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Research Article

WKB Estimates for 2 × 2 Linear Dynamic Systems on Time Scales

Gro Hovhannisyan

Kent State University, Stark Campus, 6000 Frank Avenue NW, Canton, OH 44720-7599, USA

Correspondence should be addressed to Gro Hovhannisyan, ghovhann@kent.edu

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We establish WKB estimates for 2 × 2 linear dynamic systems with a small parameter ε on a time scale unifying continuous and discrete WKB method. We introduce an adiabatic invariant for 2 × 2 dynamic system on a time scale, which is a generalization of adiabatic invariant of Lorentz's pendulum. As an application we prove that the change of adiabatic invariant is vanishing as ε approaches zero. This result was known before only for a continuous time scale. We show that it is true for the discrete scale only for the appropriate choice of graininess depending on a parameter ε. The proof is based on the truncation of WKB series and WKB estimates.

1. Adiabatic invariant of dynamic systems on time scales

Consider the following system with a small parameter $\varepsilon > 0$ on a time scale:

$$v^{\Delta}(t) = A(t\varepsilon)v(t), \tag{1.1}$$

where v^{Δ} is the delta derivative, $v(t)$ is a 2-vector function, and

$$A(t\varepsilon) = A(\tau) = \begin{pmatrix} a_{11}(\tau) & \varepsilon^{k}a_{12}(\tau) \\ \varepsilon^{-k}a_{21}(\tau) & a_{22}(\tau) \end{pmatrix}, \quad \tau = t\varepsilon,\ k \text{ is an integer.} \tag{1.2}$$

WKB method [1, 2] is a powerful method of the description of behavior of solutions of (1.1) by using asymptotic expansions. It was developed by Carlini (1817), Liouville, Green (1837) and became very useful in the development of quantum mechanics in 1920 [1, 3]. The discrete WKB approximation was introduced and developed in [4–8].

The calculus of times scales was initiated by Aulbach and Hilger [9–11] to unify the discrete and continuous analysis.

In this paper, we are developing WKB approximations for the linear dynamic systems on a time scale to unify the discrete and continuous WKB theory. Our formulas for WKB series are based on the representation of fundamental solutions of dynamic system (1.1) given in [12]. Note that the WKB estimate (see (2.21) below) has double asymptotical character and it shows that the error could be made small by either $\varepsilon \to 0$, or $t \to \infty$.

It is well known [13, 14] that the change of adiabatic invariant of harmonic oscillator is vanishing with the exponential speed as ε approaches zero, if the frequency is an analytic function.

In this paper, we prove that for the discrete harmonic oscillator (even for a harmonic oscillator on a time scale) the change of adiabatic invariant approaches zero with the power speed when the graininess depends on a parameter ε in a special way.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. If $\mathbb{T}$ has a left-scattered minimum m, then $\mathbb{T}^k = \mathbb{T} - m$, otherwise $\mathbb{T}^k = \mathbb{T}$. Here we consider the time scales with $t \geq t_0$, and $\sup \mathbb{T} = \infty$.

For $t \in \mathbb{T}$, we define forward jump operator

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}. \tag{1.3}$$

The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{1.4}$$

If $\sigma(t) > t$, we say that t is right scattered. If $t < \infty$ and $\sigma(t) = t$, then t is called right dense.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ define the delta (see [10, 11]) derivative $f^{\Delta}(t)$ to be the number (provided it exists) with the property that for given any $\epsilon > 0$, there exist a $\delta > 0$ and a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \tag{1.5}$$

for all $s \in U$.

For any positive ε define auxilliary "slow" time scales

$$T_{\varepsilon} = \{\varepsilon t = \tau,\ t \in \mathbb{T}\} \tag{1.6}$$

with forward jump operator and graininess function

$$\sigma_1(\tau) = \inf\{s\varepsilon \in \mathbb{T}_{\varepsilon},\ s\varepsilon > \tau\}, \quad \mu_1(\tau) = \varepsilon\mu(t), \quad \tau = t\varepsilon. \tag{1.7}$$

Further frequently we are suppressing dependence on $\tau = t\varepsilon$ or t. To distinguish the differentiation by t or τ we show the argument of differentiation in parenthesizes: $f^{\Delta}(t) = f^{\Delta_t}(t)$ or $f^{\Delta}(\tau) = f^{\Delta_\tau}(\tau)$.

Assuming $A, \theta_j \in C^1_{\mathrm{rd}}$ (see [10] for the definition of rd-differentiable function), denote

$$\begin{gathered} \operatorname{Tr} A(\tau) = a_{11}(\tau) + a_{22}(\tau), \quad \det A(\tau) = a_{11}(\tau)a_{22}(\tau) - a_{12}(\tau)a_{21}(\tau), \\ \lambda(\tau) = \frac{\sqrt{[\operatorname{Tr} A(\tau)]^2 - 4|A(\tau)|}}{2a_{12}(\tau)}, \end{gathered} \tag{1.8}$$

$$\operatorname{Hov}_j(t) = \theta_j^2(t) - \theta_j(t)\operatorname{Tr} A(\tau) + \det A(\tau) - \varepsilon a_{12}(\tau)(1 + \mu\theta_j)\left(\frac{a_{11} - \theta_j}{a_{12}}\right)^{\Delta}(\tau), \tag{1.9}$$

