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Research Article

Existence and Multiple Solutions for Nonlinear Second-Order Discrete Problems with Minimum and Maximum

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Consider the multiplicity of solutions to the nonlinear second-order discrete problems with minimum and maximum: $\Delta^2 u(k-1) = f(k, u(k), \Delta u(k))$, $k \in \mathbb{T}$, $\min\{u(k) : k \in \widehat{\mathbb{T}}\} = A$, $\max\{u(k) : k \in \widehat{\mathbb{T}}\} = B$, where $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$, $a, b \in \mathbb{N}$ are fixed numbers satisfying $b \geq a+2$, and $A, B \in \mathbb{R}$ are satisfying $B > A$, $\mathbb{T} = \{a+1, \dots, b-1\}$, $\widehat{\mathbb{T}} = \{a, a+1, \dots, b-1, b\}$.

1. Introduction

Let $a, b \in \mathbb{N}$, $a+2 \leq b$, $\mathbb{T} = \{a+1, \dots, b-1\}$, $\widehat{\mathbb{T}} = \{a, a+1, \dots, b-1, b\}$. Let

$$\widehat{\mathbb{E}} := \{u \mid u : \widehat{\mathbb{T}} \longrightarrow \mathbb{R}\}, \tag{1.1}$$

and for $u \in \widehat{\mathbb{E}}$, let

$$\|u\|_{\widehat{\mathbb{E}}} = \max_{k \in \widehat{\mathbb{T}}} |u(k)|. \tag{1.2}$$

Let

$$\mathbb{E} := \{u \mid u : \mathbb{T} \longrightarrow \mathbb{R}\}, \tag{1.3}$$

and for $u \in \mathbb{E}$, let

$$\|u\|_{\mathbb{E}} = \max_{k \in \mathbb{T}} |u(k)|. \tag{1.4}$$

It is clear that the above are norms on $\widehat{\mathbb{E}}$ and $\mathbb{E}$, respectively, and that the finite dimensionality of these spaces makes them Banach spaces.

In this paper, we discuss the nonlinear second-order discrete problems with minimum and maximum:

$$\Delta^2 u(k-1) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}, \tag{1.5}$$

$$\min\{u(k) : k \in \widehat{\mathbb{T}}\} = A, \qquad \max\{u(k) : k \in \widehat{\mathbb{T}}\} = B, \tag{1.6}$$

where $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $a, b \in \mathbb{N}$ are fixed numbers satisfying $b \geq a + 2$ and $A, B \in \mathbb{R}$ satisfying $B > A$.

Functional boundary value problem has been studied by several authors [1–7]. But most of the papers studied the differential equations functional boundary value problem [1–6]. As we know, the study of difference equations represents a very important field in mathematical research [8–12], so it is necessary to investigate the corresponding difference equations with nonlinear boundary conditions.

Our ideas arise from [1, 3]. In 1993, Brykalov [1] discussed the existence of two different solutions to the nonlinear differential equation with nonlinear boundary conditions

$$\begin{gathered} x'' = h(t, x, x'), \quad t \in [a, b], \\ \min\{u(t) : t \in [a, b]\} = A, \qquad \max\{u(t) : t \in [a, b]\} = B, \end{gathered} \tag{1.7}$$

where h is a bounded function, that is, there exists a constant $M > 0$, such that $|h(t, x, x')| \leq M$. The proofs in [1] are based on the technique of monotone boundary conditions developed in [2]. From [1, 2], it is clear that the results of [1] are valid for functional differential equations in general form and for some cases of unbounded right-hand side of the equation (see [1, Remark 3 and (5)], [2, Remark 2 and (8)]).

In 1998, Staněk [3] worked on the existence of two different solutions to the nonlinear differential equation with nonlinear boundary conditions

$$\begin{gathered} x''(t) = (Fx)(t), \quad \text{a.e. } t \in [0, 1], \\ \min\{u(t) : t \in [a, b]\} = A, \qquad \max\{u(t) : t \in [a, b]\} = B, \end{gathered} \tag{1.8}$$

where F satisfies the condition that there exists a nondecreasing function $f : [0, \infty) \to (0, \infty)$ satisfying $\int_0^\infty (ds/f(s)) \geq b - a$, $\int_0^\infty (s/f(s))ds = \infty$, such that

$$|(Fu)(t)| \leq f(|u'(t)|). \tag{1.9}$$

It is not difficult to see that when we take $F(u(t)) = h(t, u, u')$, (1.8) is to be (1.7), and F may not be bounded.

But as far as we know, there have been no discussions about the discrete problems with minimum and maximum in literature. So, we use the Borsuk theorem [13] to discuss the existence of two different solutions to the second-order difference equation boundary value problem (1.5), (1.6) when f satisfies

$(H1)$ $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and there exist $p : \mathbb{T} \to \mathbb{R}$, $q : \mathbb{T} \to \mathbb{R}$, $r : \mathbb{T} \to \mathbb{R}$, such that

$$|f(k, u, v)| \leq p(k)|u| + q(k)|v| + r(k), \quad (k, u, v) \in \mathbb{T} \times \mathbb{R}^2, \tag{1.10}$$

where $\Gamma := 1 - (b - a)\sum_{i=a+1}^{b-1}|p(i)| - \sum_{i=a+1}^{b-1}|q(i)| > 0$.

In our paper, we assume $\sum_{s=k}^{l} u(s) = 0$, if $l < k$.

2. Preliminaries

Definition 2.1. Let $\gamma : \widehat{\mathbb{E}} \to \mathbb{R}$ be a functional. γ is increasing if

$$x, y \in \widehat{\mathbb{E}} : x(k) < y(k), \quad \text{for } k \in \widehat{\mathbb{T}} \Longrightarrow \gamma(x) < \gamma(y). \tag{2.1}$$

Set

$$\mathscr{A} = \{\gamma \mid \gamma : \widehat{\mathbb{E}} \longrightarrow \mathbb{R} \text{ is continuous and increasing}\}, \qquad \mathscr{A}_0 = \{\gamma \mid \gamma \in \mathscr{A}, \gamma(0) = 0\}. \tag{2.2}$$

Remark 2.2. Obviously, $\min\{u(k) : k \in \widehat{\mathbb{T}}\}$, $\max\{u(k) : k \in \widehat{\mathbb{T}}\}$ belong to $\mathscr{A}_0$. Now, if we take

$$C = B - A, \qquad \omega(u) = \min\{u(k) : k \in \widehat{\mathbb{T}}\}, \tag{2.3}$$

then boundary condition (1.6) is equal to

$$\omega(u) = A, \qquad \max\{u(k) : k \in \widehat{\mathbb{T}}\} - \min\{u(k) : k \in \widehat{\mathbb{T}}\} = C. \tag{2.4}$$

So, in the rest part of this paper, we only deal with BVP (1.5), (2.4).

