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Research Article

Positive Solutions for Multiparameter Semipositone Discrete Boundary Value Problems via Variational Method

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We study the existence, multiplicity, and nonexistence of positive solutions for multiparameter semipositone discrete boundary value problems by using nonsmooth critical point theory and subsuper solutions method.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots\}$, $\mathbb{Z}(a,b) = \{a, a+1, \ldots, b\}$, when $a \leq b$.

In this paper, we consider the multiparameter semipositone discrete boundary value problem

$$\begin{gathered} -\Delta^2 u(t-1) = \lambda f(u(t)) + \mu g(u(t)), \quad t \in \mathbb{Z}(1,N), \\ u(0) = 0, \quad u(N+1) = 0, \end{gathered} \tag{1.1}$$

where $\lambda, \mu > 0$ are parameters, $N \geq 4$ is a positive integer, $\Delta u(t) = u(t+1) - u(t)$ is the forward difference operator, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $f : [0, +\infty) \to \mathbb{R}$ is a continuous positive function satisfying $f(0) > 0$, and $g : [0, +\infty) \to \mathbb{R}$ is continuous and eventually strictly positive with $g(0) < 0$.

We notice that for fixed $\mu > 0$, $\lambda f(0) + \mu g(0) < 0$ whenever $\lambda > 0$ is sufficiently small. We call (1.1) a semipositone problem. Semipositone problems are derived from [1], where Castro and Shivaji initially called them nonpositone problems, in contrast with the terminology positone problems, put forward by Keller and Cohen in [2], where the nonlinearity was positive and monotone. Semipositone problems arise in bulking of mechanical systems, design of suspension bridges, chemical reactions, astrophysics, combustion, and management of natural resources; for example, see [3–6].

In general, studying positive solutions for semipositone problems is more difficult than that for positone problems. The difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the nonlinear term is negative as well as positive. However, many methods have been applied to deal with semipositone problems, the usual approaches are quadrature method, fixed point theory, subsuper solutions method, and degree theory. We refer the readers to the survey papers [7, 8] and references therein.

Due to its importance, in recent years, continuous semipositone problems have been widely studied by many authors, see [9–15]. However, we noticed that there were only a few papers on discrete semipositone problems. One can refer to [16–18]. In these papers, semipositone discrete boundary value problems with one parameter were discussed, and subsuper solutions method and fixed point theory were used to study them. To the authors' best knowledge, there are no results established on semipositone discrete boundary value problems with two parameters. Here we want to present a different approach to deal with this topic. In [11], Costa et al. applied the nonsmooth critical point theory developed by Chang [19] to study the existence and multiplicity results of a class of semipositone boundary value problems with one parameter. We think it is also an efficient tool in dealing with the semipositone discrete boundary value problems with two parameters.

Our main objective in this paper is to apply the nonsmooth critical point theory to deal with the positive solutions of semipositone problem (1.1). More precisely, we define the discontinuous nonlinear terms

$$f_1(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ f(s) & \text{if } s > 0, \end{cases} \qquad g_1(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ g(s) & \text{if } s > 0. \end{cases} \tag{1.2}$$

Now we consider the slightly modified problem

$$\begin{gathered} -\Delta^2 u(t-1) = \lambda f_1(u(t)) + \mu g_1(u(t)), \quad t \in \mathbb{Z}(1,N), \\ u(0) = 0, \quad u(N+1) = 0. \end{gathered} \tag{1.3}$$

Just to be on the convenient side, we define $h(s) = \lambda f(s) + \mu g(s)$, $h_1(s) = \lambda f_1(s) + \mu g_1(s)$, $H(s) = \lambda F(s) + \mu G(s)$, $H_1(s) = \lambda F_1(s) + \mu G_1(s)$, where $F(s) = \int_0^s f(\tau)d\tau$, $G(s) = \int_0^s g(\tau)d\tau$,

$$F_1(s) = \int_0^s f_1(\tau)d\tau = \begin{cases} 0 & \text{if } s \leq 0, \\ F(s) & \text{if } s > 0, \end{cases} \qquad G_1(s) = \int_0^s g_1(\tau)d\tau = \begin{cases} 0 & \text{if } s \leq 0 \\ G(s) & \text{if } s > 0. \end{cases} \tag{1.4}$$

We will prove in Section 3 that the sets of positive solutions of (1.1) and (1.3) do coincide. Moreover, any nonzero solution of (1.3) is nonnegative.

Our main results are as follows.

Theorem 1.1. *Suppose that there are constants $C_1 > 0$, $\alpha > 1$, and $\beta > 2$ such that when $s > 0$ is large enough,*

$$f(s) < C_1 s^{\alpha}, \tag{1.5}$$

$$sf(s) \geq \beta F(s) > 0, \tag{1.6}$$

$$\lim_{s \to +\infty} \frac{g(s)}{s} = 0. \tag{1.7}$$

Then for fixed $\mu > 0$, there is a $\overline{\lambda} > 0$ such that for $\lambda \in (0, \overline{\lambda})$, problem (1.3) *has a nontrivial nonnegative solution. Hence problem* (1.1) *has a positive solution.*

Remark 1.2. By (1.6), there are constants $C_2, C_3 > 0$ such that for any $s \geq 0$,

$$F(s) \geq C_2 s^{\beta} - C_3. \tag{1.8}$$

Equations (1.6) and (1.8) imply that

$$\lim_{s \to +\infty} \frac{f(s)}{s} = +\infty, \tag{1.9}$$

which shows that f is superlinear at infinity.

