,n} + u_{m,n+1}u_{m+1,n} + u_{m,n}u_{m,n+1} = 0. \tag{3.15}$$

While this equation is linearisable, it does *not* have confined singularities. Indeed, if at some lattice position we have $u_{m,n} = 0$ (which is perfectly possible given the appropriate initial conditions), iterating (3.15) we find that $u_{k,n} = 0$ for all $k \geq m$. On the other hand, since (3.15) is linearisable, we expect the growth of the sequence of its iterates to be linear. This turns out to be indeed the case. Taking initial conditions $u_{0,0} =$ const., $u_{0,n} = a(n) + b(n)p/q$, $u_{m,0} = c(m) + f(m)r/s$ (with a, b, c, f arbitrary functions of their argument) and computing the global homogeneous degree $d_{m,n}$ in p, q, r, s, we find that $d_{m,n} = m + 2$ for $m > 0$.

Generalising (3.15) is quite straightforward. It suffices to start from a linearisable equation for v of higher order (of which several examples do exist). Next, we couple v to a linear equation of the form $f(v_{m,n})u_{m+1,n} + g(v_{m,n})u_{m,n+1} + h(v_{m,n})u_{m,n} = 0$ where f, g, h are first degree in v, and using the first equation we eliminate v. We surmise that the equation for u will in general have unconfined singularities. However, this has to be examined on a per-case basis since there does not seem to exist a general argument for the singularity structure of the final equation.

4. Ultradiscrete systems

Before proceeding to the analysis of ultradiscrete systems, it is interesting to spend a few lines again on their discrete counterparts focusing on the notion of singularity. Given a mapping of the form $x_{n+1} = f(x_n, x_{n-1})$, we are in the presence of a singularity whenever $\partial x_{n+1}/\partial x_{n-1} = 0$, that is x_{n+1} "loses" its dependence on x_{n-1}. When this is due to a particular choice of initial conditions, we are referring to this singularity as a *movable* one. Movable singularities may be bad, for integrability, because they may lead, after a few mapping iterations, to an indeterminate form $(0/0, \infty - \infty, \ldots)$ or propagate indefinitely. In the former case, provided we can lift the indeterminacy while recovering the lost degree of freedom (using an argument of continuity with respect to the initial conditions), we are talking about a confined singularity. The typical singularity pattern in the case of confined singularities is the following: the solution is regular for all values of the index n up to some value n_s, then a singularity appears and propagates up to n_c whereupon it disappears and the solution is again regular for all values of the index larger than n_c. In some cases, we are in presence of the reciprocal situation. The solution is singular for all values of $n < n_s$, becomes regular between n_s and n_c, and is again singular for $n > n_c$. This singularity is called weakly confined by Takenawa [37] and is considered to be compatible with integrability. At the limit where there exists no interval where the solution may be regular, and the solution is singular throughout, we are in the presence of what we call a "fixed" singularity (which again does not hinder integrability).

Joshi and Lafortune [19] have transposed these notions to the ultradiscrete case and proposed an analogue to the singularity confinement property. In the ultradiscrete systems, the nonlinearity is mediated by terms involving the max operator. Typically one is in presence of terms like $\max(X_n, 0)$. When, depending on the initial conditions, the value of X_n crosses zero, the result of the $\max(X_n, 0)$ operation becomes discontinuous: when X is slightly smaller than 0 the result is zero, while for $X > 0$ the result is X. It is this discontinuity that plays the role of the singularity since it leads nonanalyticity. Typically, if we put $X = \epsilon$, a term $\mu = \max(\epsilon, 0)$ propagates with the iterations of the mapping and perpetuates the discontinuity *unless* by some coincidence it disappears. This disappearance is the equivalent of the singularity confinement for ultradiscrete systems. Joshi and Lafortune have introduced an algorithmic method for testing the confinement property for ultradiscrete systems, linked it to integrability, and reproduced results on ultradiscrete Painlevé equations by initially deautonomising ultradiscrete mappings.

However and in analogy to the discrete case, there exist nonintegrable systems with confined singularities and integrable systems with unconfined singularities. In [31], we obtained a mapping which did pass the confinement test while having a positive algebraic entropy:

$$x_{n+1} = x_{n-1}\left(x_n + \frac{1}{x_n}\right). \tag{4.1}$$

The main advantage of this mapping over the examples of [15] is that it is multiplicative and by choosing the appropriate initial data one can restrict the solution to positive values. In that case, the ultradiscretisation of (4.1) is straightforward. We find

$$X_{n+1} = X_{n-1} + |X_n|, \tag{4.2}$$

where we have preferred to introduce the absolute value of X instead of its equivalent $\max(X,0)+\max(-X,0)$. We will examine the behaviour of a singularity appearing at, say, $n=1$ where $X_1=\epsilon$, while X_0 is regular. We again use the identity $\mu \equiv \max(\epsilon,0)=(|\epsilon|+\epsilon)/2$ and distinguish two different sectors $X_0<0$ and $X_0>0$. In the first case ($X_0<0$), we find the sequence

$$\begin{aligned}
&\vdots\\
X_{-3} &= 3X_0,\\
X_{-2} &= 2X_0-\epsilon,\\
X_{-1} &= X_0+\epsilon,\\
&X_0,\\
X_1 &= \epsilon,\\
X_2 &= X_0-\epsilon+2\mu,\\
X_3 &= -X_0+2\epsilon-2\mu,\\
X_4 &= \epsilon,\\
X_5 &= -X_0+\epsilon,\\
&\vdots
\end{aligned} \tag{4.3}$$