$$Q_0(\tau) = \frac{\operatorname{Hov}_1 - \operatorname{Hov}_2}{\theta_1 - \theta_2}, \qquad Q_1(\tau) = \frac{(\theta_1 - a_{11})\operatorname{Hov}_2 - (\theta_2 - a_{11})\operatorname{Hov}_1}{a_{12}(\theta_1 - \theta_2)}, \tag{1.10}$$

$$K(\tau) = 2\mu(t)\max_{j=1,2}\left[\left(1 + \left|\frac{e_j}{e_{3-j}}\right|\right)\left(\left|\frac{2\operatorname{Hov}_j}{\theta_1 - \theta_2}\right| + \left|\frac{\varepsilon a_{12}(1 + \mu\theta_j)}{\theta_1 - \theta_2}\left(\frac{a_{11} - \theta_j}{a_{12}}\right)^{\Delta}(\tau)\right|\right) + |\theta_j|\right], \tag{1.11}$$

where $j = 1, 2, \theta_{1,2}(t)$ are unknown phase functions, $\|\cdot\|$ is the Euclidean matrix norm, and $\{e_j(t)\}_{j=1,2}$ are the exponential functions on a time scale [10, 11]:

$$e_j(t) \equiv e_{\theta_j}(t, t_0) = \exp \int_{t_0}^{t} \lim_{p \searrow \mu(s)} \frac{\log(1 + p\theta_j(s))\Delta s}{p} < \infty, \quad j = 1, 2. \tag{1.12}$$

Using the ratio of Wronskians formula proposed in [15] we introduce a new definition of adiabatic invariant of system (1.1)

$$J(t, \theta, v, \varepsilon) = -\frac{[v_1(t)(\theta_1(t) - a_{11}(\tau) - v_2(t)a_{12}(\tau)][v_1(t)(\theta_2(t) - a_{11}(\tau)) - v_2 a_{12}(\tau)]}{(\theta_1 - \theta_2)^2(t) e_{\theta_1}(t) e_{\theta_2}(t)}, \tag{1.13}$$

Theorem 1.1. *Assume $a_{12}(\tau) \neq 0$, $A, \theta \in C^1_{\mathrm{rd}}(\mathbb{T}_\varepsilon)$, and for some positive number β and any natural number m conditions*

$$|1 + \mu(\mathrm{Tr}A + Q_0) + \mu^2(\det A + \theta_1 Q_0 - \mathrm{Hov}_1)|(\tau) \geq \beta, \quad \forall \tau \in \mathbb{T}_\varepsilon, \tag{1.14}$$

$$K(\tau) \leq \text{const}, \quad \forall \tau \in \mathbb{T}_\varepsilon, \tag{1.15}$$

$$\int_{t\varepsilon}^{\infty} \left(1 + \left|\frac{e_j}{e_{3-j}}\right|\right) \left|\frac{\mathrm{Hov}_j}{\theta_1 - \theta_2}\right|(\tau)\Delta\tau \leq C_0 \varepsilon^{m+1}, \quad j = 1, 2, \tag{1.16}$$

are satisfied, where the positive parameter ε is so small that

$$0 \leq \frac{2C_0(1 + K(\tau))}{\beta}\varepsilon^m \leq 1. \tag{1.17}$$

Then for any solution $v(t)$ of (1.1) and for all $t_1, t_2 \in \mathbb{T}$, the estimate

$$J(v, \varepsilon) \equiv |J(t_1, v, \varepsilon) - J(t_2, v, \varepsilon)| \leq C_3 \varepsilon^m \tag{1.18}$$

is true for some positive constant C_3.

Checking condition (1.16) of Theorem 1.1 is based on the construction of asymptotic solutions in the form of WKB series

$$v(t) = C_1 e_{\theta_1}(t, t_0) + C_2 e_{\theta_2}(t, t_0), \tag{1.19}$$

where $\tau = t\varepsilon$, and

$$\theta_{1,2}(t) = \sum_{j=0}^{\infty} \varepsilon^j \zeta_{j\pm}(\tau), \qquad \theta^{\Delta}_{1,2}(t) = \sum_{k=0}^{\infty} \varepsilon^{k+1} \zeta^{\Delta}_{k\pm}(\tau). \tag{1.20}$$

Here the functions $\zeta_{0+}(\tau)$, $\zeta_{0-}(\tau)$ are defined as

$$\zeta_{0\pm}(\tau) = \frac{\mathrm{Tr}A}{2} \pm a_{12}\lambda, \qquad \zeta_{1\pm}(\tau) = -\frac{1 + \mu\zeta_{0\pm}}{2\lambda}\left(\lambda \mp \frac{a_{11} - a_{22}}{2a_{12}}\right)^{\Delta}(\tau), \tag{1.21}$$

where $\lambda(\tau)$ is defined in (1.8), and $\zeta_{k+}(\tau)$, $\zeta_{k-}(\tau)$, $k = 2,3,\ldots$ are defined by recurrence relations

$$\zeta_{k\pm}(\tau) = \mp\frac{(1+\mu\zeta_{0\pm})}{2\lambda}\left(\frac{\zeta_{k-1\pm}}{a_{12}}\right)^{\Delta}(\tau) + \mp\sum_{j=1}^{k-1}\frac{\zeta_{j\pm}}{2\lambda}\left[\frac{\zeta_{k-j\pm}}{a_{12}} + \mu\left(\frac{\zeta_{k-1-j\pm} - a_{11}\delta_{j,k-1}}{a_{12}}\right)^{\Delta}(\tau)\right], \quad (1.22)$$

δ_{jk} is the Kroneker symbol ($\delta_{jk} = 1$, if $k = j$, and $\delta_{kj} = 0$ otherwise).

