Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 128602, 27 pages
doi:10.1155/2009/128602

Research Article

Global Behavior of Solutions to Two Classes of Second-Order Rational Difference Equations

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Received 9 December 2008; Accepted 7 July 2009

Recommended by Ondrej Dosly

For nonnegative real numbers α, β, γ, A, B, and C such that $B+C>0$ and $\alpha+\beta+\gamma>0$, the difference equation $x_{n+1}=(\alpha+\beta x_n+\gamma x_{n-1})/(A+Bx_n+Cx_{n-1})$, $n=0,1,2,\dots$ has a unique positive equilibrium. A proof is given here for the following statements: (1) *For every choice of positive parameters* α, β, γ, A, B, and C, *all solutions to the difference equation* $x_{n+1}=(\alpha+\beta x_n+\gamma x_{n-1})/(A+Bx_n+Cx_{n-1})$, $n=0,1,2,\dots,x_{-1},x_0\in[0,\infty)$ *converge to the positive equilibrium or to a prime period-two solution*. (2) *For every choice of positive parameters* α, β, γ, B, and C, *all solutions to the difference equation* $x_{n+1}=(\alpha+\beta x_n+\gamma x_{n-1})/(Bx_n+Cx_{n-1})$, $n=0,1,2,\dots$, x_{-1}, $x_0\in(0,\infty)$ *converge to the positive equilibrium or to a prime period-two solution*.

1. Introduction and Main Results

In their book [1], Kulenović and Ladas initiated a systematic study of the difference equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\dots, \tag{1.1}$$

for nonnegative real numbers $\alpha,\ \beta,\ \gamma,\ A,\ B,$ and C such that $B+C>0$ and $\alpha+\beta+\gamma>0$, and for nonnegative or positive initial conditions $x_{-1},\ x_0$. Under these conditions, (1.1) has a unique positive equilibrium. One of their main ideas in this undertaking was to make the task more manageable by considering separate cases when one or more of the parameters in (1.1) is zero. The need for this strategy is made apparent by cases such as the well-known *Lyness Equation* [2–4].

$$x_{n+1}=\frac{\alpha+x_n}{x_{n-1}}, \tag{1.2}$$

whose dynamics differ significantly from other equations in this class. There are a total of 42 cases that arise from (1.1) in the manner just discussed, under the hypotheses $B + C > 0$ and $\alpha + \beta + \gamma > 0$. The recent publications [5, 6] give a detailed account of the progress up to 2007 in the study of dynamics of the class of equations (1.1). After a sustained effort by many researchers (for extensive references, see [5, 6]), there are some cases that have resisted a complete analysis. We list them as follows in normalized form, as presented in [5, 6]:

$$x_{n+1} = \frac{\alpha + x_n}{A + x_{n-1}}, \tag{1.3}$$

$$x_{n+1} = \frac{\alpha + x_n}{x_n + Cx_{n-1}}, \tag{1.4}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{x_{n-1}}, \tag{1.5}$$

$$x_{n+1} = \frac{\alpha + x_n}{A + Bx_n + x_{n-1}}, \tag{1.6}$$

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{A + Bx_n + x_{n-1}}, \tag{1.7}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_{n-1}}, \tag{1.8}$$

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-1}}{Bx_n + x_{n-1}}, \tag{1.9}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + Bx_n + x_{n-1}}. \tag{1.10}$$

The dynamics of (1.7) has been settled recently in [7, 8]. Global attractivity of the positive equilibrium of (1.3) has been proved recently in [9]. Since (1.6) can be reduced to (1.3) through a change of variables [10], global behavior of solutions to (1.6) is also settled. Equation (1.5) is another equation that can be reduced to (1.3), through the change of variables $x_n = y_n + \gamma$ [11].

Ladas and coworkers [1, 5, 6] have posed a series of conjectures on these equations. One of them is the following.