Lemma 2.3. *Suppose $c, d \in \mathbb{N}$, $c < d$, $u = (u(c), u(c+1), \ldots, u(d))$. If there exist $\eta_1, \eta_2 \in \{c, c+1, \ldots, d-1, d\}$, $\eta_1 < \eta_2$, such that $u(\eta_1)u(\eta_2) \le 0$, then*

$$\begin{aligned}
|u(k)| &\le (d-c) \max_{k \in \{c, \ldots, \eta_2 - 1\}} |\Delta u(k)|, \quad k \in \{c, \ldots, \eta_1\}, \\
|u(k)| &\le (d-c) \max_{k \in \{\eta_1, \ldots, \eta_2 - 1\}} |\Delta u(k)|, \quad k \in \{\eta_1 + 1, \ldots, \eta_2\}, \\
|u(k)| &\le (d-c) \max_{k \in \{\eta_1, \ldots, d-1\}} |\Delta u(k)|, \quad k \in \{\eta_2 + 1, \ldots, d\}.
\end{aligned} \tag{2.5}$$

Furthermore, one has

$$\max_{k \in \{c, \ldots, d\}} |u(k)| \le (d-c) \max_{k \in \{c, \ldots, d-1\}} |\Delta u(k)|. \tag{2.6}$$

Proof. Without loss of generality, we suppose $u(\eta_1) \le 0 \le u(\eta_2)$.

(i) For $k \le \eta_1 < \eta_2$, we have

$$u(k) = u(\eta_1) - \sum_{i=k}^{\eta_1 - 1} \Delta u(i), \qquad u(k) = u(\eta_2) - \sum_{i=k}^{\eta_2 - 1} \Delta u(i). \tag{2.7}$$

Then

$$-\sum_{i=k}^{\eta_2 - 1} \Delta u(i) \le u(k) \le -\sum_{i=k}^{\eta_1 - 1} \Delta u(i). \tag{2.8}$$

Furthermore,

$$|u(k)| \le \max\left\{\left|\sum_{i=k}^{\eta_2 - 1} \Delta u(i)\right|, \left|\sum_{i=k}^{\eta_1 - 1} \Delta u(i)\right|\right\}, \tag{2.9}$$

which implies

$$|u(k)| \le (d-c) \max_{k \in \{c, \ldots, \eta_2 - 1\}} |\Delta u(k)|. \tag{2.10}$$

(ii) For $\eta_1 < k \leq \eta_2$, we get

$$u(k) = u(\eta_1) + \sum_{i=\eta_1}^{k-1} \Delta u(i), \qquad u(k) = u(\eta_2) - \sum_{i=k}^{\eta_2-1} \Delta u(i). \tag{2.11}$$

Then

$$-\sum_{i=k}^{\eta_2-1} \Delta u(i) \leq u(k) \leq \sum_{i=\eta_1}^{k-1} \Delta u(i). \tag{2.12}$$

Furthermore,

$$|u(k)| \leq \max\left\{ \left|\sum_{i=k}^{\eta_2-1} \Delta u(i)\right|, \left|\sum_{i=\eta_1}^{k-1} \Delta u(i)\right| \right\}, \tag{2.13}$$

which implies

$$|u(k)| \leq (d-c) \max_{k \in \{\eta_1, \ldots, \eta_2-1\}} |\Delta u(k)|. \tag{2.14}$$

(iii) For $\eta_1 < \eta_2 < k$, we have

$$u(k) = u(\eta_1) + \sum_{i=\eta_1}^{k-1} \Delta u(i), \qquad u(k) = u(\eta_2) + \sum_{i=\eta_2}^{k-1} \Delta u(i). \tag{2.15}$$

Then

$$\sum_{i=\eta_2}^{k-1} \Delta u(i) \leq u(k) \leq \sum_{i=\eta_1}^{k-1} \Delta u(i). \tag{2.16}$$

Furthermore,

$$|u(k)| \leq \max\left\{ \left|\sum_{i=\eta_2}^{k-1} \Delta u(i)\right|, \left|\sum_{i=\eta_1}^{k-1} \Delta u(i)\right| \right\}, \tag{2.17}$$

which implies

$$|u(k)| \leq (d-c) \max_{k \in \{\eta_1, \ldots, d-1\}} |\Delta u(k)|. \tag{2.18}$$

In particular, it is not hard to obtain

$$\max_{k \in \{c, \ldots, d\}} |u(k)| \leq (d-c) \max_{k \in \{c, \ldots, d-1\}} |\Delta u(k)|. \tag{2.19}$$

□

Similarly, we can obtain the following lemma.

Lemma 2.4. *Suppose $c, d \in \mathbb{N}$, $c < d$, $u = (u(c), u(c+1), \ldots, u(d))$. If there exists $\eta_1 \in \{c, c+1, \ldots, d-1, d\}$ such that $u(\eta_1) = 0$, then*

$$\begin{aligned} |u(k)| &\le (d-c) \max_{k\in\{c,\ldots,\eta_1-1\}} |\Delta u(k)|, \quad k \in \{c,\ldots,\eta_1\}, \\ |u(k)| &\le (d-c) \max_{k\in\{\eta_1,\ldots,d-1\}} |\Delta u(k)|, \quad k \in \{\eta_1+1,\ldots,d\}. \end{aligned} \tag{2.20}$$

In particular, one has

$$\max_{k\in\{c,\ldots,d\}} |u(k)| \le (d-c) \max_{k\in\{c,\ldots,d-1\}} |\Delta u(k)|. \tag{2.21}$$

Lemma 2.5. *Suppose $\gamma \in \mathcal{A}_0$, $c \in [0,1]$. If $u \in \widehat{\mathbb{E}}$ satisfies*

$$\gamma(u) - c\gamma(-u) = 0, \tag{2.22}$$

then there exist $\xi_0, \xi_1 \in \widehat{\mathbb{T}}$, such that $u(\xi_0) \le 0 \le u(\xi_1)$.

Proof. We only prove that there exists $\xi_0 \in \widehat{\mathbb{T}}$, such that $u(\xi_0) \le 0$, and the other can be proved similarly.