Denote

$$Z_1(\tau) = Z(\zeta_{0+}(\tau)), \qquad Z_2(\tau) = Z(\zeta_{0-}(\tau)), \quad (1.23)$$

$$\begin{aligned} Z(\zeta_0) &= a_{12}(1+\mu\zeta_0)\left(\frac{\zeta_m}{a_{12}}\right)^{\Delta} \\ &\quad + \sum_{j=1}^{m}\zeta_j\left[\zeta_{m+1-j} + \varepsilon\zeta_{m+2-j} + a_{12}\mu\left(\frac{\zeta_{m-j} - a_{11}\delta_{j,m} + \varepsilon\zeta_{m+1-j}}{a_{12}}\right)^{\Delta}(\tau)\right]. \end{aligned} \quad (1.24)$$

In the next Theorem 1.2 by truncating series (1.20):

$$\theta_1(t) = \sum_{k=0}^{m}\varepsilon^k\zeta_{k+}, \qquad \theta_2(t) = \sum_{k=0}^{m}\varepsilon^k\zeta_{k-}, \quad (1.25)$$

where $\zeta_{k\pm}(t)$, $k = 1,2,\ldots,m$ are given in (1.21) and (1.22), we deduce estimate (1.16) from condition (1.26) below given directly in the terms of matrix $A(\tau)$.

Theorem 1.2. *Assume that* $a_{12}(\tau)\neq 0$, $A,\theta \in C^1_{\mathrm{rd}}(\mathbb{T}_\varepsilon)$, *and conditions* (1.14), (1.15), (1.17), *and*

$$\int_{t\varepsilon}^{\infty}\left(1 + \left|\frac{e_j}{e_{3-j}}\right|\right)\left|\frac{Z_j(\tau)}{\theta_1 - \theta_2}\right|\Delta\tau \le C_0, \quad j = 1,2, \quad (1.26)$$

are satisfied. Then, estimate (1.18) *is true.*

Note that if $a_{11} = a_{22}$, then formulas (1.21) and (1.22) are simplified:

$$\zeta_{0\pm}(\tau) = a_{11}(\tau) \pm a_{12}\lambda(\tau), \qquad \zeta_{1\pm} = -\frac{(1+\mu\zeta_{0\pm}(\tau))\lambda^{\Delta}(\tau)}{2\lambda(\tau)}, \quad (1.27)$$

where from (1.8)

$$\lambda(\tau) = \frac{\sqrt{a_{12}(\tau)a_{21}(\tau)}}{a_{12}(\tau)}. \quad (1.28)$$

Taking $m = 1$ in (1.25) and $\zeta_{0\pm}(t)$, $\zeta_{1\pm}(t)$ as in (1.21), we have

$$\theta_1(t) = \zeta_{0+}(t) + \varepsilon\zeta_{1+}(t), \qquad \theta_2(t) = \zeta_{0-}(t) + \varepsilon\zeta_{1-}(t), \quad (1.29)$$

which means that in (1.20) $\zeta_{2\pm} = \zeta_{3\pm} = \cdots = 0$, and from (1.24)

$$Z(\zeta_0) = \zeta_1^2 + a_{12}(1+\mu\zeta_0)\left(\frac{\zeta_1}{a_{12}}\right)^{\Delta} + \mu a_{12}\zeta_1\left(\frac{\zeta_0 - a_{11} + \varepsilon\zeta_1}{a_{12}}\right)^{\Delta}. \quad (1.30)$$

Example 1.3. Consider system (1.1) with $a_{11} = a_{22}$. Then for continuous time scale $\mathbb{T} = R$ we have $\mu = 0$, and by picking $m = 1$ in (1.25) we get by direct calculations $\zeta_{1+} = \zeta_{1-}$ and

$$\operatorname{Hov}(\theta_1) = \operatorname{Hov}(\theta_2) = Z(\zeta_{0+}) = Z(\zeta_{0-}). \tag{1.31}$$

In view of

$$Z_1 = Z_2 = \zeta_{1+}^2 + a_{12}\left(\frac{\zeta_{1+}}{a_{12}}\right)^{\Delta} = \left(\frac{\lambda_\tau}{\lambda}\right)^2 - 2a_{12}\left(\frac{\lambda_\tau}{a_{12}\lambda}\right)_\tau = \lambda^{1/2}(\tau)\left(a_{12}^{-1}(\tau)(\lambda^{-1/2}(\tau))_\tau\right)_\tau, \tag{1.32}$$

condition (1.26) under the assumption $\Re[\lambda] = 0$ turns to

$$\int_0^\infty \left|a_{12}^{-1}(\tau)\lambda^{-1/2}(\tau)\left(a_{12}^{-1}(\tau)(\lambda^{-1/2}(\tau))_\tau\right)_\tau\right|\Delta\tau < C_0, \tag{1.33}$$

and from Theorem 1.2 we have the following corollary.