Remark 1.3. Equation (1.7) implies that g is sublinear at infinity. Moreover, it is easy to know that

$$\lim_{s \to +\infty} \frac{G(s)}{s^2} = 0. \tag{1.10}$$

Hence G is subquadratic at infinity.

Theorem 1.4. *Suppose that the conditions of Theorem 1.1 hold. Moreover, g is increasing on $[0, +\infty)$. Then there is a $\mu^* > 0$ such that for $\mu > \mu^*$, problem* (1.1) *has at least two positive solutions for sufficiently small λ.*

Theorem 1.5. *Suppose that the conditions of Theorem 1.1 hold. Moreover, f is nondecreasing on $[0, +\infty)$. Then for fixed $\mu > 0$, problem* (1.1) *has no positive solution for sufficiently large λ.*

2. Preliminaries

In this section, we recall some basic results on variational method for locally Lipschitz functional $I : X \to \mathbf{R}$ defined on a real Banach space X with norm $\|\cdot\|$. I is called locally

Lipschitzian if for each $u \in X$, there is a neighborhood $V = V(u)$ of u and a constant $B = B(u)$ such that

$$|I(x) - I(y)| \leq B\|x - y\|, \quad \forall x, y \in V. \tag{2.1}$$

The following abstract theory has been developed by Chang [19].

Definition 2.1. For given $u, z \in X$, the generalized directional derivative of the functional I at u in the direction z is defined by

$$I^0(u; z) = \limsup_{k \to 0\, t \to 0} \frac{1}{t}[I(u + k + tz) - I(u + k)]. \tag{2.2}$$

The following properties are known:

(i) $z \to I^0(u; z)$ is subadditive, positively homogeneous, continuous, and convex;

(ii) $|I^0(u; z)| \leq B\|z\|$;

(iii) $I^0(u; -z) = (-I)^0(u; z)$.

Definition 2.2. The generalized gradient of I at u, denoted by $\partial I(u)$, is defined to be the subdifferential of the convex function $I^0(u; z)$ at $z = 0$, that is,

$$w \in \partial I(u) \subset X^* \Longleftrightarrow \langle w, z \rangle \leq I^0(u; z), \quad \forall z \in X. \tag{2.3}$$

The generalized gradient $\partial I(u)$ has the following main properties.

(1) For all $u \in X$, $\partial I(u)$ is a nonempty convex and w^*-compact subset of X^*;

(2) $\|w\|_{X^*} \leq B$ for all $w \in \partial I(u)$.

(3) If $I, J : X \to \mathbb{R}$ are locally Lipschitz functional, then

$$\partial(I + J)(u) \subset \partial I(u) + \partial J(u). \tag{2.4}$$

(4) For any $\lambda > 0$, $\partial(\lambda I)(u) = \lambda \partial I(u)$.

(5) If I is a convex functional, then $\partial I(u)$ coincides with the usual subdifferential of I in the sense of convex analysis.

(6) If I is Gâteaux differential at every point of v of a neighborhood V of u and the Gâteaux derivative is continuous, then $\partial I(u) = \{I'(u)\}$.

(7) The function

$$\zeta(u) = \min_{w \in \partial I(u)} \|w\|_{X^*} \tag{2.5}$$

exists, that is, there is a $w_0 \in \partial I(u)$ such that $\|w_0\|_{X^*} = \min_{w \in \partial I(u)} \|w\|_{X^*}$.

(8) $I^0(u; z) = \max\{\langle w, z \rangle \mid w \in \partial I(u)\}$.

(9) If I has a minimum at $u_0 \in X$, then $0 \in \partial I(u_0)$.

Definition 2.3. $u \in X$ is a critical point of the locally Lipschitz functional I if $0 \in \partial I(u)$.

Definition 2.4. I is said to satisfy Palais-Smale condition ((PS) condition for short) if any sequence $\{u_n\}$ such that $I(u_n)$ is bounded and $\zeta(u_n) = \min_{w \in \partial I(u_n)} \|w\|_{X^*} \to 0$ has a convergent subsequence.