Suppose $u(k) > 0$ for $k \in \widehat{\mathbb{T}}$. Then $\gamma(u) > \gamma(0) = 0$, $\gamma(-u) < \gamma(0) = 0$. Furthermore, $\gamma(u) - c\gamma(-u) > 0$, which contradicts with $\gamma(u) - c\gamma(-u) = 0$. □

Define functional $\phi : (v(a), v(a+1), \ldots, v(b-1)) \to \mathbb{R}$ by

$$\phi(v) = \max\left\{ \sum_{k=c}^{d-1} v(k) : c \le d,\ c, d \in \widehat{\mathbb{T}} \setminus \{b\} \right\}. \tag{2.23}$$

Lemma 2.6. *Suppose $u(k)$ is a solution of* (1.5) *and $\omega(u) = 0$. Then*

$$\min\{\phi(\Delta u), \phi(-\Delta u)\} \le \frac{(b-a)}{2\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{2.24}$$

Proof. Let

$$C_+ = \{k \mid \Delta u(k) > 0, k \in \widehat{\mathbb{T}} \setminus \{b\}\}, \qquad C_- = \{k \mid \Delta u(k) < 0, k \in \widehat{\mathbb{T}} \setminus \{b\}\}, \tag{2.25}$$

and N_{C_+} be the number of elements in C_+, N_{C_-} the number of elements in C_-.

If $C_+ = \varnothing$, then $\phi(\Delta u) = 0$; if $C_- = \varnothing$, then $\phi(-\Delta u) = 0$. Equation (2.24) is obvious.

Now, suppose $C_+ \ne \varnothing$ and $C_- \ne \varnothing$. It is easy to see that

$$\min\{N_{C_+}, N_{C_-}\} \le \frac{b-a}{2}. \tag{2.26}$$

At first, we prove the inequality

$$\phi(\Delta u) \le \frac{N_{C_+}}{\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{2.27}$$

Since $\omega(u) = 0$, by Lemma 2.5, there exist $\xi_1, \xi_2 \in \widehat{\mathbb{T}}$, $\xi_1 \le \xi_2$, such that $u(\xi_1)u(\xi_2) \le 0$. Without loss of generality, we suppose $u(\xi_1) \le 0 \le u(\xi_2)$.

For any $\alpha \in C_+$, there exits β satisfying one of the following cases:

Case 1. $\beta = \min\{k \in \widehat{\mathbb{T}} \setminus \{b\} \mid \Delta u(k) \leq 0, k > \alpha\}$,

Case 2. $\beta = \max\{k \in \widehat{\mathbb{T}} \setminus \{b\} \mid \Delta u(k) \leq 0, k < \alpha\}$.

We only prove that (2.27) holds when Case 1 occurs, (if Case 2 occurs, it can be similarly proved).

If Case 1 holds, we divide the proof into two cases.

Case 1.1. If $u(\alpha)u(\beta) \leq 0$, without loss of generality, we suppose $u(\alpha) \leq 0 \leq u(\beta)$, then by Lemma 2.3, we have

$$\left|u(k)\right| \leq (b-a) \max_{k \in \{\alpha,\dots,\beta-1\}} \left|\Delta u(k)\right|, \quad k \in \{\alpha+1,\dots,\beta\}. \tag{2.28}$$

Combining this with

$$0 \geq u(\alpha) = u(\beta) - \sum_{i=\alpha}^{\beta-1} \Delta u(i) \geq -\sum_{i=\alpha}^{\beta-1} \Delta u(i), \tag{2.29}$$

we have

$$\left|u(k)\right| \leq (b-a) \max_{k \in \{\alpha,\dots,\beta-1\}} \left|\Delta u(k)\right|, \quad k \in \{\alpha,\dots,\beta\}. \tag{2.30}$$

At the same time, for $k \in \{\alpha,\dots,\beta-1\}$, we have $\Delta u(k) > 0$ and

$$\Delta u(k) = \Delta u(\beta) - \sum_{i=k+1}^{\beta} \Delta^2 u(i-1), \qquad \Delta u(k) = \Delta u(\alpha) + \sum_{i=\alpha+1}^{k} \Delta^2 u(i-1). \tag{2.31}$$

For $k = \beta$, we get

$$0 \geq \Delta u(\beta) = \Delta u(\alpha) + \sum_{i=\alpha+1}^{\beta} \Delta^2 u(i-1) \geq \sum_{i=\alpha+1}^{\beta} \Delta^2 u(i-1). \tag{2.32}$$

So, for $k \in \{\alpha,\dots,\beta\}$,

$$\begin{aligned}
\left|\Delta u(k)\right| &\leq \max\left\{ \sum_{i=\alpha+1}^{k} \left|\Delta^2 u(i-1)\right|, \sum_{i=k+1}^{\beta} \left|\Delta^2 u(i-1)\right| \right\} \\
&\leq \sum_{i=\alpha+1}^{\beta} \left|\Delta^2 u(i-1)\right| \\
&= \sum_{i=\alpha+1}^{\beta} \left|f\left(i, u(i), \Delta u(i)\right)\right| \\
&\leq \sum_{i=\alpha+1}^{\beta} \left(\left|p(i)\right|\left|u(i)\right| + \left|q(i)\right|\left|\Delta u(i)\right| + \left|r(i)\right|\right) \\
&\leq \sum_{i=a+1}^{b-1} \left(\left|p(i)\right|(b-a) \max_{k \in \{\alpha,\dots,\beta-1\}} \left|\Delta u(k)\right| + \left|q(i)\right| \max_{k \in \{\alpha,\dots,\beta\}} \left|\Delta u(k)\right| + \left|r(i)\right|\right).
\end{aligned} \tag{2.33}$$

Thus

$$\left|\Delta u(\alpha)\right| \leq \max_{k \in \{\alpha,\dots,\beta\}} \left|\Delta u(k)\right| \leq \frac{1}{\Gamma} \sum_{i=a+1}^{b-1} \left|r(i)\right|. \tag{2.34}$$

Case 1.2 ($u(\alpha)u(\beta) \geq 0$). Without loss of generality, we suppose $u(\alpha) \geq 0,\ u(\beta) \geq 0$. Then ξ_1 will be discussed in different situations.