Corollary 1.4. *Assume that* $a_{12}^{-1} \in C^1[0,\infty)$, $\lambda \in C^2[0,\infty)$, $\Re[\lambda(\tau)] \equiv 0$, $a_{11}(\tau) \equiv a_{22}(\tau)$, *and* (1.33) *is satisfied. Then for* $\varepsilon \le 1/C_0$ *estimate* (1.18) *with* $m = 1$ *is true for all solutions* $v(t)$ *of system* (1.1) *on continuous time scale* $\mathbb{T} = R$.

If $a_{12} = 1$, *then* (1.33) *turns to*

$$\int_{t_0\varepsilon}^\infty |\lambda^{-1/2}(\tau)(\lambda^{-1/2}(\tau))_{\tau\tau}|\Delta\tau < C_0, \tag{1.34}$$

and for $\lambda(\tau) = \sqrt{a_{21}} = i\tau^{-2\gamma}$ *it is satisfied for any real* γ.

If $\lambda(\tau)$ *is an analytic function, then it is known (see [13]) that the change of adiabatic invariant approaches zero with exponential speed as* ε *approaches zero.*

Example 1.5. Consider harmonic oscillator on a discrete time scale $\mathbb{T} = \varepsilon\mathbb{Z}$,

$$u^{\Delta\Delta}(t) + w^2(t\varepsilon)u(t) = 0, \quad t \in \varepsilon\mathbb{Z}, \tag{1.35}$$

which could be written in form (1.1), where

$$A = \begin{pmatrix} 0 & 1 \\ -w^2(t\varepsilon) & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} u \\ u^\Delta \end{pmatrix}. \tag{1.36}$$

Choosing $m = 1$ from formulas (1.27) and (1.29) we have $\lambda(\tau) = iw(\tau)$, and

$$\begin{aligned} \theta_1(t) &= \zeta_{0+} + \varepsilon\zeta_{1+} = iw(\tau) - \frac{\varepsilon w^\Delta(\tau)}{2w(\tau)} - \frac{i\varepsilon\mu w^\Delta(\tau)}{2}, \quad \tau = t\varepsilon, \\ \theta_2(t) &= \zeta_{0-} + \varepsilon\zeta_{1-} = -iw(\tau) - \frac{\varepsilon w^\Delta(\tau)}{2w(\tau)} + \frac{i\varepsilon\mu w^\Delta(\tau)}{2}. \end{aligned} \tag{1.37}$$

From (1.13) we get

$$J(t,v,\varepsilon) = \frac{[v_2(t) + iw(\tau)v_1(t)][v_2(t) - iw(\tau)v_1(t)]}{(2w(\tau) - \varepsilon\mu(t)w^\Delta(\tau))^2 e_{\theta_1}(t)e_{\theta_2}(t)}, \tag{1.38}$$

or

$$J(t,u,\varepsilon) = \frac{\left(u^{\Delta}(t)\right)^2 + w^2(\tau)u^2(t)}{\left(2w(\tau) - \varepsilon\mu(t)w^{\Delta}(\tau)\right)^2 e_{\eta}}, \tag{1.39}$$

$$\eta = \theta_1 + \theta_2 + \mu\theta_1\theta_2 = -\frac{\varepsilon w^{\Delta}(\tau)}{w} + \frac{\mu\left(\varepsilon w^{\Delta}\right)^2}{4w^2} + \mu\left(w - \frac{\varepsilon\mu w^{\Delta}}{2}\right)^2. \tag{1.40}$$

If we choose

$$w(\tau) = \frac{a\varepsilon^2}{\tau^2} + \frac{b\varepsilon^3}{\tau^3} = \frac{a}{t^2} + \frac{b}{t^3}, \qquad \lambda(\tau) = \sqrt{a_{21}(\tau)} = iw(\tau), \tag{1.41}$$

then all conditions of Theorem 1.2 are satisfied (see proof of Example 1.5 in the next section) for any real numbers $b, a \neq 0$, and estimate (1.18) with $m = 1$ is true.

Note that for continuous time scale we have $\mu = 0$, and (1.39) turns to the formula of adiabatic invariant for Lorentz's pendulum ([13]):

$$J(t,v,\varepsilon) = \frac{u_t^2(t) + w^2(t\varepsilon)u^2(t)}{4w(t\varepsilon)}. \tag{1.42}$$

2. WKB series and WKB estimates

Fundamental system of solutions of (1.1) could be represented in form

$$v(t) = \Psi(t)(C + \delta(t)), \tag{2.1}$$

where $\Psi(t)$ is an approximate fundamental matrix function and $\delta(t)$ is an error vector function.

Introduce the matrix function

$$H(t) = \left(1 + \mu(t)\Psi^{-1}(t)\Psi^{\Delta}(t)\right)^{-1}\Psi^{-1}(t)\left(A(t)\Psi(t) - \Psi^{\Delta}(t)\right). \tag{2.2}$$

In [16], the following theory was proved.

Theorem 2.1. *Assume there exists a matrix function $\Psi(t) \in C^1_{\mathrm{rd}}(\mathbb{T}_\infty)$ such that $\|H\| \in R^+_{\mathrm{rd}}$, the matrix function $\Psi + \mu\Psi^{\nabla}$ is invertible, and the following exponential function on a time scale is bounded:*

$$e_{\|H(t)\|}(\infty, t) = \exp\int_t^\infty \lim_{p\searrow\mu(s)} \frac{\log(1 + p\|H(s)\|)\Delta s}{p} < \infty. \tag{2.3}$$

Then every solution of (1.1) *can be represented in form* (2.1) *and the error vector function* $\delta(t)$ *can be estimated as*

$$\|\delta(t)\| \leq \|C\|\left(e_{\|H\|}(\infty, t) - 1\right), \tag{2.4}$$

where $\|\cdot\|$ is the Euclidean vector (or matrix) norm.