Conjecture 1.1 (Ladas et al.). *For* (1.9) *and* (1.10)*, every solution converges to the positive equilibrium or to a prime period-two solution.*

In this article, we prove this conjecture. Our main results are the following.

Theorem 1.2. *For every choice of positive parameters α, β, γ, A, B, and C, all solutions to the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots, \; x_{-1}, x_0 \in [0, \infty) \tag{1.11}$$

converge to the positive equilibrium or to a prime period-two solution.

Theorem 1.3. *For every choice of positive parameters $\alpha,\ \beta,\ \gamma,\ B$, and C, all solutions to the difference equation*

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}},\quad n=0,1,2,\ldots,\ x_{-1},x_0\in(0,\infty) \tag{1.12}$$

converge to the positive equilibrium or to a prime period-two solution.

A reduction of the number of parameters of (1.12) is obtained with the change of variables $x_n=(\gamma/C)y_n$, which yields the equation

$$y_{n+1}=\frac{r+py_n+y_{n-1}}{qy_n+y_{n-1}},\quad n=0,1,2,\ldots,\ y_{-1},y_0\in(0,\infty), \tag{1.13}$$

where $r=\alpha C/\gamma^2$, $p=\beta/\gamma$, and $q=B/C$.

The number of parameters of (1.11) can also be reduced, which we proceed to do next. Consider the following affine change of variables which is helpful to reduce number of parameters and simplify calculations:

$$x_n=\left(\frac{\gamma}{C}+\frac{A}{B+C}\right)y_n-\frac{A}{B+C}. \tag{1.14}$$

With (1.14), (1.11) may now be rewritten as

$$y_{n+1}=\frac{r+py_n+y_{n-1}}{qy_n+y_{n-1}},\quad n=0,1,2,\ldots,\ y_{-1},y_0\in[L,\infty), \tag{1.15}$$

where

$$\begin{gathered}p=\frac{AB+(B+C)\beta}{AC+(B+C)\gamma},\\ q=\frac{B}{C},\\ r=\frac{C(B+C)\left(B\alpha+C\alpha-A\beta-A\gamma\right)}{\left(\gamma(B+C)+AC\right)^2},\\ L=\frac{AC}{AC+(B+C)\gamma}.\end{gathered} \tag{1.16}$$

Theorems 1.2 and 1.3 can be reformulated in terms of the parameters p, q, and r as follows.

Theorem 1.4. *Let $\alpha,\ \beta,\ \gamma,\ A,\ B$, and C be positive numbers, and let $p,\ q,\ r$, and L be given by relations* (1.16). *Then every solution to* (1.15) *converges to the unique equilibrium or to a prime period-two solution.*

Theorem 1.5. *Let p, q, and r be positive numbers. Then every solution to* (1.13) *converges to the unique equilibrium or to a prime period-two solution.*

In this paper we prove Theorems 1.4 and 1.5; Theorems 1.2 and 1.3 follow as an immediate corollary.

The two main differences between (1.15) and (1.13) are the set of initial conditions, and the possibility of having a negative value of r in (1.15), while only positive values of r are allowed in (1.13). Nevertheless, for both (1.15) and (1.13) the unique equilibrium has the formula:

$$\overline{y} = \frac{p+1+\sqrt{(p+1)^2+4r(q+1)}}{2(q+1)}. \tag{1.17}$$

Although it is not possible to prove Theorem 1.2 as a simple corollary to Theorem 1.3, the changes of variables leading to Theorems 1.4 and 1.5 will result in proofs to the former theorems that are greatly simplified.

Our main results Theorems 1.2 and 1.3 imply that when prime period-two solutions to (1.11) or (1.13) do not exist, then the unique equilibrium is a global attractor. We have not treated here certain questions about the global dynamics of (1.11) and (1.13), such as the character of the prime period-two solutions to either equation, or even for more general rational second-order equations, when such solutions exist. This matter has been treated in [12].