Case 1.2.1 ($\xi_1 < \alpha \leq \beta$). By Lemma 2.3 (we take $\eta_1 = \xi_1,\ \eta_2 = \alpha,\ d = \beta$), it is not difficult to see that

$$\left|u(k)\right| \leq (b-a) \max_{k\in\{\xi_1,\dots,\beta-1\}} \left|\Delta u(k)\right|, \quad k \in \left\{\xi_1+1,\dots,\beta\right\}. \tag{2.35}$$

For $k = \xi_1$, we have

$$0 \geq u\left(\xi_1\right) = u(\alpha) - \sum_{i=\xi_1}^{\alpha-1} \Delta u(i) \geq -\sum_{i=\xi_1}^{\alpha-1} \Delta u(i). \tag{2.36}$$

So, we get

$$\left|u(k)\right| \leq (b-a) \max_{k\in\{\xi_1,\dots,\beta-1\}} \left|\Delta u(k)\right|, \quad k \in \left\{\xi_1,\dots,\beta\right\}. \tag{2.37}$$

At the same time, for $k \in \{\alpha,\dots,\beta\}$,

$$\Delta u(k) = \Delta u(\beta) - \sum_{i=k+1}^{\beta} \Delta^2 u(i-1), \qquad \Delta u(k) = \Delta u(\alpha) + \sum_{i=\alpha+1}^{k} \Delta^2 u(i-1). \tag{2.38}$$

Combining this with $\Delta u(\beta) \leq 0,\ \Delta u(\alpha) > 0$, we have

$$\begin{aligned}
\left|\Delta u(k)\right| &\leq \max\left\{ \sum_{i=k+1}^{\beta} \left|\Delta^2 u(i-1)\right|, \sum_{i=\alpha+1}^{k} \left|\Delta^2 u(i-1)\right| \right\} \\
&\leq \sum_{i=\alpha+1}^{\beta} \left|\Delta^2 u(i-1)\right| \\
&\leq \sum_{i=\alpha+1}^{\beta} \left(\left|p(i)\right|\left|u(i)\right| + \left|q(i)\right|\left|\Delta u(i)\right| + \left|r(i)\right|\right) \\
&\leq \sum_{i=a+1}^{b-1} \left(\left|p(i)\right|(b-a) \max_{k\in\{\xi_1,\dots,\beta-1\}} \left|\Delta u(k)\right| + \left|q(i)\right| \max_{k\in\{\alpha+1,\dots,\beta\}} \left|\Delta u(k)\right| + \left|r(i)\right|\right),
\end{aligned} \tag{2.39}$$

for $k \in \{\alpha,\dots,\beta\}$.

Also, for $k \in \{\xi_1,\dots,\alpha-1\}$, we have $\Delta u(k) > 0$ and

$$\Delta u(k) = \Delta u(\beta) - \sum_{i=k+1}^{\beta} \Delta^2 u(i-1), \qquad \Delta u(k) = \Delta u(\alpha) - \sum_{i=k+1}^{\alpha} \Delta^2 u(i-1). \tag{2.40}$$

Similarly, we get

$$\left|\Delta u(k)\right| \leq \sum_{i=a+1}^{b-1} \left(\left|p(i)\right|(b-a) \max_{k\in\{\xi_1,\dots,\beta-1\}} \left|\Delta u(k)\right| + \left|q(i)\right| \max_{k\in\{\xi_1+1,\dots,\beta\}} \left|\Delta u(k)\right| + \left|r(i)\right|\right). \tag{2.41}$$

By (2.39) and (2.41), for $k \in \{\xi_1, \dots, \beta\}$,

$$\left|\Delta u(k)\right| \leq \sum_{i=a+1}^{b-1} \left(\left|p(i)\right|(b-a) \max_{k\in\{\xi_1,\dots,\beta\}} \left|\Delta u(k)\right| + \left|q(i)\right| \max_{k\in\{\xi_1,\dots,\beta\}} \left|\Delta u(k)\right| + \left|r(i)\right| \right). \tag{2.42}$$

Then

$$\left|\Delta u(\alpha)\right| \leq \max_{k\in\{\xi_1,\dots,\beta\}} \left|\Delta u(k)\right| \leq \frac{1}{\Gamma} \sum_{i=a+1}^{b-1} \left|r(i)\right|. \tag{2.43}$$

Case 1.2.2 ($\alpha \leq \xi_1 < \beta$). By Lemma 2.3 (we take $c = \alpha,\ \eta_1 = \xi_1,\ \eta_2 = \beta$), it is easy to obtain that

$$\left|u(k)\right| \leq (b-a) \max_{k\in\{\alpha,\dots,\beta-1\}} \left|\Delta u(k)\right|, \quad k \in \{\alpha, \dots, \beta\}. \tag{2.44}$$

At the same time, for $k \in \{\alpha, \dots, \beta\}$,

$$\Delta u(k) = \Delta u(\beta) - \sum_{i=k+1}^{\beta} \Delta^2 u(i-1), \qquad \Delta u(k) = \Delta u(\alpha) + \sum_{i=\alpha+1}^{k} \Delta^2 u(i-1). \tag{2.45}$$

Together with $\Delta u(\beta) \leq 0$, $\Delta u(\alpha) > 0$, we have

$$\begin{aligned}
\left|\Delta u(k)\right| &\leq \max\left\{ \sum_{i=k+1}^{\beta} \left|\Delta^2 u(i-1)\right|, \sum_{i=\alpha+1}^{k} \left|\Delta^2 u(i-1)\right| \right\} \\
&\leq \sum_{i=\alpha+1}^{\beta} \left|\Delta^2 u(i-1)\right| \\
&\leq \sum_{i=\alpha+1}^{\beta} \left(\left|p(i)\right|\left|u(i)\right| + \left|q(i)\right|\left|\Delta u(i)\right| + \left|r(i)\right| \right) \\
&\leq \sum_{i=a+1}^{b-1} \left(\left|p(i)\right|(b-a) \max_{k\in\{\alpha,\dots,\beta-1\}} \left|\Delta u(k)\right| + \left|q(i)\right| \max_{k\in\{\alpha,\dots,\beta\}} \left|\Delta u(k)\right| + \left|r(i)\right| \right).
\end{aligned} \tag{2.46}$$

Thus

$$\left|\Delta u(\alpha)\right| \leq \max_{k\in\{\alpha,\dots,\beta\}} \left|\Delta u(k)\right| \leq \frac{1}{\Gamma} \sum_{i=a+1}^{b-1} \left|r(i)\right|. \tag{2.47}$$