Remark 2.2. If $\mu(t) \geq 0$, then from (2.4) we get

$$\|\delta(t)\| \leq \|C\|\left(e^{\int_t^\infty \|H(s)\|\Delta s} - 1\right). \tag{2.5}$$

Proof of Remark 2.2. Indeed if $x \geq 0$, the function $f(x) = x - \log(1+x)$ is increasing, so $f(x) \geq f(0)$, $\log(1+x) \leq x$, and from $p \geq 0$, $\|H(t)\| \geq 0$ we get

$$\frac{\log(1+p\|H(s)\|)}{p} \leq \|H(s)\|, \tag{2.6}$$

and by integration

$$\int_t^\infty \lim_{p \searrow \mu(s)} \frac{\log(1+p\|H(s)\|)}{p} \Delta s \leq \int_t^\infty \|H(s)\|\Delta s, \tag{2.7}$$

or

$$e_{\|H\|}(t,\infty) - 1 \leq -1 + \exp\int_t^\infty \|H(s)\|\Delta s. \tag{2.8}$$

□

Note that from the definition

$$\sigma_1(\tau) = \varepsilon\sigma(t), \qquad \mu_1(\tau) = \varepsilon\mu(t), \qquad q^\Delta(t) = \varepsilon q^{\Delta_\tau}(\tau). \tag{2.9}$$

Indeed

$$\begin{gathered}
\varepsilon\sigma(t) = \varepsilon \inf_{s\in\mathbb{T}}\{s, s > t\} = \inf_{\varepsilon s\in\mathbb{T}_\varepsilon}\{\varepsilon s, s > t\} = \inf_{\varepsilon s\in\mathbb{T}_\varepsilon}\{\varepsilon s, \varepsilon s > \varepsilon t\} = \sigma_1(\varepsilon t) = \sigma_1(\tau), \\
\sigma_1(\tau) = \varepsilon\sigma(t), \quad \mu_1(\tau) = \sigma_1(t\varepsilon) - \varepsilon t = \varepsilon(\sigma(t) - t) = \varepsilon\mu(t), \\
q(\varepsilon\sigma(t)) = q(t\varepsilon) + \varepsilon\mu(t)q^{\Delta_\tau}(\tau) = q(t\varepsilon) + \mu(t)q^\Delta(t).
\end{gathered} \tag{2.10}$$

If $a_{12}(\tau) \neq 0$, then the fundamental matrix $\Psi(t)$ in (2.1) is given by (see [12])

$$\Psi(t) = \begin{pmatrix} e_{\theta_1}(t) & e_{\theta_2}(t) \\ U_1(t)e_{\theta_1}(t) & U_2(t)e_{\theta_2}(t) \end{pmatrix}, \qquad U_j(t) = \frac{\theta_j(t) - a_{11}(t)}{a_{12}(t)}. \tag{2.11}$$

Lemma 2.3. *If conditions* (1.14), (1.15) *are satisfied, then*

$$\|H(t)\| \leq \frac{2(1+K(\tau))}{\beta} \max_{j=1,2}\left[\left(1 + \left|\frac{e_j(t)}{e_{3-j}(t)}\right|\right)\left|\frac{\mathrm{Hov}_j(t)}{\theta_1(t) - \theta_2(t)}\right|\right], \quad t \in \mathbb{T}, \tag{2.12}$$

where the functions $\mathrm{Hov}_j(t), K(\tau)$ *are defined in* (1.9), (1.11).

Proof. Denote

$$\Omega = 1 + \mu\Psi^{-1}\Psi^\Delta, \qquad M = \Psi^{-1}(A\Psi - \Psi^\Delta). \tag{2.13}$$

By direct calculations (see [12]), we get from (2.11)

$$M = \frac{1}{\theta_1 - \theta_2}\begin{pmatrix} -\mathrm{Hov}_1 & -\dfrac{e_2 \mathrm{Hov}_2}{e_1} \\ \dfrac{e_1 \mathrm{Hov}_1}{e_2} & \mathrm{Hov}_2 \end{pmatrix}, \qquad \Psi^{\Delta}\Psi^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + Q_1 & a_{22} + Q_0 \end{pmatrix}. \tag{2.14}$$

Using (2.14), we get

$$\det \Omega = \det(\Psi\Omega\Psi^{-1}) = \det(1 + \mu\Psi^{\Delta}\Psi^{-1}) = 1 + \mu(Q_0 + \mathrm{Tr}A) + \mu^2(\det A + a_{11}Q_0 - a_{12}Q_1), \tag{2.15}$$

and from (1.14)

$$\begin{gathered} |\det(\Omega)| = |1 + \mu(Q_0 + \mathrm{Tr}A) + \mu^2(\det A + a_{11}Q_0 - a_{12}Q_1)| \geq \beta > 0, \\ \|\Omega^{-1}\| = \frac{\|\Omega^{\mathrm{co}}\|}{|\det \Omega|} \leq \frac{\|\Omega\|}{|\det \Omega|} \leq \frac{\|\Omega\|}{\beta}, \quad H = \Omega^{-1}M, \\ \Psi^{-1}A\Psi = \frac{1}{\theta_1 - \theta_2}\begin{pmatrix} -\theta_1^2 + \theta_1 \mathrm{Tr}A - \det A & -\dfrac{e_2(\theta_2^2 - \theta_2 \mathrm{Tr}A + \det A)}{e_1} \\ \dfrac{e_1(\theta_1^2 - \theta_1 \mathrm{Tr}A + \det A)}{e_2} & \theta_2^2 - \theta_2 \mathrm{Tr}A + \det A \end{pmatrix} + \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \\ \|M\| \leq 2\max_{j=1,2}\left[\left(1 + \left|\frac{e_j}{e_{3-j}}\right|\right)\left|\frac{\mathrm{Hov}_j}{\theta_1 - \theta_2}\right|\right]. \end{gathered} \tag{2.16}$$