This work is organized as follows. The main results are stated in Section 1. Results from literature which are used here are given in Section 2 for convenience. In Section 3, it is shown that either every solution to (1.15) converges to the equilibrium or there exists an invariant and attracting interval I with the property that the function $f(x,y)$ associated with the difference equation is coordinatewise strictly monotonic on $I \times I$. In Section 4, a global convergence result is obtained for (1.13) over a specific range of parameters and for initial conditions in an invariant compact interval. Theorem 1.4 is proved in Section 5, and the proof of Theorem 1.5 is given in Section 6. Tables 1 and 2 include computer algebra system code for performing certain calculations that involve polynomials with a large number of terms (over 365 000 in one case). These computer calculations are used to support certain statements in Section 4. Finally, we refer the reader to [1] for terminology and definitions that concern difference equations.

2. Results from Literature

The results in this subsection are from literature, and they are given here for easy reference. The first result is a reformulation of [1, Theorems 1.4.5–1.4.8].

Theorem 2.1 (see [1, 13]). *Suppose that a continuous function $f : [a,b]^2 \to [a,b]$ satisfies one of (i)–(iv):*

(i) *$f(x,y)$ is nondecreasing in x, y, and*

$$\forall (m,M) \in [a,b]^2, \quad \bigl(f(m,m)=m,\ f(M,M)=M\bigr) \Longrightarrow m = M; \tag{2.1}$$

(ii) $f(x, y)$ *is nonincreasing in* x, y, *and*

$$\forall(m, M) \in [a, b]^2, \quad (f(m, m) = M,\ f(M, M) = m) \Longrightarrow m = M; \tag{2.2}$$

(iii) $f(x, y)$ *is nonincreasing in* x *and nondecreasing in* y, *and*

$$\forall(m, M) \in [a, b]^2, \quad (f(m, M) = M,\ f(M, m) = m) \Longrightarrow m = M; \tag{2.3}$$

(iv) $f(x, y)$ *is nondecreasing in* x *and nonincreasing in* y, *and*

$$\forall(m, M) \in [a, b]^2, \quad (f(M, m) = M,\ f(m, M) = m) \Longrightarrow m = M. \tag{2.4}$$

Then $y_{n+1} = f(y_n, y_{n-1})$ *has a unique equilibrium in* $[a, b]$, *and every solution with initial values in* $[a, b]$ *converges to the equilibrium.*

The following result is [1, Theorem A.0.8].

Theorem 2.2. *Suppose that a continuous function* $f : [a, b]^3 \to [a, b]$ *is nonincreasing in all variables, and*

$$\forall(m, M) \in [a, b]^3, \quad (f(m, m, m) = M,\ f(m, m, m) = M) \Longrightarrow m = M. \tag{2.5}$$

Then $y_{n+1} = f(y_n, y_{n-1}, y_{n-2})$ *has a unique equilibrium in* $[a, b]$, *and every solution with initial values in* $[a, b]$ *converges to the equilibrium.*

Theorem 2.3 (see [14]). *Let* I *be a set of real numbers, and let* $F : I \times I \to I$ *be a function* $F(u, v)$ *which decreases in* u *and increases in* v. *Then for every solution* $\{x_n\}_{n=-1}^{\infty}$ *of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{2.6}$$

the subsequences $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *of even and odd terms do exactly one of the following.*

(i) *They are both monotonically increasing.*

(ii) *They are both monotonically decreasing.*

(iii) *Eventually, one of them is monotonically increasing and the other is monotonically decreasing.*

Theorem 2.3 has this corollary.

Corollary 2.4 (see [14]). *If* I *is a compact interval, then every solution of* (2.6) *converges to an equilibrium or to a prime period-two solution.*

Theorem 2.5 (see [15]). *Assume the following conditions hold.*

(i) $h \in C[(0,\infty)\times(0,\infty),(0,\infty)]$.

(ii) $h(x,y)$ *is decreasing in* x *and strictly decreasing in* y.

(iii) $xh(x,x)$ *is strictly increasing in* x.