Case 1.2.3 ($\alpha < \beta \leq \xi_1$). Without loss of generality, we suppose $\beta < \xi_1$ (when $\beta = \xi_1$, by Lemma 2.4, it can be proved similarly). Then from Lemma 2.3 (we take $c = \alpha,\ \eta_1 = \beta,\ \eta_2 = \xi_1$), it is not difficult to see that

$$\left|u(k)\right| \leq (b-a) \max_{k\in\{\alpha,\dots,\xi_1-1\}} \left|\Delta u(k)\right|, \quad k \in \left\{\alpha, \dots, \xi_1\right\}. \tag{2.48}$$

For $k \in \{\alpha, \dots, \beta-1\}$, we have

$$\Delta u(k) = \Delta u(\beta) - \sum_{i=k+1}^{\beta} \Delta^2 u(i-1). \tag{2.49}$$

Together with $\Delta u(\beta) \leq 0$ and $\Delta u(k) > 0$, for $k \in \{\alpha, \dots, \beta - 1\}$, we get

$$
\begin{aligned}
|\Delta u(k)| &\leq \sum_{i=k+1}^{\beta} |\Delta^2 u(i-1)| \\
&= \sum_{i=k+1}^{\beta} |f(i, u(i), \Delta u(i))| \\
&\leq \sum_{i=k+1}^{\beta} (|p(i)||u(i)| + |q(i)||\Delta u(i)| + |r(i)|) \\
&\leq \sum_{i=k+1}^{\beta} \Big(|p(i)|(b-a) \max_{k\in\{\alpha,\dots,\xi_1-1\}} |\Delta u(k)| + |q(i)| \max_{k\in\{\alpha,\dots,\xi_1-1\}} |\Delta u(k)| + |r(i)|\Big),
\end{aligned}
\tag{2.50}
$$

for $k \in \{\alpha, \dots, \beta - 1\}$.

Also, for $k \in \{\beta, \dots, \xi_1\}$, we have

$$
\Delta u(k) = \Delta u(\alpha) + \sum_{i=\alpha+1}^{k} \Delta^2 u(i-1), \qquad \Delta u(k) = \Delta u(\beta) + \sum_{i=\beta+1}^{k} \Delta^2 u(i-1). \tag{2.51}
$$

This being combined with $\Delta u(\beta) \leq 0$, $\Delta u(\alpha) > 0$, we get

$$
\begin{aligned}
|\Delta u(k)| &\leq \max\left\{ \sum_{i=\alpha+1}^{k} |\Delta^2 u(i-1)|, \sum_{i=\beta+1}^{k} |\Delta^2 u(i-1)| \right\} \\
&\leq \sum_{i=\alpha+1}^{\xi_1} |\Delta^2 u(i-1)| \\
&\leq \sum_{i=\alpha+1}^{\xi_1} \Big(|p(i)| \max_{k\in\{\alpha,\dots,\xi_1-1\}} |\Delta u(k)| + |q(i)| \max_{k\in\{\alpha,\dots,\xi_1\}} |\Delta u(k)| + |r(i)|\Big).
\end{aligned}
\tag{2.52}
$$

From (2.50) and (2.52),

$$
|\Delta u(\alpha)| \leq \max_{k\in\{\alpha,\dots,\xi_1\}} |\Delta u(k)| \leq \frac{1}{\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{2.53}
$$

At last, from Case 1 and Case 2, we obtain

$$
\Delta u(k) \leq \frac{1}{\Gamma} \sum_{i=a+1}^{b-1} |r(i)|, \quad k \in C_+. \tag{2.54}
$$

Then by the definition of ϕ and (2.54),

$$
\phi(\Delta u) \leq \sum_{k\in C_+} \Delta u(k) \leq \frac{\sum_{i=a+1}^{b-1} |r(i)|}{\Gamma} \sum_{k\in C_+} \leq \frac{N_{C_+}}{\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{2.55}
$$

Similarly, we can prove

$$\phi(-\Delta u) \le \frac{N_{C_-}}{\Gamma} \sum_{i=a+1}^{b-1} \left| r(i) \right|. \tag{2.56}$$

From (2.26), (2.55), and (2.56), the assertion is proved. □

Remark 2.7. It is easy to see that ϕ is continuous, and

$$\max\{u(k) : k \in \widehat{\mathbb{T}}\} - \min\{u(k) : k \in \widehat{\mathbb{T}}\} = \max\{\phi(\Delta u), \phi(-\Delta u)\}. \tag{2.57}$$

Lemma 2.8. *Let C be a positive constant as in* (2.3), *ω as in* (2.3), *ϕ as in* (2.23). *Set*

$$\begin{aligned}\Omega = \{&(u,\alpha,\beta) \mid (u,\alpha,\beta) \in \widehat{\mathbb{E}} \times \mathbb{R}^2, \|u\|_{\widehat{\mathbb{E}}} < (C+1)(b-a), \\ &|\alpha| < (C+1)(b-a), |\beta| < C+1\}.\end{aligned} \tag{2.58}$$

Define $\Gamma_i : \overline{\Omega} \to \widehat{\mathbb{E}} \times \mathbb{R}^2$ $(i = 1,2)$:

$$\begin{aligned}\Gamma_1(u,\alpha,\beta) &= (\alpha + \beta(k-a), \alpha + \omega(u), \beta + \phi(\Delta u) - C), \\ \Gamma_2(u,\alpha,\beta) &= (\alpha + \beta(k-a), \alpha + \omega(u), \beta + \phi(-\Delta u) - C).\end{aligned} \tag{2.59}$$

Then

$$D(I - \Gamma_i, \Omega, 0) \ne 0, \quad i = 1,2, \tag{2.60}$$

where D denotes Brouwer degree, and I the identity operator on $\widehat{\mathbb{E}} \times \mathbb{R}^2$.

Proof. Obviously, Ω is a bounded open and symmetric with respect to $\theta \in \Omega$ subset of Banach space $\widehat{\mathbb{E}} \times \mathbb{R}^2$.