So by using (1.9), we have

$$\begin{gathered} \|\Psi^{-1}A\Psi\| \leq 2\max_{j=1,2}\left[\left(1 + \left|\frac{e_j}{e_{3-j}}\right|\right)\left(\left|\frac{\mathrm{Hov}_j}{\theta_1 - \theta_2}\right| + \left|\frac{\varepsilon a_{12}(1 + \mu\theta_j)[(a_{11} - \theta_j)/a_{12}]^{\Delta}(\tau)}{\theta_1 - \theta_2}\right|\right) + |\theta_j|\right] \\ \|\Omega\| = \|1 + \mu(\Psi^{-1}A\Psi - M)\| \leq 1 + \mu(\|\Psi^{-1}A\Psi\| + \|M\|). \end{gathered} \tag{2.17}$$

From (2.2), (2.13), (2.17), we get (2.12) in view of

$$\|H\| \leq \|\Omega^{-1}\| \cdot \|M\| \leq \frac{\|\Omega\|}{\beta}\|M\| \leq \frac{1 + K}{\beta}\|M\|. \tag{2.18}$$

□

Proof of Theorem 1.1. From (1.16) changing variable of integration $\tau = \varepsilon s$, we get

$$\int_t^{\infty}\|M(s)\|\Delta s \leq \int_t^{\infty} 2\max_{j=1,2}\left(1 + \left|\frac{e_j(s)}{e_{3-j}(s)}\right|\right)\left|\frac{\mathrm{Hov}_j(s)}{\theta_1(s) - \theta_2(s)}\right|\Delta s \leq 2C_0\varepsilon^m, \quad j = 1, 2. \tag{2.19}$$

So using (2.12), we get

$$\int_t^{\infty}\|H(s)\|\Delta s \leq \int_t^{\infty}\frac{1 + K(\varepsilon s)}{\beta}\|M(s)\|\Delta s \leq cC_0\varepsilon^m. \tag{2.20}$$

From this estimate and (2.5), we have

$$\|\delta(t)\| \leq \|C\|\left(e^{\int_t^\infty \|H(s)\|\Delta s} - 1\right) \leq \|C\|\left(e^{C_0 c\varepsilon^m} - 1\right) \leq e\|C\|C_0 c\varepsilon^m, \tag{2.21}$$

where ε is so small that (1.17) is satisfied. The last estimate follows from the inequality $e^x - 1 \leq ex$, $x \in [0,1]$. Indeed because $g(x) = ex + 1 - e^x$ is increasing for $0 \leq x \leq 1$, we have $g(x) \geq g(0)$.

Further from (2.1), (2.11), we have

$$v_1 = (C_1 + \delta_1)e_{\theta_1} + (C_2 + \delta_2)e_{\theta_2}, \qquad v_2 = (C_1 + \delta_1)U_1 e_{\theta_1} + (C_2 + \delta_2)U_2 e_{\theta_2}. \tag{2.22}$$

Solving these equation for $C_j + \delta_j$, we get

$$C_1 + \delta_1 = \frac{v_1 U_2 - v_2}{(U_2 - U_1)e_{\theta_1}}, \qquad C_2 + \delta_2 = \frac{v_2 - v_1 U_1}{(U_2 - U_1)e_{\theta_2}}. \tag{2.23}$$

By multiplication (see (1.12)), we get

$$\begin{gathered} J(t) = (C_1 + \delta_1(t))(C_2 + \delta_2(t)) = C_1C_2 + C_2\delta_1(t) + C_1\delta_2(t) + \delta_1(t)\delta_2(t), \\ J(t_1) - J(t_2) = C_2(\delta_1(t_1) - \delta_1(t_2)) + C_1(\delta_2(t_1) - \delta_2(t_2)) + \delta_1(t_1)\delta_2(t_1) - \delta_1(t_2)\delta_2(t_2), \end{gathered} \tag{2.24}$$

and using estimate (2.21), we have

$$|J(t_1, \theta, v, \varepsilon) - J(t_2, \theta, v, \varepsilon)| \leq C_3 \varepsilon^m. \tag{2.25}$$

□

Proof of Theorem 1.2. Let us look for solutions of (1.1) in the form

$$v(t) = \Psi(t)C, \tag{2.26}$$

where Ψ is given by (2.11), and functions θ_j are given via WKB series (1.20).