Lemma 2.5 (see [19, Mountain Pass Theorem]). *Let X be a real Hilbert space and let I be a locally Lipschitz functional satisfying (PS) condition. Suppose that $I(0) = 0$ and that the following hold.*

(i) *There exist constants $\rho > 0$ and $a > 0$ such that $I(u) \geq a$ if $\|u\| = \rho$.*

(ii) *There is an $e \in X$ such that $\|e\| > \rho$ and $I(e) \leq 0$.*

Then I possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)), \tag{2.6}$$

where

$$\Gamma = \{g \in C([0,1], X) \mid \gamma(0) = 0,\ \gamma(1) = e\}. \tag{2.7}$$

Next we give the definitions of the subsolution and the supersolution of the following boundary value problem:

$$\begin{gathered} -\Delta^2 u(t-1) = \mu g(u(t)), \quad t \in \mathbb{Z}(1, N), \\ u(0) = 0, \quad u(N+1) = 0. \end{gathered} \tag{2.8}$$

Definition 2.6. If $u_1(t)$, $t \in \mathbb{Z}(0, N+1)$ satisfies the following conditions:

$$\begin{gathered} -\Delta^2 u_1(t-1) \leq \mu g(u_1(t)), \quad t \in \mathbb{Z}(1, N), \\ u_1(0) \leq 0, \quad u_1(N+1) \leq 0, \end{gathered} \tag{2.9}$$

then u_1 is called a subsolution of problem (2.8).

Definition 2.7. If $u_2(t), t \in \mathbb{Z}(0, N+1)$ satisfies the following conditions:

$$\begin{gathered} -\Delta^2 u_2(t-1) \geq \mu g(u_2(t)), \quad t \in \mathbb{Z}(1, N), \\ u_2(0) \geq 0, \quad u_2(N+1) \geq 0, \end{gathered} \tag{2.10}$$

then u_2 is called a supersolution of problem (2.8).

Lemma 2.8. *Suppose that there exist a subsolution u_1 and a supersolution u_2 of problem* (2.8) *such that $u_1(t) \le u_2(t)$ in $\mathbb{Z}(1,N)$. Then there is a solution $\breve{u}$ of problem* (2.8) *such that $u_1(t) \le \breve{u}(t) \le u_2(t)$ in $\mathbb{Z}(1,N)$.*

Remark 2.9. If (2.8) is replaced by (1.1), then we have similar definitions and results as Definitions 2.6, 2.7, and Lemma 2.8

3. Proof of main results

Let E be the class of the functions $u : \mathbb{Z}(0,N+1) \to \mathbb{R}$ such that $u(0) = u(N+1) = 0$. Equipped with the usual inner product and the usual norm

$$(u,v) = \sum_{t=1}^{N}(u(t),v(t)), \quad \|u\| = \left(\sum_{t=1}^{N}u^2(t)\right)^{1/2}, \tag{3.1}$$

E is an N-dimensional Hilbert space. Define the functional J on E as

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{t=1}^{N+1}\left[(\Delta u(t-1))^2 - 2H_1(u(t))\right] \\ &= \frac{1}{2}u^T Au - \sum_{t=1}^{N}H_1(u(t)) = K(u) - \sum_{t=1}^{N}H_1(u(t)), \end{aligned} \tag{3.2}$$

where $u = \{u(1),u(2),\dots,u(N)\}$, $K(u) = (1/2)u^T Au$ and

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{N\times N}. \tag{3.3}$$

Clearly, H_1 is a locally Lipschitz function and $J(u)$ is a locally Lipschitz functional on E. By a simple computation, we obtain

$$\frac{\partial}{\partial u(t)}K(u) = 2u(t) - u(t+1) - u(t-1) = -\Delta^2 u(t-1). \tag{3.4}$$

By [19, Theorem 2.2], the critical point of the functional $J(u)$ is a solution of the inclusion

$$-\Delta^2 u(t-1) \in \left[\underline{h_1}(u(t)), \overline{h_1}(u(t))\right], \quad t \in \mathbb{Z}(1,N), \tag{3.5}$$

where $\underline{h_1}(s) = \min[h_1(s+0),\ h_1(s-0)]$, $\overline{h_1}(s) = \max[h_1(s+0), h_1(s-0)]$.

Remark 3.1. We can show that $\underline{h_1}(s) = \overline{h_1}(s) = \lambda f(s) + \mu g(s)$ for $s > 0$, $\underline{h_1}(s) = \overline{h_1}(s) = 0$ for $s < 0$. For fixed μ and sufficiently small λ, $\lambda f(0) + \mu g(0) < 0$. Then $\underline{h_1}(0) = \lambda f(0) + \mu g(0)$, $\overline{h_1}(0) = 0$.

Remark 3.2. If $u > 0$, then the above inclusion becomes

$$-\Delta^2 u(t-1) = \lambda f(u(t)) + \mu g(u(t)), \quad t \in \mathbb{Z}(1, N). \tag{3.6}$$

It is clear that A is a positive definite matrix. Let $\eta_{\max} > 0$, $\eta_{\min} > 0$ be the largest and smallest eigenvalue of A, respectively. Denote by $u^- = \max\{-u, 0\}$. Let $P_1 = \{t \in \mathbb{Z}(1, N) \mid u(t) \le 0\}$, $P_2 = \{t \in \mathbb{Z}(1, N) \mid u(t) > 0\}$. Notice that $u^-(t) = 0$ for $t \in P_2$ and $f_1(u(t)) = 0$ for $t \in P_1$. Then

$$\sum_{t=1}^{N} f_1(u(t)) u^-(t) = \sum_{t \in P_1} f_1(u(t)) u^-(t) + \sum_{t \in P_2} f_1(u(t)) u^-(t) = 0. \tag{3.7}$$