Define $H, G : [0,1] \times \overline{\Omega} \to \widehat{\mathbb{E}} \times \mathbb{R}^2$

$$\begin{aligned}H(\lambda,u,\alpha,\beta) = (&\alpha + \beta(k-a), \alpha + \omega(u) - (1-\lambda)\omega(-u), \beta + \phi(\Delta u) \\ &- \phi((\lambda-1)\Delta u) - \lambda C), \\ G(\lambda,u,\alpha,\beta) = (&u,\alpha,\beta) - H(\lambda,u,\alpha,\beta).\end{aligned} \tag{2.61}$$

For $(u,\alpha,\beta) \in \overline{\Omega}$,

$$\begin{aligned}G(1,u,\alpha,\beta) &= (u,\alpha,\beta) - (\alpha + \beta(k-a), \alpha + \omega(u), \beta + \phi(\Delta u) - C) \\ &= (I - \Gamma_1)(u,\alpha,\beta).\end{aligned} \tag{2.62}$$

By Borsuk theorem, to prove $D(I - \Gamma_1, \Omega, 0) \ne 0$, we only need to prove that the following hypothesis holds.

(a) $G(0,\cdot,\cdot,\cdot)$ is an odd operator on $\overline{\Omega}$, that is,

$$G(0,-u,-\alpha,-\beta) = -G(0,u,\alpha,\beta), \quad (u,\alpha,\beta) \in \overline{\Omega}; \tag{2.63}$$

(b) H is a completely continuous operator;

(c) $G(\lambda, u, \alpha, \beta) \neq 0$ for $(\lambda, u, \alpha, \beta) \in [0,1] \times \partial\Omega$.

First, we take $(u, \alpha, \beta) \in \overline{\Omega}$, then

$$\begin{aligned} G(0, -u, -\alpha, -\beta) &= (-u, -\alpha, -\beta) - \big(-\alpha - \beta(k-a), -\alpha + \omega(-u) - \omega(u), -\beta + \phi(-\Delta u) - \phi(\Delta u)\big) \\ &= -\big((u, \alpha, \beta) - (\alpha + \beta(k-a), \alpha + \omega(u) - \omega(-u), \beta + \phi(\Delta u) - \phi(-\Delta u))\big) \\ &= -G(0, u, \alpha, \beta). \end{aligned} \tag{2.64}$$

Thus (a) is asserted.

Second, we prove (b).

Let $(\lambda_n, u_n, \alpha_n, \beta_n) \subset [0,1] \times \overline{\Omega}$ be a sequence. Then for each $n \in \mathbb{Z}^+$ and the fact $k \in \widehat{\mathbb{T}}$, $|\lambda_n| \leq 1$, $|\alpha_n| \leq (C+1)(b-a)$, $|\beta_n| \leq C+1$, $\|u\|_{\widehat{\mathbb{E}}} \leq (C+1)(b-a)$. The Bolzano-Weiestrass theorem and $\widehat{\mathbb{E}}$ is finite dimensional show that, going if necessary to subsequences, we can assume $\lim_{n\to\infty}\lambda_n = \lambda_0$, $\lim_{n\to\infty}\alpha_n = \alpha_0$, $\lim_{n\to\infty}\beta_n = \beta_0$, $\lim_{n\to\infty}u_n = u$. Then

$$\begin{aligned} \lim_{n\to\infty} H(\lambda_n, u_n, \alpha_n, \beta_n) &= \lim_{n\to\infty} \big(\alpha_n + \beta_n(k-a), \lambda_n + \omega(u_n) - (1-\lambda_n)\omega(-u_n), \\ &\qquad \beta_n + \phi(\Delta u_n) - \phi((\lambda_n - 1)(\Delta u_n)) - \lambda_n C\big) \\ &= \big(\alpha_0 + \beta_0(k-a), \lambda_0 + \omega(u) - (1-\lambda_0)\omega(-u), \\ &\qquad \beta_0 + \phi(\Delta u) - \phi((\lambda_0 - 1)(\Delta u)) - \lambda_0 C\big). \end{aligned} \tag{2.65}$$

Since ω and ϕ are continuous, H is a continuous operator. Then H is a completely continuous operator.

At last, we prove (c).

Assume, on the contrary, that

$$H(\lambda_0, u_0, \alpha_0, \beta_0) = (u_0, \alpha_0, \beta_0), \tag{2.66}$$

for some $(\lambda_0, u_0, \alpha_0, \beta_0) \in [0,1] \times \partial\Omega$. Then

$$\alpha_0 + \beta_0(k-a) = u_0(k), \quad k \in \widehat{\mathbb{T}}, \tag{2.67}$$

$$\omega(u_0) - (1-\lambda_0)\omega(-u_0) = 0, \tag{2.68}$$

$$\phi(\Delta u_0) - \phi((\lambda_0 - 1)\Delta u_0) = \lambda_0 C. \tag{2.69}$$

By (2.67) and Lemma 2.5 (take $u = u_0$, $c = 1 - \lambda_0$), there exists $\xi \in \widehat{\mathbb{T}}$, such that $u_0(\xi) \leq 0$. Also from (2.67), we have $u_0(\xi) = \alpha_0 + \beta_0(\xi - a)$, then we get

$$u_0(k) = u_0(\xi) + \beta_0(k - \xi), \tag{2.70}$$

$$u_0(k) \leq \beta_0(k - \xi), \quad k \in \widehat{\mathbb{T}}. \tag{2.71}$$

Case 1. If $\beta_0 = 0$, then $u_0(k) \le 0$. Now, we claim $u_0(k) \equiv 0,\ k \in \widehat{\mathbb{T}}$. In fact, $u_0(k) \le 0$ and (2.68) show that there exists $k_0 \in \widehat{\mathbb{T}}$ satisfying $u_0(k_0) = 0$. This being combined with $\Delta u_0(k) = \beta_0 = 0$,

$$u_0(k) \equiv 0, \quad k \in \widehat{\mathbb{T}}. \tag{2.72}$$

So, $\alpha_0 = u_0(a) = 0$, which contradicts with $(u_0, \alpha_0, \beta_0) \in \partial\Omega$.