Substituting series (1.20) in (1.9), we get

$$\begin{aligned} \operatorname{Hov}(\theta_1) = &\sum_{r,j=0}^{\infty} (\zeta_r \varepsilon^r)(\zeta_j \varepsilon^j) - \operatorname{Tr}(A)\sum_{r=0}^{\infty} \zeta_r \varepsilon^r + \det A \\ &+ a_{12}\varepsilon\left(1 + \mu\sum_{r=0}^{\infty}\zeta_r\varepsilon^r\right)\left(\frac{\sum_{j=0}^{\infty}\zeta_j\varepsilon^j - a_{11}}{a_{12}}\right)^{\Delta}(\tau), \end{aligned} \tag{2.27}$$

or

$$\operatorname{Hov}(\theta_1) \equiv \sum_{k=0}^{\infty} b_k(\tau)\varepsilon^k. \tag{2.28}$$

To make $\operatorname{Hov}(\theta_1)$ asymptotically equal zero or $\operatorname{Hov}(\theta_1) \equiv 0$ we must solve for ζ_k the equations

$$b_k(\tau) = 0, \quad k = 0, 1, 2 \ldots. \tag{2.29}$$

By direct calculations from the first quadratic equation

$$b_0 = \zeta_0^2 - \zeta_0 \mathrm{Tr} A + \det(A) = 0, \tag{2.30}$$

and the second one

$$b_1(\tau) = 2\zeta_1\zeta_0 - \zeta_1 \mathrm{Tr} A + a_{12}(1+\mu\zeta_0)\left(\frac{\zeta_0 - a_{11}}{a_{12}}\right)^{\Delta} = 0, \tag{2.31}$$

we get two solutions $\zeta_{j\pm}$ given by (1.21) and (1.22). Note that

$$\begin{gathered}\frac{\zeta_{0+} - a_{11}}{a_{12}} = \frac{a_{22} - a_{11}}{2a_{12}} + \lambda, \qquad \frac{\zeta_{0-} - a_{11}}{a_{12}} = \frac{a_{22} - a_{11}}{2a_{12}} - \lambda, \\ \zeta_{1+} - \zeta_{1-} = a_{12}\mu\lambda^{\Delta} + \frac{2+\mu \mathrm{Tr} A}{2\lambda}\left(\frac{a_{11} - a_{22}}{2a_{12}}\right)^{\Delta}.\end{gathered} \tag{2.32}$$

Furthermore from $(k+1)$th equation

$$\begin{aligned}b_k &= (2\zeta_0 - \mathrm{Tr} A)\zeta_k + a_{12}(1+\mu\zeta_0)\left(\frac{\zeta_{k-1}}{a_{12}}\right)^{\Delta} \\ &\quad + \sum_{j=1}^{k-1}\zeta_j\left[\zeta_{k-j} + a_{12}\mu\left(\frac{\zeta_{k-1-j} - a_{11}\delta_{j,k-1}}{a_{12}}\right)^{\Delta}(\tau)\right] = 0,\end{aligned} \tag{2.33}$$

we get recurrence relations (1.22).

In view of Theorem 1.1, to prove Theorem 1.2 it is enough to deduce condition (1.16) from (1.26). By truncation of series (1.20) or by taking

$$\zeta_{k+} = \zeta_{k-} = 0, \quad k = m+1, m+2, \ldots, \tag{2.34}$$

we get (1.25). Defining $\zeta_{j\pm}$, $j = 1, 2, \ldots, m$ as in (1.21) and (1.22), we have

$$\begin{gathered}b_0 = b_1 = \cdots = b_{m-1} = b_m = b_{m+3} = b_{m+4} = \cdots = 0, \\ b_{m+1} = a_{12}(1+\mu\zeta_0)\left(\frac{\zeta_m}{a_{12}}\right)^{\Delta} + \sum_{j=1}^{m}\zeta_j\left[\zeta_{m+1-j} + a_{12}\mu\left(\frac{\zeta_{m-j} - a_{11}\delta_{j,m}}{a_{12}}\right)^{\Delta}(\tau)\right], \\ b_{m+2} = \sum_{j=1}^{m}\zeta_j\left[\zeta_{m+2-j} + a_{12}\mu\left(\frac{\zeta_{m+1-j}}{a_{12}}\right)^{\Delta}(\tau)\right].\end{gathered} \tag{2.35}$$

Now (1.16) follows from (1.26) in view of

$$\mathrm{Hov}(\theta_k) = \varepsilon^{m+1}(b_{m+1} + b_{m+2}\varepsilon) = \varepsilon^{m+1}Z_k, \quad k = 1, 2. \tag{2.36}$$

□

Note that from (1.13) and the estimates

$$\begin{aligned}
\log|1+p\theta| &\leq \log\sqrt{1+2p\Re(\theta)+p^2|\theta|^2} \leq \frac{1}{2}\left|2p\Re(\theta)+p^2|\theta|^2\right|,\\
\log|1+p\theta| &\leq \log\sqrt{1+2p\Re(\theta)+p^2|\theta|^2} \leq \sqrt{\left|2p\Re(\theta)+p^2|\theta|^2\right|},
\end{aligned} \tag{2.37}$$

it follows

$$|e_\theta(t,t_0)| \leq \exp\int_{t_0}^{t}\left|\Re[\theta(s)]+\frac{\mu(s)|\theta(s)|^2}{2}\right|\Delta s, \tag{2.38}$$

$$|e_\theta(t,t_0)| \leq \exp\int_{t_0}^{t}\sqrt{|\theta(s)|^2+\frac{2\Re[\theta(s)]}{\mu(s)}}\,\Delta s, \quad \mu(s)>0. \tag{2.39}$$

