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Research Article

On the Difference Equation $x_{n+1} = (\alpha x_n + \beta x_{n-1})e^{-x_n}$

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We study a discrete delay *Mosquito population equation*. Firstly, we study the stability of the equilibria of the system and the existence of period-two bifurcation by analyzing the characteristic equation. Secondly, the direction and stability of the bifurcation are determined by using the normal form theory. Finally, some computer simulations are performed to illustrate the analytical results found.

1. Introduction and preliminaries

Recently, there has been a great interest in studying nonlinear difference equations and systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economy, psychology, sociology, and so forth. Such equations also appear naturally as discrete analogues of differential equations which model various biological and economical systems [1–4]. In this paper, we study the following discrete delay *Mosquito population equation* [1]:

$$x_{n+1} = \left(\alpha x_n + \beta x_{n-1}\right)e^{-x_n}, \quad x_0, x_1 > 0,\ n = 1, 2, 3, \ldots, \tag{1.1}$$

where

$$\alpha \in (0, 1), \qquad \beta \in (0, \infty). \tag{1.2}$$

The equilibrium points of (1.1) are solutions of the following equation

$$x^* = \left(\alpha x^* + \beta x^*\right)e^{-x^*}. \tag{1.3}$$

It is easy to see that $x^* = 0$ is always a equilibrium to (1.1), and (1.1) has an unique positive equilibrium $x^* = \ln(\alpha + \beta)$, when $\alpha + \beta > 1$.

By the well-known linear stability theorem, it is easy to know that the zero equilibrium of (1.1) is asymptotically stable when $\alpha + \beta < 1$ (see [1–3]), and unstable when $\alpha + \beta > 1$, and a fold bifurcation takes place when $\alpha + \beta = 1$.

But "a question " of mathematics and biology is whether stable and sustained oscillation possible for (1.1), when $\alpha+\beta > 1$, increases. In the present paper, we provide a detailed analysis of these questions. Regarding β as a parameter, by analyzing the characteristic equation and applying the local Hopf theory (see, e.g., Kuznetsov [5] or Wiggins [6]), we investigate the stability of the equilibria and existence of period-two bifurcation. More specifically, we give a bifurcation set in (α, β)-plane, from which one can see how the parameters α and β affect the dynamics of (1.1). Furthermore, using the normal form theory, we drive a formula for determining the direction of the period-two bifurcation and the stability of the period-two solution bifurcation from the positive equilibrium E^*.

2. Stability and existence of bifurcation

Set $u_n = x_n, v_n = x_{n-1}$, then (1.1) becomes

$$\begin{aligned} u_{n+1} &= \left(\alpha u_n + \beta v_n\right) e^{-u_n}, \\ v_{n+1} &= u_n, \end{aligned} \tag{2.1}$$

which, in turn, defines the two-dimensional discrete-time dynamical system,

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} (\alpha u + \beta v) e^{-u} \\ u \end{pmatrix} = G(U, \alpha, \beta), \tag{2.2}$$

where $U = (u, v)^T$. The map always has the fixed point $E^0 = (x^0, x^0)^T = (0, 0)^T$. For $\alpha + \beta > 1$, a unique nontrivial positive fixed point $E^* = (x^*, x^*)^T$ appears, with the coordinates

$$x^* = \ln(\alpha + \beta). \tag{2.3}$$

The Jacobian matrix of the map (2.2) evaluated at the nontrivial fixed point is given by

$$dG(E^*, \alpha, \beta) = \begin{pmatrix} \dfrac{\alpha}{\alpha + \beta} - x^* & \dfrac{\beta}{\alpha + \beta} \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

with the characteristic equation

$$\lambda^2 - \left(\frac{\alpha}{\alpha + \beta} - x^*\right)\lambda - \frac{\beta}{\alpha + \beta} = 0. \tag{2.5}$$

Regarding β as a parameter, it is easy to know that the equation

$$\ln(\alpha + \beta) = \frac{2\alpha}{\alpha + \beta} \tag{2.6}$$

has the unique solution $\beta = \beta_0(\alpha) = \beta_0$.

Theorem 2.1. *Suppose that* $\alpha + \beta > 1$.

(1) *If* $\beta < \beta_0(\alpha)$, *then* E^* *is asymptotically stable.*

(2) *If* $\beta > \beta_0(\alpha)$, *then* E^* *is unstable.*

(3) *The bifurcation of a period-two solution occurs at* $\beta = \beta_0(\alpha)$, *that is, system* (2.1) *has a unique period-two solution bifurcating from the equilibrium* E^*.