Similarly, $g_1(u(t)) = 0$ for $t \in P_1$. Hence

$$\sum_{t=1}^{N} g_1(u(t)) u^-(t) = \sum_{t \in P_1} g_1(u(t)) u^-(t) + \sum_{t \in P_2} g_1(u(t)) u^-(t) = 0. \tag{3.8}$$

Lemma 3.3. *If u is a solution of* (1.3)*, then $u \ge 0$. Moreover, either $u > 0$ in $\mathbb{Z}(1, N)$, or $u = 0$ everywhere.*

Proof. It is not difficult to see that $(\Delta u^-(t) + \Delta u(t)) \Delta u^-(t) \le 0$ for $t \in \mathbb{Z}(0, N)$. In fact, no matter that $\Delta u(t) \ge 0$ or $\Delta u(t) < 0$, the former inequality holds. Hence $\Delta u^-(t) \cdot \Delta u(t) \le -(\Delta u^-(t))^2$.

If u is a solution of (1.3), then we have

$$\begin{aligned} 0 &= \sum_{t=1}^{N} \left[\Delta^2 u(t-1) + \lambda f_1(u(t)) + \mu g_1(u(t))\right] u^-(t) \\ &= -\sum_{t=1}^{N+1} \Delta u(t-1) \Delta u^-(t-1) + \sum_{t=1}^{N} [\lambda f_1(u(t)) + \mu g_1(u(t))] u^-(t) \\ &\ge \sum_{t=1}^{N+1} (\Delta u^-(t-1))^2 = (u^-)^T A u^- \ge \eta_{\min} \|u^-\|^2. \end{aligned} \tag{3.9}$$

So $u^- = 0$. Hence $u \ge 0$. If $u(t) = 0$, then

$$u(t+1) + u(t-1) = \Delta^2 u(t-1) = -\lambda f_1(u(t)) - \mu g_1(u(t)) = -\lambda f_1(0) - \mu g_1(0) = 0. \tag{3.10}$$

Therefore $u(t+1) = u(t-1) = 0$. It follows that $u = 0$ everywhere. □

Lemma 3.4. *If* (1.6) *and* (1.7) *hold, then $h_1(s)s \ge \beta_0 H_1(s)$ for large $s > 0$, where $\beta_0 \in (2, \beta)$.*

Proof. Notice that $h_1(s)s \geq \beta_0 H_1(s)$ is equivalent to $h(s)s \geq \beta_0 H(s)$ if $s > 0$. To prove that $h(s)s \geq \beta_0 H(s)$ for large $s > 0$, it suffices to show that

$$\underline{\lim}_{s \to +\infty} \frac{h(s)s}{\beta_0 H(s)} > 1. \tag{3.11}$$

By (1.6), for large $s > 0$, we have

$$\frac{\beta_0 F(s)}{f(s)s} \leq \frac{\beta_0}{\beta}. \tag{3.12}$$

Hence, if $s > 0$ is large, then

$$\frac{h(s)s}{\beta_0 H(s)} = \frac{\lambda f(s)s + \mu g(s)s}{\beta_0(\lambda F(s) + \mu G(s))} = \frac{1 + \mu g(s)/\lambda f(s)}{\beta_0 F(s)/f(s)s + \beta_0 \mu G(s)/\lambda f(s)s} \geq \frac{1 + \mu g(s)/\lambda f(s)}{\beta_0/\beta + \beta_0 \mu G(s)/\lambda f(s)s}. \tag{3.13}$$

Taking inferior limit on both sides of the above inequality, we have

$$\underline{\lim}_{s \to +\infty} \frac{h(s)s}{\beta_0 H(s)} \geq \underline{\lim}_{s \to +\infty} \frac{1 + \mu g(s)/\lambda f(s)}{\beta_0/\beta + \beta_0 \mu G(s)/\lambda f(s)s} \geq \frac{\underline{\lim}_{s \to +\infty}(1 + \mu g(s)/\lambda f(s))}{\overline{\lim}_{s \to +\infty}(\beta_0/\beta + \beta_0 \mu G(s)/\lambda f(s)s)}. \tag{3.14}$$

Since f is superlinear and g is sublinear, $\lim_{s \to +\infty}(\mu g(s)/\lambda f(s)) = 0$. Then $\underline{\lim}_{s \to +\infty}(1 + \mu g(s)/\lambda f(s)) = \lim_{u \to +\infty}(1 + \mu g(s)/\lambda f(s)) = 1$. Moreover, since G is subquadratic and f is superlinear, $\lim_{u \to +\infty}(G(s)/f(s)s) = \lim_{s \to +\infty}((G(s)/s^2)/(f(s)s/s^2)) = 0$. Therefore, $\overline{\lim}_{s \to +\infty}(\beta_0/\beta + \beta_0 \mu G(s)/\lambda f(s)s) = \lim_{s \to +\infty}(\beta_0/\beta + \beta_0 \mu G(s)/\lambda f(s)s) = \beta_0/\beta$. From the above results, we can conclude that $\underline{\lim}_{s \to +\infty}(h(s)s/\beta_0 H(s)) \geq \beta/\beta_0 > 1$. □