Proof of Example 1.5. From (1.37), (1.41), we have

$$\begin{gathered}
\theta_1-\theta_2 = i\left(2w(\tau)-\varepsilon\mu w^\Delta(\tau)\right), \qquad \theta_1+\theta_2 = -\frac{\varepsilon w^\Delta(\tau)}{w}, \qquad \theta_1\theta_2 = \frac{\left(\varepsilon w^\Delta\right)^2}{4w^2}+\left(w-\frac{\varepsilon\mu w^\Delta}{2}\right)^2,\\
\eta_1(t) = \frac{\theta_1-\theta_2}{1+\mu\theta_2} = \frac{2ia}{t^2}+O(t^{-3}), \qquad \eta_2(t) = \frac{\theta_2-\theta_1}{1+\mu\theta_1} = \frac{-2ia}{t^2}+O(t^{-3}), \quad \tau\longrightarrow\infty,
\end{gathered} \tag{2.40}$$

and using (2.39), we get

$$\left|\frac{e_{\theta_1}}{e_{\theta_2}}\right| \leq |e_{\eta_1}| \leq \text{const}, \qquad \left|\frac{e_{\theta_2}}{e_{\theta_1}}\right| \leq |e_{\eta_2}| \leq \text{const}. \tag{2.41}$$

Further for $\tau\to\infty$

$$\begin{aligned}
\zeta_{1\pm} &= -\frac{\lambda^\Delta}{2\lambda}\mp\frac{\lambda^\Delta}{2} = \frac{1}{\tau}+\frac{b\varepsilon-3a\mu}{2a\tau^2}+\frac{1}{\tau^3}\left(2\mu^2-\frac{3b\varepsilon\mu}{2a}-\frac{b^2\varepsilon^2}{2a^2}\pm ia\varepsilon^2\right)+O(\tau^{-4}),\\
Z_1 &= \zeta_{1+}^2+\zeta_{1+}^\Delta+\varepsilon\zeta_{1+}^\Delta\zeta_{1+}+O(\tau^{-4}) = \frac{\mu-\varepsilon}{\tau^3}+O(\tau^{-4}) = O(\tau^{-4}), \qquad Z_2 = Z_1+O(\tau^{-4}).
\end{aligned} \tag{2.42}$$

So if $\mu=\varepsilon$, then (1.26) and all other conditions of Theorem 1.2 are satisfied, and (1.18) is true with $m=1$. □

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References

[1] M. Fröman and P. O. Fröman, *JWKB-Approximation. Contributions to the Theory*, North-Holland, Amsterdam, The Netherlands, 1965.

[2] M. H. Holmes, *Introduction to Perturbation Methods*, vol. 20 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 1995.

[3] G. D. Birkhoff, "Quantum mechanics and asymptotic series," *Bulletin of the American Mathematical Society*, vol. 39, no. 10, pp. 681–700, 1933.
[4] P. A. Braun, "WKB method for three-term recursion relations and quasienergies of an anharmonic oscillator," *Theoretical and Mathematical Physics*, vol. 37, no. 3, pp. 1070–1081, 1979.
[5] O. Costin and R. Costin, "Rigorous WKB for finite-order linear recurrence relations with smooth coefficients," *SIAM Journal on Mathematical Analysis*, vol. 27, no. 1, pp. 110–134, 1996.
[6] R. B. Dingle and G. J. Morgan, "WKB methods for difference equations—I," *Applied Scientific Research*, vol. 18, pp. 221–237, 1967.
[7] J. S. Geronimo and D. T. Smith, "WKB (Liouville-Green) analysis of second order difference equations and applications," *Journal of Approximation Theory*, vol. 69, no. 3, pp. 269–301, 1992.
[8] P. Wilmott, "A note on the WKB method for difference equations," *IMA Journal of Applied Mathematics*, vol. 34, no. 3, pp. 295–302, 1985.
[9] B. Aulbach and S. Hilger, "Linear dynamic processes with inhomogeneous time scale," in *Nonlinear Dynamics and Quantum Dynamical Systems (Gaussig, 1990)*, vol. 59 of *Mathematical Research*, pp. 9–20, Akademie, Berlin, Germany, 1990.
[10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Application*, Birkhäuser, Boston, Mass, USA, 2001.
[11] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
[12] G. Hovhannisyan, "Asymptotic stability for 2×2 linear dynamic systems on time scales," *International Journal of Difference Equations*, vol. 2, no. 1, pp. 105–121, 2007.
[13] J. E. Littlewood, "Lorentz's pendulum problem," *Annals of Physics*, vol. 21, no. 2, pp. 233–242, 1963.
[14] W. Wasow, "Adiabatic invariance of a simple oscillator," *SIAM Journal on Mathematical Analysis*, vol. 4, no. 1, pp. 78–88, 1973.
[15] G. Hovhannisyan and Y. Taroyan, "Adiabatic invariant for N-connected linear oscillators," *Journal of Contemporary Mathematical Analysis*, vol. 31, no. 6, pp. 47–57, 1997.
[16] G. Hovhannisyan, "Error estimates for asymptotic solutions of dynamic equations on time scales," in *Proceedings of the 6th Mississippi State–UBA Conference on Differential Equations and Computational Simulations*, vol. 15 of *Electronic Journal of Differential Equations Conference*, pp. 159–162, Southwest Texas State University, San Marcos, Tex, USA, 2007.