Proof. By the linear stability theorem, we know that the necessary and sufficient condition for both roots of (2.5) to have absolute value less than one is $\ln(\alpha+\beta) < 2\alpha/(\alpha+\beta)$, that is, $\beta < \beta_0(\alpha)$, so the stability statements are true.

The next proof shows the existence of a period-two solution. Let $u_n = u'_n + x^*$, $v_n = v'_n + x^*$, then there are

$$u'_{n+1} = \left[\alpha u'_n + \beta v'_n + (\alpha+\beta)x^*\right] e^{-x^*-u'_n} - x^*, \qquad u'_{n+1} = v'_n. \tag{2.7}$$

By (2.4) and (2.6), we know that when $\beta = \beta_0$, the Jacobian of the new map at $\overline{U} = (0,0)^T$ is

$$A = dG(\overline{U}, \alpha, \beta_0) = \begin{pmatrix} \dfrac{-\alpha}{\alpha+\beta_0} & \dfrac{\beta_0}{\alpha+\beta_0} \\ 1 & 0 \end{pmatrix} \tag{2.8}$$

and has eigenvalues -1 and $\beta_0/(\alpha+\beta_0)$. The eigenvalue -1 whit corresponding eigenvector $Y = (1,-1)^T$. Note that 1 is not an eigenvalue of A.

A straightforward calculation shows that

$$\text{Range}\,(I+A) = \text{span}\left(\frac{\beta_0}{\alpha+\beta_0}, 1\right)^T. \tag{2.9}$$

Now,

$$\begin{aligned} \frac{d}{d\beta}\left(dG(\overline{U}, \alpha, \beta)\right)\Big|_{\beta=\beta_0} &= \begin{pmatrix} \dfrac{-2\alpha-\beta_0}{(\alpha+\beta_0)^2} & \dfrac{-\beta_0}{(\alpha+\beta_0)^2} \\ 0 & 0 \end{pmatrix}, \\ \frac{d}{d\beta}\left(dG(\overline{U}, \alpha, \beta)\right)\Big|_{\beta=\beta_0} Y &= \left(\frac{-2\alpha}{(\alpha+\beta_0)^2}, 0\right)^T \notin \text{span}\left(\frac{\beta_0}{\alpha+\beta_0}, 1\right)^T. \end{aligned} \tag{2.10}$$

By the period-doubling bifurcation theorem (Stuart and Humphries, [7, page 41, Theorem 1.4.5]), the bifurcation of a period-two solution occurs. □

3. Direction of bifurcation of the period-two cycle

In the previous section, we have shown that the system (2.1) undergoes a period-two bifurcation at the positive equilibrium E^* when $\beta = \beta_0$. In this section, by using the normal form method for discrete system introduced by Kuznetsov [5] or Wiggins [6], we will study the direction and stability of the period-two bifurcation.

We can write system (2.2) as

$$U \longmapsto AU + F(U), \quad U \in \Re^2, \tag{3.1}$$

where $F(U) = O(\|U\|^2)$ is a smooth function. As before, its Taylor expansion is represented in the form

$$F(U) = \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U) + O(\|U\|^4), \tag{3.2}$$

where

$$\begin{aligned} B(U,U) &= \left(b_0(U,U),0\right)^T, \\ C(U,U,U) &= \left(c_0(U,U,U),0\right)^T, \end{aligned} \tag{3.3}$$

$$\begin{aligned} b_0(\phi,\psi) &= \frac{-\beta_0}{\alpha+\beta_0}\left(\phi_1\psi_2 + \phi_2\psi_1\right), \\ c_0(\phi,\psi,\eta) &= \frac{\alpha}{\alpha+\beta_0}\left(\phi_1\psi_1\eta_1\right) + \frac{\beta_0}{\alpha+\beta_0}\left(\phi_1\psi_2\eta_1 + \phi_1\psi_1\eta_2 + \phi_2\psi_1\eta_1\right). \end{aligned} \tag{3.4}$$

Let $q \in \Re^2$ is the eigenvector of A with eigenvalue -1, let $p \in \Re^2$ be the adjoint eigenvector, that is, $A^T p = -p$, where A^T is the transposed matrix. So, from (2.6), we know that $q = (1,-1)^T$, and $p = D(1,-\beta_0/(\alpha+\beta_0))^T$.

Normalize p with respect to q such that $\langle p,q\rangle = 1$, where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\Re^2$, we have

$$q = (1,-1)^T, \qquad p = \frac{\alpha+\beta_0}{\alpha+2\beta_0}\left(1, \frac{-\beta_0}{\alpha+\beta_0}\right)^T. \tag{3.5}$$

Let W^{su} denote a linear eigenspace of A corresponding to all eigenvalues other than -1, we know that $y \in W^{su}$ if and only if $\langle p,y\rangle = 0$.