Lemma 3.5. *If* (1.6) *and* (1.7) *hold, then J satisfies (PS) condition.*

Proof. Notice that $E^* = E$. Let $L(u) = \sum_{t=1}^{N} H_1(u(t))$. From [19, Theorem 2.2], for any given $w \in \partial L(u) \subset E^*$, we have $w(t) \in [\underline{h_1}(u(t)), \overline{h_1}(u(t))]$. Then

$$w(t) = \lambda f_1(u(t)) + \mu g_1(u(t)) \quad \text{if } u(t) \neq 0, \qquad w(t) \in [\lambda f(0) + \mu g(0), 0] \quad \text{if } u(t) = 0. \tag{3.15}$$

Therefore

$$\langle w, u \rangle = \sum_{t=1}^{N} h_1(u(t))u(t), \ \forall w \in \partial L(u). \tag{3.16}$$

By Lemma 3.4, there is a constant $M > 0$ such that $L(u) \leq (1/\beta_0)\langle w, u\rangle + M$ for $u \in \mathbf{R}^N$. Suppose that $\{u_n\}$ is a sequence such that $J(u_n)$ is bounded and $\zeta(u_n) \to 0$ as $n \to \infty$. Then by Properties (3) and (7) in Definition 2.2, there are $C > 0$ and $w_n \in \partial L(u_n)$ such that $|J(u_n)| \leq C$ and

$$\left|\langle \partial K(u_n) - w_n, u_n\rangle\right| \leq \|u_n\| \quad \text{for sufficiently large } n. \tag{3.17}$$

It implies that

$$u_n^T A u_n - \langle w_n, u_n\rangle \geq -\|u_n\|. \tag{3.18}$$

Hence

$$\begin{aligned} C &\geq \frac{1}{2} u_n^T A u_n - L(u_n) \\ &\geq \frac{1}{2} u_n^T A u_n - \frac{1}{\beta_0}\langle w_n, u_n\rangle - M \\ &= \left(\frac{1}{2} - \frac{1}{\beta_0}\right) u_n^T A u_n + \frac{1}{\beta_0}\left[u_n^T A u_n - \langle w_n, u_n\rangle\right] - M \\ &\geq \left(\frac{1}{2} - \frac{1}{\beta_0}\right)\eta_{\min}\|u_n\|^2 - \frac{1}{\beta_0}\|u_n\| - M. \end{aligned} \tag{3.19}$$

This implies that $\{u_n\}$ is bounded. Since E is finite dimensional, $\{u_n\}$ has a convergent subsequence in E. □

Lemma 3.6. *For fixed $\mu > 0$, there exist $\rho > 0$ and $\overline{\lambda} > 0$ such that if $\lambda \in (0, \overline{\lambda})$, then $J(u) \geq (\eta_{\min} M_1^2/16)\lambda^{-2/(\alpha-1)}$ for $\|u\| = \rho$.*

Proof. By (1.5) and (1.7), there are $C_4, C_5 > 0$ such that

$$F_1(s) \leq \frac{C_1|s|^{\alpha+1}}{\alpha+1} + C_4, \quad \forall s \in \mathbf{R}, \tag{3.20}$$

$$G_1(s) \leq \frac{\eta_{\min}}{4\mu}|s|^2 + C_5, \quad \forall s \in \mathbf{R}. \tag{3.21}$$

The equivalence of norm on E implies that there exists $C_6 > 0$ such that $\|u\|_{\alpha+1} \leq C_6\|u\|$, where $\|u\|_{\alpha+1} = (\sum_{t=1}^N |u(t)|^{\alpha+1})^{1/(\alpha+1)}$. Let $M_1 = (\eta_{\min}(\alpha+1)/8C_1C_6^{\alpha+1})^{1/(\alpha-1)}$ and $\rho = M_1\lambda^{-1/(\alpha-1)}$. Let

$\|u\| = \rho$. It follows from (3.20) and (3.21) that there is $\overline{\lambda} > 0$ such that if $\lambda \in (0, \overline{\lambda})$, then

$$
\begin{aligned}
J(u) &= \frac{1}{2} u^T A u - \sum_{t=1}^{N} H_1(u(t)) \\
&\geq \frac{1}{2}\eta_{\min}\|u\|^2 - \frac{\lambda C_1}{\alpha+1}\sum_{t=1}^{N}|u(t)|^{\alpha+1} - \lambda C_4 N - \frac{\eta_{\min}}{4\mu}\cdot\mu\sum_{t=1}^{N}|u(t)|^2 - \mu C_5 N \\
&\geq \frac{1}{4}\eta_{\min}\|u\|^2 - \frac{\lambda C_1 C_6^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1} - \lambda C_4 N - \mu C_5 N \\
&= \lambda^{-2/(\alpha-1)}\left(\frac{\eta_{\min} M_1^2}{8} - \lambda^{(\alpha+1)/(\alpha-1)} C_4 N - \lambda^{2/(\alpha-1)} \mu C_5 N\right) \\
&\geq \frac{\eta_{\min} M_1^2}{16}\lambda^{-2/(\alpha-1)}.
\end{aligned}
\tag{3.22}
$$