Case 2. If $\beta_0 > 0$, then from (2.67), $\Delta u_0(k) > 0$, and the definition of ϕ, we have

$$\phi(\Delta u_0) - \phi\big((\lambda_0 - 1)\Delta u_0\big) = \beta_0(b-a). \tag{2.73}$$

Together with (2.69), we get $\phi(\beta_0) = \lambda_0 C$, and

$$\beta_0 = \frac{\lambda_0 C}{b-a} < C + 1. \tag{2.74}$$

Furthermore, $\Delta u_0(k) > 0$ shows that $u_0(k)$ is strictly increasing. From (2.68) and Lemma 2.5, there exist $\xi_0, \xi_1 \in \widehat{\mathbb{T}}$ satisfying $u_0(\xi_0) \le 0 \le u_0(\xi_1)$. Thus, $u_0(a) \le 0 \le u_0(b)$. It is not difficult to see that

$$u_0(a) = u_0\big(\xi_1\big) - \sum_{k=a}^{\xi_1 - 1} \Delta u_0(k) \ge -\sum_{k=a}^{\xi_1 - 1} \Delta u_0(k), \tag{2.75}$$

that is,

$$\big|u_0(a)\big| \le \left| -\sum_{k=a}^{\xi_1 - 1} \Delta u_0(k) \right| < (C+1)(b-a). \tag{2.76}$$

Similarly, $|u_0(b)| < (C+1)(b-a)$, then we get $\|u_0\|_{\widehat{\mathbb{E}}} < (C+1)(b-a)$ and $|\alpha_0| = |u_0(a)| < (C+1)(b-a)$, which contradicts with $(u_0, \alpha_0, \beta_0) \in \partial\Omega$.

Case 3. If $\beta_0 < 0$, then from (2.67), we get $\Delta u_0(k) = \beta_0 < 0$ and

$$\phi(\Delta u_0) - \phi\big((\lambda_0 - 1)\Delta u_0\big) = \big(1 - \lambda_0\big)\beta_0(b-a). \tag{2.77}$$

By (2.69), we have

$$\big(1 - \lambda_0\big)\beta_0(b-a) = \lambda_0 C. \tag{2.78}$$

If $\lambda_0 = 0$, then $\beta_0(b-a) = 0$. Furthermore, $\beta_0 = 0$, which contradicts with $\beta_0 < 0$.

If $\lambda_0 = 1$, then $\lambda_0 C = 0$. Furthermore, $C = 0$, which contradicts with $C > 0$.

If $\lambda \in (0,1)$, then $(1-\lambda_0)\beta_0(b-a) < 0,\ \lambda_0 C > 0,$ a contradiction.

Then (c) is proved.

From the above discussion, the conditions of Borsuk theorem are satisfied. Then, we get

$$D(I - \Gamma_1, \Omega, 0) \ne 0. \tag{2.79}$$

Set

$$\begin{aligned} H(\lambda, u, \alpha, \beta) = \big(&\alpha + \beta(k-a), \alpha + \omega(u) - (1-\lambda)\omega(-u), \\ &\beta + \phi(-\Delta u) - \phi\big((1-\lambda)\Delta u\big) - \lambda C\big). \end{aligned} \tag{2.80}$$

Similarly, we can prove

$$D(I - \Gamma_2, \Omega, 0) \ne 0. \tag{2.81}$$

□

3. The main results

Theorem 3.1. *Suppose* $(H1)$ *holds. Then* (1.5) *and* (1.6) *have at least two different solutions when* $A = 0$ *and*

$$C > \frac{b-a}{2\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{3.1}$$

Proof. Let $A = 0,\ C > ((b-a)/2\Gamma)\sum_{i=a+1}^{b-1}|r(i)|$. Consider the boundary conditions

$$\omega(u) = 0, \qquad \phi(\Delta u) = C, \tag{3.2}$$

$$\omega(u) = 0, \qquad \phi(-\Delta u) = C. \tag{3.3}$$

Suppose $u(k)$ is a solution of (1.5). Then from Remark 2.7,

$$\max\{u(k) : k \in \widehat{\mathbb{T}}\} - \min\{u(k) : k \in \widehat{\mathbb{T}}\} = \max\{\phi(\Delta u), \phi(-\Delta u)\}. \tag{3.4}$$

Now, if (1.5) and (3.2) have a solution $u_1(k)$, then Lemma 2.6 and (3.2) show that $\phi(-\Delta u_1) < C$ and

$$\max\{u_1(k) : k \in \widehat{\mathbb{T}}\} - \min\{u_1(k) : k \in \widehat{\mathbb{T}}\} = C. \tag{3.5}$$

So, $u_1(k)$ is a solution of (1.5) and (2.4), that is, $u_1(k)$ is a solution of (1.5) and (1.6).

Similarly, if (1.5), (3.3) have a solution $u_2(k)$, then $\phi(\Delta u_2) < C$ and

$$\max\{u_2(k) : k \in \widehat{\mathbb{T}}\} - \min\{u_2(k) : k \in \widehat{\mathbb{T}}\} = C. \tag{3.6}$$

So, $u_2(k)$ is a solution of (1.5) and (2.4).

Furthermore, since $\phi(\Delta u_1) = C$ and $\phi(\Delta u_2) < C,\ u_1 \neq u_2$.

Next, we need to prove BVPs (1.5), (3.2), and (1.5) and (3.3) have solutions, respectively.

Set

$$\begin{aligned}\Omega = \{(u,\alpha,\beta) \mid (u,\alpha,\beta) \in \widehat{\mathbb{E}} \times \mathbb{R}^2, \|u\|_{\widehat{\mathbb{E}}} < (C+1)(b-a),\\ |\alpha| < (C+1)(b-a), |\beta| < C+1\}.\end{aligned} \tag{3.7}$$

Define operator $S_1 : [0,1] \times \overline{\Omega} \to \widehat{\mathbb{E}} \times \mathbb{R}^2$,

$$S_1(\lambda,u,\alpha,\beta) = \left(\alpha + \beta(k-a) + \lambda\sum_{i=a}^{k-1}\sum_{l=a+1}^{i} f(l,u(l),\Delta u(l)), \alpha + \omega(u), \beta + \phi(\Delta u) - C\right). \tag{3.8}$$

Obviously,

$$S_1(0,u,\alpha,\beta) = \Gamma_1(u,\alpha,\beta), \quad (u,\alpha,\beta) \in \overline{\Omega}. \tag{3.9}$$

Consider the parameter equation

$$S_1(\lambda,u,\alpha,\beta) = (u,\alpha,\beta), \quad \lambda \in [0,1]. \tag{3.10}$$

Now, we prove (3.10) has a solution, when $\lambda = 1$.

By Lemma 2.8, $D(I-\Gamma_1,\Omega,0)\neq 0$. Now we prove the following hypothesis.