(iv) *The equation*

$$x_{n+1} = x_n h(x_n, x_{n-1}), \quad n = 0,1,\ldots \tag{2.7}$$

has a unique positive equilibrium $\overline{x}$.

Then $\overline{x}$ *is a global attractor of all positive solutions of* (2.7).

3. Existence of an Invariant and Attracting Interval

In this section we prove a proposition which is key for later developments. We will need the function

$$f(x,y) := \frac{r+px+y}{qx+y}, \quad x,y \in [L,\infty), \tag{3.1}$$

associated to (1.15).

Proposition 3.1. *At least one of the following statements is true.*

(A) *Every solution to* (1.15) *converges to the equilibrium.*

(B) *There exist* m^*, M^* *with* $L < m^* < M^*$ *such that the following is true.*

  (i) $[m^*,M^*]$ *is an invariant interval for* (1.15), *that is,* $f([m^*,M^*]\times[m^*,M^*]) \subset [m^*,M^*]$.

  (ii) *Every solution to* (1.15) *eventually enters* $[m^*,M^*]$.

  (iii) $f(x,y)$ *is coordinatewise strictly monotonic on* $[m^*,M^*]^2$.

The proof of Proposition 3.1 will be given at the end of the section, after we prove several lemmas. The next lemma states that the function $f(\cdot,\cdot)$ associated to (1.15) is bounded.

Lemma 3.2. *There exist positive constants* $\mathcal{L}$ *and* $\mathcal{U}$ *such that* $L < \mathcal{L}$ *and*

$$\mathcal{L} \le f(x,y) \le \mathcal{U}, \quad x,y \in [L,\infty). \tag{3.2}$$

In particular,

$$f([\mathcal{L},\mathcal{U}]\times[\mathcal{L},\mathcal{U}]) \subset [\mathcal{L},\mathcal{U}]. \tag{3.3}$$

Proof. The function

$$\tilde{f}(x,y) = \frac{\alpha+\beta x+\gamma y}{A+Bx+Cy}, \quad (x,y)\in(0,\infty)^2, \tag{3.4}$$

associated to (1.11) is bounded:

$$\frac{\min\{\alpha,\beta,\gamma\}}{\max\{A,B,C\}} \le \frac{\alpha+\beta x+\gamma y}{A+Bx+Cy} \le \frac{\max\{\alpha,\beta,\gamma\}}{\min\{A,B,C\}}, \quad (x,y)\in(0,\infty)^2. \tag{3.5}$$

Set $\tilde{\mathcal{L}} := \min\{\alpha,\beta,\gamma\}/\max\{A,B,C\}$ and $\tilde{\mathcal{U}} := \max\{\alpha,\beta,\gamma\}/\min\{A,B,C\}$. The affine change of coordinates (1.14) maps the rectangular region $[\tilde{\mathcal{L}},\tilde{\mathcal{U}}]^2$ onto a rectangular region $[\mathcal{L},\mathcal{U}]^2$ which satisfies (3.2) and (3.3). □

Lemma 3.3. *If $p=q$, then every solution to* (1.15) *converges to the unique equilibrium.*

Proof. If $p=q$, then $D_1f(x,y) = (-pr/(px+y)^2)$ and $D_2f(x,y) = -(r/(px+y)^2)$. Thus, depending on the sign of r, the function $f(x,y)$ is either nondecreasing in both coordinates, or nonincreasing in both coordinates on $[L,\infty)$. By Lemma 3.2, all solutions $\{y_n\}_{n=-1}^{\infty}$ satisfy $y_n \in [\mathcal{L},\mathcal{U}]$ for $n\ge 1$. A direct algebraic calculation may be used to show that all solutions $(m,M)\in[\mathcal{L},\mathcal{U}]$ of either one of the systems of equations

$$\begin{aligned} M &= f(M,M),\\ m &= f(m,m), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} M &= f(m,m),\\ m &= f(M,M) \end{aligned} \tag{3.7}$$

necessarily satisfy $m=M$. In either case, the hypotheses (i) or (ii) of Theorem 2.1 are satisfied, and the conclusion of the lemma follows. □

We will need the following elementary result, which is given here without proof.