□

Lemma 3.7. *There is an $e \in E$ such that $\|e\| > \rho$ and $J(e) < 0$.*

Proof. It follows from Remark 1.2 that $F(s) \geq C_2 s^{\beta} - C_3$ for $s > 0$. By the equivalence of the norms on E, there exists $C_7 > 0$ such that $\|u\|_{\beta} \geq C_7\|u\|$, where $\|u\|_{\beta} = (\sum_{t=1}^{N}|u(t)|^{\beta})^{1/\beta}$. Let v_1 be the eigenfunction to the principal eigenvalue η_1 of

$$
\begin{gathered}
-\Delta^2 u(t-1) = \eta u(t), \quad t \in \mathbb{Z}(1, N), \\
u(0) = 0, \quad u(N+1) = 0
\end{gathered}
\tag{3.23}
$$

with $v_1 > 0$ and $\|v_1\| = 1$. Let

$$
G_m = \min\{G(u) \mid u \in [0, +\infty)\}. \tag{3.24}
$$

Clearly $G_m < 0$. Since $\beta > 2$, for $k > 0$,

$$
\begin{aligned}
J(kv_1) &= \frac{1}{2}k^2 v_1^T A v_1 - \lambda\sum_{t=1}^{N} F(kv_1(t)) - \mu\sum_{t=1}^{N} G(kv_1(t)) \\
&\leq \frac{\eta_{\max}}{2}k^2 - \lambda C_2 (C_7 k)^{\beta} + \lambda C_3 N - \mu G_m N \\
&\longrightarrow -\infty \quad \text{as } k \longrightarrow +\infty.
\end{aligned}
\tag{3.25}
$$

Hence there is a $k_1 > \rho$ such that $J(k_1 v_1) < 0$. Let $e = k_1 v_1$. Then $\|e\| > \rho$ and $J(e) < 0$. The second condition of Mountain Pass theorem is verified. □

Proof of Theorem 1.1. Clearly, $J(0) = 0$. Lemma 3.5 implies that J satisfies (PS) condition. It follows from Lemmas 3.6, 3.7, and 2.5 that J has a nontrivial critical point $\hat{u}$ such that $J(\hat{u}) \geq (\eta_{\min} M_1^2/16)\lambda^{-2/(\alpha-1)}$. By Lemma 3.3 and Remark 3.2, $\hat{u}$ is a positive solution of (1.1). The proof is complete. □

Proof of Theorem 1.4. We will apply the subsuper solutions method to prove the multiplicity results.

Firstly, we will prove that there exists $\mu^* > 0$ such that if $\mu > \mu^*$, then the following boundary value problem

$$\begin{gathered} -\Delta^2 u(t-1) = \mu g(u(t)), \quad t \in \mathbb{Z}(1,N), \\ u(0) = 0, \quad u(N+1) = 0 \end{gathered} \tag{3.26}$$

has a positive solution $\underline{u}$. In fact, since $g(u)$ is increasing on $[0,+\infty)$ and eventually strictly positive, $g(u) \geq -C_8$ for $u \geq 0$ and some $C_8 > 0$. Let r_1 be the eigenfunction to the principal eigenvalue μ_1 of

$$\begin{gathered} -\Delta^2 u(t-1) = \mu u(t), \quad t \in \mathbb{Z}(1,N), \\ u(0) = 0, \quad u(N+1) = 0 \end{gathered} \tag{3.27}$$

with $r_1 > 0$ and $\|r_1\| = 1$.

Notice that $\mu_1 = 2 - 2\cos(\pi/(N+1))$ and $r_1(t) = \sin(\pi t/(N+1))$ (see [20]). Let $C_9 > 0$ be a constant such that $C_9 \leq 2\sin^2(\pi/(N+1))\cos(2\pi/(N+1))$. For $t \in Q_1 = \{t \in \mathbb{Z}(1,N) \mid t = 1 \text{ or } t = N\}$, $N \geq 4$, we have $(\Delta r_1(t))^2 + (\Delta r_1(t-1))^2 - 2\mu_1 r_1^2(t) = 2\sin^2(\pi/(N+1))\cos(2\pi/(N+1)) \geq C_9 > 0$.