Now, we can decompose any vector $U \in \Re^2$ as

$$\begin{aligned} U &= zq + y, \\ z &= \langle p,U\rangle, \\ y &= U - \langle p,U\rangle q. \end{aligned} \tag{3.6}$$

In the coordinates (z,y), the map (3.1) can be written as

$$\begin{aligned} \tilde{z} &= -z + \langle p, F(zq+y)\rangle, \\ \tilde{y} &= Ay + F(zq+y) - \langle p, F(zq+y)\rangle q. \end{aligned} \tag{3.7}$$

Using Taylor expansion, we can write (3.7) in the form as following:

$$\begin{aligned} \tilde{z} &= -z + \frac{1}{2}\sigma z^2 + z\langle\overline{\alpha}, y\rangle + \frac{1}{6}\delta z^3 + \cdots, \\ \tilde{y} &= Ay + \frac{1}{2}\overline{\beta}z^2 + \cdots, \end{aligned} \tag{3.8}$$

where $z \in \Re$, $y \in \Re^2$, $\sigma, \delta \in \Re$, $\overline{\alpha}, \overline{\beta} \in \Re^2$. σ, δ, and $\overline{\beta}$ are given as following:

$$\sigma = \langle p, B(q,q)\rangle, \qquad \delta = \langle p, C(q,q,q)\rangle, \qquad \overline{\beta} = B(q,q) - \langle p, B(q,q)\rangle q, \tag{3.9}$$

and the scalar product $\langle \overline{\alpha}, y\rangle$ can be expressed as

$$\langle \overline{\alpha}, y\rangle = \langle p, B(q,y)\rangle. \tag{3.10}$$

The center manifold of (3.8) has the representation

$$y = V(z) = \frac{1}{2}\omega_2 z^2 + O(z^3). \tag{3.11}$$

Substituting this expansion into the second equation of (3.8), using the first equation and the invariance of the center manifold, we get the following linear equation for ω_2:

$$(A - E)\omega_2 + \overline{\beta} = 0. \tag{3.12}$$

The matrix $(A - E)$ is invertible because $\lambda = 1$ is not an eigenvalue of A. Thus, (3.12) can be solved directly giving

$$\omega_2 = -(A - E)^{-1}\overline{\beta}, \tag{3.13}$$

and the restriction of (3.8) to the center manifold takes the form

$$\widetilde{z} = -z + \frac{1}{2}\sigma z^2 + \frac{1}{6}\left(\delta - 3\left\langle p, B\left(q, (A - E)^{-1}\overline{\beta}\right)\right\rangle\right)z^3 + O(z^4). \tag{3.14}$$

Using (3.9), we can write the restricted map as

$$\widetilde{z} = -z + a(0)z^2 + b(0)z^3 + O(z^4), \tag{3.15}$$

with

$$\begin{aligned} a(0) &= \frac{1}{2}\langle p, B(q,q)\rangle, \\ b(0) &= \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{4}\left(\langle p, B(q,q)\rangle\right)^2 - \frac{1}{2}\left\langle p, B\left(q, (A - E)^{-1}B(q,q)\right)\right\rangle. \end{aligned} \tag{3.16}$$

The map (3.15) can be transformed to the normal form

$$\widetilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4), \tag{3.17}$$

where

$$c(0) = a^2(0) + b(0). \tag{3.18}$$

Thus, the critical normal form coefficient $c(0)$, that determines the nondegeneracy of the flip bifurcation and allows us to predict the direction of bifurcation of the period-two cycle, is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\left\langle p, B\left(q, (A - E)^{-1}B(q,q)\right)\right\rangle. \tag{3.19}$$

From (3.2), (3.4), and (3.5), we get

$$c(0) = \frac{\alpha - 3\beta_0}{6(\alpha + 2\beta_0)}. \tag{3.20}$$

Because $\alpha - 3\beta_0 < 0$, so we get $c(0) < 0$.