(a) $S_1(\lambda,u,\alpha,\beta)$ is a completely continuous operator;

(b)

$$S_1(\lambda,u,\alpha,\beta)\neq(u,\alpha,\beta),\qquad (\lambda,u,\alpha,\beta)\in[0,1]\times\partial\Omega. \tag{3.11}$$

Since $\widehat{\mathbb{E}}$ is finite dimensional, $S_1(\lambda,u,\alpha,\beta)$ is a completely continuous operator. Suppose (b) is not true. Then,

$$S_1(\lambda_0,u_0,\alpha_0,\beta_0)=(u_0,\alpha_0,\beta_0), \tag{3.12}$$

for some $(\lambda_0,u_0,\alpha_0,\beta_0)\in[0,1]\times\partial\Omega$. Then

$$u_0(k)=\alpha_0+\beta_0(k-a)+\lambda_0\sum_{i=a}^{k-1}\sum_{l=a+1}^{i}f(l,u(l),\Delta u(l)), \tag{3.13}$$

$$\omega(u_0)=0, \tag{3.14}$$

$$\phi(\Delta u_0)=C. \tag{3.15}$$

From (3.13), $u_0(k)$ is a solution of second-order difference equation $\Delta^2u(k-1)=\lambda_0f(k,u(k),\Delta u(k))$. By Remark 2.7, $\max_{k\in\widehat{\mathbb{T}}\setminus\{b\}}|\Delta u_0(k)|\leq C<C+1$. And from (3.14), there exist $\xi_0,\xi_1\in\widehat{\mathbb{T}}$, such that $u_0(\xi_0)\leq 0\leq u_0(\xi_1)$. Now, we can prove it in two cases.

Case 1. If there exists $\xi\in\widehat{\mathbb{T}}$, such that $u_0(\xi)=0$, then

(i) for all $k\in\{k,\ldots,\xi\}$,

$$|u_0(k)|=\left|u_0(\xi)-\sum_{i=k}^{\xi-1}\Delta u_0(i)\right|\leq\sum_{i=k}^{\xi-1}|\Delta u_0(i)|<(C+1)(b-a). \tag{3.16}$$

(ii) For all $k\in\{\xi+1,\ldots,b\}$,

$$|u_0(k)|=\left|u_0(\xi)+\sum_{i=\xi}^{k-1}\Delta u_0(i)\right|\leq\sum_{i=\xi}^{k-1}|\Delta u_0(i)|<(C+1)(b-a). \tag{3.17}$$

Case 2. If $\forall k\in\widehat{\mathbb{T}}$, $u_0(k)\neq 0$. Set

$$C_+=\{k\mid u_0(k)>0,\ k\in\widehat{\mathbb{T}}\},\qquad C_-=\{k\mid u_0(k)<0,\ k\in\widehat{\mathbb{T}}\},$$
$$k_0=\max C_+,\qquad k_1=\min C_-. \tag{3.18}$$

(i) For $k\in C_+$, if $k<k_1$, then

$$u_0(k)=u_0(k_1)-\sum_{i=k}^{k_1-1}\Delta u_0(i)<-\sum_{i=k}^{k_1-1}\Delta u_0(i), \tag{3.19}$$

that is,

$$|u_0(k)|<\sum_{i=k}^{k_1-1}|\Delta u_0(i)|<(C+1)(b-a). \tag{3.20}$$

For $k > k_1$,

$$u_0(k) = u_0(k_1) + \sum_{i=k_1}^{k-1} \Delta u_0(i) < \sum_{i=k_1}^{k-1} \Delta u_0(i), \tag{3.21}$$

then

$$|u_0(k)| < (C+1)(b-a). \tag{3.22}$$

(ii) Similarly, we can prove $|u_0(k)| < (C+1)(b-a)$ for $k \in C_-$.

Combining Case 1 with Case 2, we get

$$|u_0(k)| < (C+1)(b-a), \quad k \in \widehat{\mathbb{T}}. \tag{3.23}$$

Moreover, $\alpha_0 = u_0(a)$, $\beta_0 = \Delta u_0(a)$, so,

$$|\alpha_0| \le \|u_0\|_{\widehat{\mathbb{E}}} < (C+1)(b-a), \qquad |\beta_0| < C+1, \tag{3.24}$$

which contradicts with $(u_0, \alpha_0, \beta_0) \in \partial\Omega$.

Similarly, consider the operator $S_2 : [0,1] \times \overline{\Omega} \to \widehat{\mathbb{E}} \times \mathbb{R}^2$,

$$S_2(\lambda, u, \alpha, \beta) = \left(\alpha + \beta(k-a) + \sum_{i=a}^{k-1} \sum_{l=a+1}^{i} f(l, u(l), \Delta u(l)), \alpha + \omega(u), \beta + \phi(-\Delta u) - C \right), \tag{3.25}$$

we can obtain a solution of BVP (1.5) and (3.3). □

Theorem 3.2. *Suppose* $(H1)$ *holds. Then* (1.5) *and* (1.6) *have at least two different solutions when* $A, B \in \mathbb{R}$ *and*

$$C > \frac{b-a}{2\Gamma} \sum_{i=a+1}^{b-1} |r(i)|. \tag{3.26}$$

Proof. Obviously, $\omega(A) = A$. Set

$$\widetilde{\omega}(u) = \omega(u + A) - A. \tag{3.27}$$

Then $\widetilde{\omega}(0) = 0$. Define continuous function $f_1 : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$,

$$f_1(k, u(k), \Delta u(k)) = f(k, v(k), \Delta v(k)), \quad v(k) = u(k) + A. \tag{3.28}$$

Then

$$\begin{aligned} |f_1(k, u(k), \Delta u(k))| &= |f(k, v(k), \Delta v(k))| \\ &\le p(k)|v(k)| + q(k)|\Delta v(k)| + r(k) \\ &\le p(k)|u(k)| + q(k)|\Delta u(k)| + r(k) + p(k)A. \end{aligned} \tag{3.29}$$

Set $\widetilde{r}(k) = r(k) + p(k)A$. Then f_1 satisfies $(H1)$.

By Theorem 3.1,

$$\Delta^2 u(k-1) = f_1(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}, \tag{3.30}$$

$$\widetilde{\omega}(u) = 0, \qquad \max\{u(k) : k \in \widehat{\mathbb{T}}\} - \min\{u(k) : k \in \widehat{\mathbb{T}}\} = B - A := C \tag{3.31}$$

have at least two difference solutions $u_1(k)$, $u_2(k)$. Since $u(k)$ is a solution of (3.30), if and only if $u(k) + A$ is a solution of (1.5), we see that

$$u_i(k) = \widetilde{u}_i(k) + A, \quad i = 1, 2 \tag{3.32}$$

are two different solutions of (1.5) and (2.4), then $u_i(k)$ are the two different solutions of (1.5) and (1.6). □

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