We will verify that $\psi = (\mu C_8/C_9) r_1^2$ is a subsolution of (3.26) for μ large. Notice that

$$\begin{aligned} -\Delta^2 r_1^2(t-1) &= 2r_1^2(t) - r_1^2(t+1) - r_1^2(t-1) \\ &= 2r_1^2(t) - (r_1(t) + \Delta r_1(t))^2 - (r_1(t) - \Delta r_1(t-1))^2 \\ &= 2\mu_1 r_1^2(t) - \left(\Delta r_1(t)\right)^2 - \left(\Delta r_1(t-1)\right)^2. \end{aligned} \tag{3.28}$$

On the other hand, for $t \in Q_1$, we have $(\Delta r_1(t))^2 + (\Delta r_1(t-1))^2 - 2\mu_1 r_1^2(t) \geq C_9$, which implies that

$$\frac{C_8}{C_9}\left[2\mu_1 r_1^2(t) - \left(\Delta r_1(t)\right)^2 - \left(\Delta r_1(t-1)\right)^2\right] - g(\psi(t)) \leq 0. \tag{3.29}$$

Then for $t \in Q_1$, $-\Delta^2\psi(t-1) \leq \mu g(\psi(t))$. Next, for $t \in \mathbb{Z}(1,N) \setminus Q_1$, we have $r_1(t) \geq \overline{r}$ for some $\overline{r} > 0$ and $(C_8/C_9) r_1^2(t) \geq C_{10}$ for some $C_{10} = (C_8/C_9)\overline{r}^2 > 0$. Hence $\psi(t) = (\mu C_8/C_9) r_1^2(t) \geq \mu C_{10}$. Since g is increasing and eventually strictly positive, there is a $\mu^* > 0$ such that if $\mu > \mu^*$ and $t \in \mathbb{Z}(1,N) \setminus Q_1$,

$$g(\psi(t)) \geq \frac{C_8}{C_9} \cdot 2\mu_1 \geq \frac{C_8}{C_9}\left[2\mu_1 r_1^2(t) - \left(\Delta r_1(t)\right)^2 - \left(\Delta r_1(t-1)\right)^2\right]. \tag{3.30}$$

Hence for $t \in \mathbf{Z}(1,N) \setminus Q_1$, $-\Delta^2\psi(t-1) \le \mu g(\psi(t))$. Notice that $r_1(0)=0, r_1(N+1)=0$. Then $\psi(0)=0, \psi(N+1)=0$. So we have

$$
\begin{gathered}
-\Delta^2\psi(t-1) \le \mu g(\psi(t)), \quad t \in \mathbb{Z}(1,N), \\
\psi(0) \le 0, \quad \psi(N+1) \le 0,
\end{gathered}
\tag{3.31}
$$

that is, ψ is a subsolution of (3.26).

Now we look for the supersolution of (3.26). Let z be a solution of

$$
\begin{gathered}
-\Delta^2 u(t-1) = 1, \quad t \in \mathbb{Z}(1,N), \\
u(0) = 0, \quad u(N+1) = 0.
\end{gathered}
\tag{3.32}
$$

Then $z(s) = \sum_{t=1}^{N} G(s,t) = (1/(N+1))\{\sum_{t=1}^{s-1}[(N+1)-s]t + \sum_{t=s}^{N} s[(N+1)-t]\} = s[(N+1)-s]/2$, where

$$
G(s,t) = \begin{cases} \dfrac{t[(N+1)-s]}{N+1}, & 0 \le t \le s-1, \\[2ex] \dfrac{s[(N+1)-t]}{N+1}, & s \le t \le N+1. \end{cases}
\tag{3.33}
$$

Clearly, $z(s) > 0$ for $s \in \mathbb{Z}(1,N)$, $z(0) = 0, z(N+1) = 0$. Define $\phi = \mu\sigma z$, where $\sigma > 0$ is large enough so $\phi > \psi$ in $\mathbb{Z}(1,N)$ and

$$
\frac{g(\mu\sigma z)}{\sigma} < 1.
\tag{3.34}
$$

This is possible since g is a sublinear function. So

$$
\begin{gathered}
-\Delta^2\phi(t-1) \ge \mu g(\phi(t)), \quad t \in \mathbb{Z}(1,N), \\
\phi(0) \ge 0, \quad \phi(N+1) \ge 0,
\end{gathered}
\tag{3.35}
$$

which shows that ϕ is a supersolution of (3.26). Therefore, by Lemma 2.8, there is a solution $\underline{u}$ of (3.26) such that $\psi \le \underline{u} \le \phi$.

Secondly, we will prove that $\underline{u}$ is a subsolution of (1.1). Since $\lambda > 0$ and $f > 0$, it follows that

$$
\begin{gathered}
-\Delta^2\underline{u}(t-1) \le \lambda f(\underline{u}(t)) + \mu g(\underline{u}(t)), \quad t \in \mathbb{Z}(1,N), \\
\underline{u}(0) \le 0, \quad \underline{u}(N+1) \le 0,
\end{gathered}
\tag{3.36}
$$

which implies that $\underline{u}$ is a subsolution of (1.1).

Lastly, we will look for the supersolution of (1.1) and prove the existence of positive solution of (1.1). Let z be as in (3.32). Notice that g is sublinear. Define $\overline{u} = \xi z$, where $\xi > 0$ is independent of λ and large enough so that $\overline{u} \geq \underline{u}$ in $\mathbb{Z}(1, N)$ and

$$\mu \frac{g(\xi z(t))}{\xi} < \frac{1}{2}. \tag{3.37}$$

Let $\lambda > 0$ be so small that

$$\lambda \frac{f(\xi z(t))}{\xi} < \frac{1}{2}. \tag{3.38}$$

Then

$$\begin{gathered} -\Delta^2 \overline{u}(t-1) = \xi \geq \lambda f(\overline{u}(t)) + \mu g(\overline{u}(t)), \quad t \in \mathbb{Z}(1, N), \\ \overline{u}(0) \geq 0, \quad \overline{u}(N+1) \geq 0. \end{gathered} \tag{3.39}$$

Hence $\overline{u}$ is a supersolution of (1.1). Thus, by Remark 2.9, problem (1.1) has a solution $\tilde{u}$ such that $\underline{u} \leq \tilde{u} \leq \overline{u}$ for $\mu > \mu^*$ and λ small, which is positive for $t \in \mathbb{Z}(1, N)$.