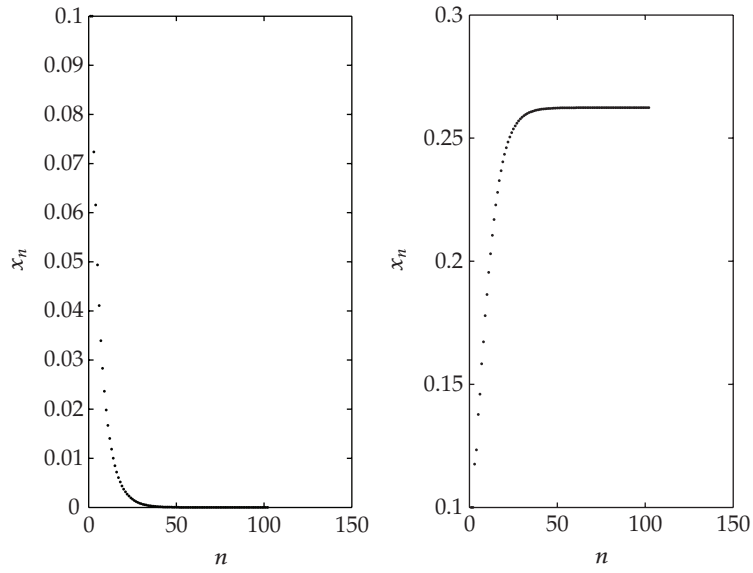

Figure 1: $x_0 = 0.1$, $x_1 = 0.1$. (a) $\alpha = 0.5$, $\beta = 0.3$. $\alpha + \beta < 1$; (b) $\alpha = 0.5$, $\beta = 0.8$. $\alpha + \beta > 1$ and $\beta < \beta_0$.

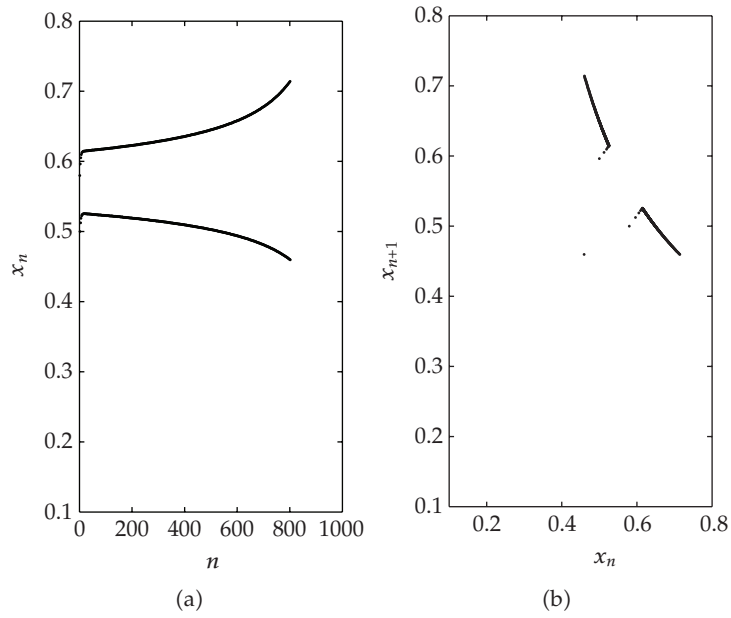

Figure 2: $\alpha = 0.5$, $\beta = 1.264$, $\beta > \beta_0$. $x_0 = 0.1$, $x_1 = 0.1$.

A general result for the direction and stability of period-two bifurcation; see for example, Wiggins [6, Chapter 3, Section 3.2, Theorem 3.2.3]. In fact, we have the following result.

Theorem 3.1. *A period-two bifurcation of* (2.1) *at* $\beta = \beta_0$ *occurs, and the unique period-two solution bifurcating from* E^* *is unstable.*

4. Numerical test

To illustrate the analytical results found, let us consider the following particular case of system (2.1). We have carried out numerical simulations on system (2.1) using Matlab with these parameter values, and for different β.

For instance, if the parameter values are chosen as $\alpha = 0.5$ and $\beta = 0.3$, we have $\alpha + \beta < 1$, then the zero solution is asymptotically stable (see Figure 1(a)). If the parameter values are chosen as $\alpha = 0.5$ and $\beta = 0.8$, we have $\alpha + \beta > 1$, then the zero solution is unstable and a new equilibrium appears. By Theorem 2.1 we know that if the parameter values are chosen as $\beta < \beta_0$, the positive equilibrium is asymptotically stable (see Figure 1(b)).

By Theorem 2.1 we know that if the parameter values are chosen as $\beta > \beta_0$, the positive equilibrium is asymptotically stable. If the parameter values are chosen such that $\beta > \beta_0$, the positive equilibrium is unstable, and the bifurcation takes place when β crosses $\beta_0(\ln(\alpha + \beta_0) = 2\alpha/(\alpha + \beta_0))$ to the right. By Theorem 3.1, the bifurcating period-two solution is unstable (see Figure 2).

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