Lemma 3.4. *Suppose $q\neq p$. The function $f(x,y)$ has continuous partial derivatives on $(L,\infty)^2$, and*

(i) *$D_1f(x,y)=0$ if and only if $y = qr/(q-p)$, and $D_1f(x,y)>0$ if and only if $(p-q)\ y > qr$;*

(ii) *$D_2f(x,y)=0$ if and only if $x=-r/(p-q)$, and $D_2f(x,y)>0$ if and only if $(q-p)x>r$.*

We will need to refer to the values K_1 and K_2 where the partial derivatives of $f(x,y)$ change sign.

Definition 3.5. If $p\neq q$, set

$$K_1 := \frac{qr}{p-q}, \qquad K_2 := \frac{-r}{p-q}. \tag{3.8}$$

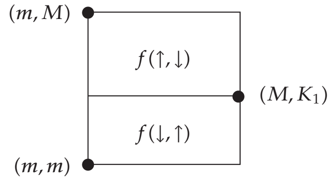

Figure 1: The arrows indicate type of coordinatewise monotonicity of $f(x,y)$ on each region.

Definition 3.6. For $L \le m \le M$, let

$$\phi(m,M) := \min\left\{f(x,y) : (x,y) \in [m,M]^2\right\},$$
$$\Phi(m,M) := \max\left\{f(x,y) : (x,y) \in [m,M]^2\right\}. \tag{3.9}$$

Lemma 3.7. *Suppose $p \ne q$. If $[m,M] \subset [\mathcal{L},\mathcal{U}]$ is an invariant interval for* (1.15) *with $m \le K_1 \le M$ or $m \le K_2 \le M$, then $m < \phi(m,M)$ or $\Phi(m,M) < M$ or $m = M = \overline{y}$.*

Proof. By definition of ϕ and Φ, $m \le \phi(m,M)$ and $\Phi(m,M) \le M$. Suppose

$$m = \phi(m,M), \qquad \Phi(m,M) = M. \tag{3.10}$$

The proof will be complete when it is shown that $m = M$. There are a total of four cases to consider: (a) $r \ge 0$ and $p > q$, (b) $r < 0$ and $p < q$, (c) $r \ge 0$ and $p < q$, and (d) $r < 0$ and $p > q$. We present the proof of case (a) only, as the proof of the other cases is similar.

If $r \ge 0$ and $p > q$, then $K_1 \in [m,M]$ and $K_2 \notin [m,M]$. Note that

$$[m,M] \times [m,M] = [m,M] \times [m,K_1] \bigcup [m,M] \times [K_1,M]. \tag{3.11}$$

By Lemma 3.4, the signs of the partial derivatives of $f(x,y)$ are constant on the interior of each of the sets $[m,M] \times [m,K_1]$ and $[m,M] \times [K_1,M]$, as shown in Figure 1.

Since $f(x,y)$ is nonincreasing in both x and y on $[m,M] \times [m,K_1]$,

$$f(M,K_1) \le f(x,y) \le f(m,m) \quad \text{for } (x,y) \in [m,M] \times [m,K_1]. \tag{3.12}$$

Similarly $f(x,y)$ is nondecreasing in x and nonincreasing in y on $[m,M] \times [K_1,M]$, hence

$$f(m,M) \le f(x,y) \le f(M,K_1) \quad \text{for } (x,y) \in [m,M] \times [K_1,M]. \tag{3.13}$$