Now we are going to find the second positive solution of problem (1.1). Notice that $\underline{u}$ and $\overline{u}$ are independent of λ. Since f is positive on $[0, +\infty)$, by the definition of f_1 we have $\sum_{t=1}^{N} F_1(u(t)) \geq 0$. Then for $u \in [\underline{u}, \overline{u}]$,

$$\begin{aligned} J(u) &= \frac{1}{2} u^T A u - \lambda \sum_{t=1}^{N} F_1(u(t)) - \mu \sum_{t=1}^{N} G_1(u(t)) \\ &\leq \frac{1}{2} u^T A u - \mu \sum_{t=1}^{N} G_1(u(t)) \leq J_0, \end{aligned} \tag{3.40}$$

where $J_0 = \max_{u \in [\underline{u}, \overline{u}]} ((1/2) u^T A u - \mu \sum_{t=1}^{N} G_1(u(t)))$. On the other hand, by Lemma 3.6, we can take appropriate $\overline{\lambda}$ such that if $\lambda \in (0, \overline{\lambda})$, then $J(u) \geq (\eta_{\min} M_1^2 / 16) \lambda^{-2/(\alpha-1)} > J_0 + 1$ for $\|u\| = \rho$. Hence by Theorem 1.1, $J(\hat{u}) > J_0$. So $\hat{u} \notin [\underline{u}, \overline{u}]$ and $\hat{u} \neq \tilde{u}$, which shows that $\hat{u}$ and $\tilde{u}$ are two different positive solutions of (1.1). The proof is complete. □

Proof of Theorem 1.5. Just to be on the contradiction side, let u be a positive solution of (1.1). Since f is superlinear and increasing, $f(0) > 0$, there are $C_{11}, C_{12} > 0$ such that for $s \geq 0$, $f(s) \geq C_{11} s + C_{12}$. Hence for $\lambda > 0$ and $s \geq 0$, $\lambda f(s) + \mu g(s) \geq \lambda (C_{11} s + C_{12}) + \mu G_m$, where G_m is the same as that of the proof of Lemma 3.7. If λ is large enough, then $\lambda C_{12} + \mu G_m \geq (1/2) \lambda C_{12}$. Therefore $\lambda f(s) + \mu g(s) \geq \lambda C_{11} s + (1/2) \lambda C_{12}$ for large $\lambda > 0$ and $s \geq 0$. Multiplying both sides of

$$-\Delta^2 y_1(t-1) = \lambda_1 y_1(t) \tag{3.41}$$

by $u(t)$ and summing it from 1 to N, we get

$$\sum_{t=1}^{N}\left(-\Delta^2 y_1(t-1)\right)u(t)=\sum_{t=1}^{N}\lambda_1 y_1(t)u(t). \tag{3.42}$$

Multiplying both sides of (1.1) by $y_1(t)$ and summing it from 1 to N, we have

$$\sum_{t=1}^{N}\left(-\Delta^2 u(t-1)\right)y_1(t)=\sum_{t=1}^{N}(\lambda f(u(t))+\mu g(u(t)))y_1(t). \tag{3.43}$$

It is easy to see that

$$\sum_{t=1}^{N}\left(-\Delta^2 u(t-1)\right)y_1(t)=\sum_{t=1}^{N}\left(-\Delta^2 y_1(t-1)\right)u(t). \tag{3.44}$$

Hence

$$\begin{gathered}\sum_{t=1}^{N}\lambda_1 y_1(t)u(t)=\sum_{t=1}^{N}(\lambda f(u(t))+\mu g(u(t)))y_1(t),\\ \sum_{t=1}^{N}\lambda_1 u(t)y_1(t)\geq\sum_{t=1}^{N}\left(\lambda C_{11}u(t)+\frac{1}{2}\lambda C_{12}\right)y_1(t),\\ \sum_{t=1}^{N}(\lambda_1-\lambda C_{12})u(t)y_1(t)\geq\sum_{t=1}^{N}\frac{1}{2}\lambda C_{12}y_1(t).\end{gathered} \tag{3.45}$$

For $\lambda>\lambda_1/C_{12}$, we obtain a contradiction. So for a given $\mu>0$, (1.1) has no positive solution if λ is large. The proof is complete. □

Example 3.8. We give an example to illustrate the result of Theorem 1.1. Let $f(u)=u^3+1$ and $g(u)=(u-1)^{2/3}-2$. Clearly, f and g satisfy the conditions of Theorem 1.1. Then problem (1.1) has at least a positive solution.

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