We can see readily that the singularity, indicated by the presence of μ, is confined (to X_2 and X_3 only). Turning to the case $X_0>0$, we find the sequence

$$\begin{aligned}
&\vdots\\
X_{-4} &= -X_0+2\mu+\epsilon,\\
X_{-3} &= -X_0+2\mu,\\
X_{-2} &= \epsilon,\\
X_{-1} &= -X_0+\epsilon,\\
&X_0,\\
X_1 &= \epsilon,\\
X_2 &= X_0+2\mu-\epsilon,\\
X_3 &= -X_0+2\mu,\\
X_4 &= 2X_0+4\mu-\epsilon,\\
&\vdots
\end{aligned} \tag{4.4}$$

In this case, we are in presence of a weakly confined solution: a regular part around $n=0$ is surrounded by unconfined singularities both for large positive and large negative n's. Thus, the ultradiscrete mapping (4.2) has confined singularities and is not integrable.

The converse situation of a mapping, which is while integrable does not possess confined singularities, does also exist. As expected, an example is to be sought among linearisable systems. In [31], we discovered the "multiplicative" linearisable mapping

$$\frac{x_{n+1}}{x_{n-1}} = a\frac{x_n+a}{x_n+1}. \tag{4.5}$$

Without loss of generality, the parameter a can be always taken larger than unity. (Indeed it suffices to reverse the direction of the evolution in which case a goes to $1/a$.) We can now ultradiscretise (4.5) to

$$X_{n+1} = X_{n-1} + A + \max(X_n, A) - \max(X_n, 0), \tag{4.6}$$

where $A > 0$. The complete description of the solution would require examining several sectors that exist, but in order to show that there exist unconfined singularities, it suffices to exhibit such a situation in one sector. It turns out that the case where X_0 has a large negative value is one leading to unconfined singularities

$$\begin{aligned}
&\vdots \\
X_{-4} &= -X_0 - 4A, \\
X_{-3} &= -4A + \epsilon, \\
X_{-2} &= X_0 - 2A, \\
X_{-1} &= -2A + \epsilon, \\
&X_0, \\
X_1 &= \epsilon, \\
X_2 &= X_0 + 2A - \mu, \\
X_3 &= 2A + \epsilon, \\
X_4 &= X_0 + 3A - \mu, \\
X_5 &= 4A + \epsilon, \\
X_6 &= X_0 + 4A - \mu, \\
X_7 &= 6A + \epsilon, \\
&\vdots
\end{aligned} \tag{4.7}$$

We remark readily that while for negative indices the solution is regular, a singularity, mediated by μ, appears for positive n's and is never confined.

While some progress towards a Nevanlinna-like theory for ultradiscrete systems has been recently accomplished [38], it is clear that more work is necessary before we can be sure where we stand concerning ultradiscrete integrability.

5. Conclusion

In the light of the examples analysed in the previous section, we can now present our ideas and beliefs on integrability and its detection. A perfect criterion of integrability would in theory be both necessary and sufficient. However, no real-life integrability criterion meets these stringent requirements. This has also to do with the loose definition of "integrability" which in some cases is used in lieu of "linearisability" or even "solvability."

In the continuous domain, the "Painlevé" criterion is of unmatched success. Systems integrable through spectral methods have the Painlevé property and its rigorous detection constitutes a reliable integrability predictor. (It goes without saying that the practical implementations of singularity analysis [39], known as the "Painlevé test," offer no such guarantees: they may miss some critical singularities to say nothing of the complexity of the task which sometimes leads to incomplete applications of the test and erroneous conclusions.) If one extends the notion of integrability so as to include systems integrable through linearisation or quadratures, then the relation to the Painlevé property is lost and the Painlevé criterion is violated by these systems.

In the case of discrete systems, the situation is more complicated. Integrability is not conditioned by the local singularity behaviour which explains why the singularity confinement test is not sufficient and one must consider the growth properties of the solution. While the algebraic entropy techniques furnish a successful test, the ideal discrete analogue of the Painlevé property is still lacking. Perhaps something along the lines of the Sakai approach [40] for the derivation of the discrete Painlevé equations could provide the answer. Just as in the continuous case, systems which are integrable through spectral method possess the singularity confinement property while the linearisable systems do not. The main difference here is that the algebraic entropy techniques constitute a reliable linearisability detector (with a few precautions as to the class of transformations allowed).

Finally, the ultradiscrete case is even less well understood. As we have shown here, there exist ultradiscrete systems with confined singularities (in the sense of Joshi and Lafortune) and which are nonintegrable. Conversely, some systems integrable through linearisation do not have confined singularities, in perfect parallel to the discrete situation. What is more worrisome here is that there exists at least one example [20] of a linearisable system which possesses an explicit invariant and still has unconfined singularities. In the ultradiscrete case, the notion of integrability through spectral methods is not yet well established. Thus we cannot make any claims concerning the confinement properties of such systems. Another difficulty has to do with the linearisable character which is assessed through the relation of the ultradiscrete system to its discrete parent and not in an intrinsic way. Moreover, the study of the growth properties of ultradiscrete systems (in particular through the equivalent of Nevanlinna theory) is at its initial phase. We hope that all these open questions will find some answers in the next few years.

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