From (3.12) and (3.13) one has

$$\phi(m,M) = f(m,M), \qquad \Phi(m,M) = f(m,m). \tag{3.14}$$

Combine (3.14) with relation (3.10) to obtain the system of equations

$$\begin{aligned} f(m, M) &= m, \\ f(m, m) &= M. \end{aligned} \tag{3.15}$$

Eliminating M from system (3.15) gives the cubic in m

$$q(q+1)m^3 + (1-pq)m^2 + (-1-p-qr)m - r = 0, \tag{3.16}$$

which has the roots

$$-\frac{1}{q}, \qquad \frac{1-p-\sqrt{(1+p)^2+4r(1+q)}}{2(1+q)}, \qquad \frac{1-p+\sqrt{(1+p)^2+4r(1+q)}}{2(1+q)}. \tag{3.17}$$

Only one root in the list (3.17) is positive, namely,

$$m = \frac{1-p+\sqrt{(1+p)^2+4r(1+q)}}{2(1+q)} = \overline{y}. \tag{3.18}$$

Substituting into one of the equations of system (3.15) one also obtains $M = \overline{y}$, which gives the desired relation $m = M = \overline{y}$. □

Definition 3.8. Let $m_0 := \mathcal{L}$, $M_0 := \mathcal{U}$, and for $\ell = 0, 1, 2, \ldots$ let $m_\ell := \phi(m_\ell, M_\ell)$, $M_\ell := \Phi(m_\ell, M_\ell)$.

By the definitions of m_ℓ, M_ℓ, $\phi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$, we have that $[m_{\ell+1}, M_{\ell+1}] \subset [m_\ell, M_\ell]$ for $\ell = 0, 1, 2, \ldots$. Thus the sequence $\{m_\ell\}$ is nondecreasing, and $\{M_\ell\}$ is nonincreasing. Let $m^* := \lim m_\ell$ and $M^* := \lim M_\ell$.

Lemma 3.9. *Suppose $p \neq q$. Either there exists $N \in \mathbb{N}$ such that $\{K_1, K_2\} \cap [m_N, M_N] = \emptyset$, or there exists $m^* = M^* = \overline{y}$.*

Proof. Arguing by contradiction, suppose $m^* < M^*$ and for all $\ell \in \mathbb{N}$, $\{K_1, K_2\} \cap [m_\ell, M_\ell] \neq \emptyset$. Since the intervals $[m_\ell, M_\ell]$ are nested and $\cap [m_\ell, M_\ell] = [m^*, M^*]$, it follows that $\{K_1, K_2\} \cap [m^*, M^*] \neq \emptyset$. By Lemma 3.7, we have

$$m^* < \phi(m^*, M^*) \quad \text{or} \quad \Phi(m^*, M^*) < M^*. \tag{3.19}$$

Continuity of the functions ϕ and Φ implies

$$\begin{aligned} \phi(m^*, M^*) &= \lim \phi(m_\ell, M_\ell) = \lim m_{\ell+1} = m^*, \\ \text{or} \quad \Phi(m^*, M^*) &= \lim \Phi(m_\ell, M_\ell) = \lim M_{\ell+1} = M^*. \end{aligned} \tag{3.20}$$

Statements (3.19) and (3.20) give a contradiction. □

Proof of Proposition 3.1. Suppose that statement (A) is not true. By Lemma 3.3, one must have $p \neq q$. Note that if $\{y_\ell\}$ is a solution to (1.15), then $y_{\ell+1} \in [m_\ell, M_\ell]$ for $\ell = 0, 1, 2, \ldots$. If $m^* = M^*$, since $m_\ell \to m^*$ and $M_\ell \to M^*$, we have $y_\ell \to \overline{y}$, but this is statement (A) which we are negating. Thus $m^* < M^*$, and by Lemma 3.9 there exists $N \in \mathbb{N}$ such that $\{K_1, K_2\} \cap [m_N, M_N] = \emptyset$; so $f(x, y)$ is coordinatewise monotonic on $[m_N, M_N]$. The set $[m_N, M_N]$ is invariant, and every solution enters $[m_N, M_N]$ starting at least with the term with subindex $N+1$. We have shown that if statement (A) is not true, then statement (B) is necessarily true. This completes the proof of the proposition. □

4. Equation (1.13) with $r \geq 0$, $p > q$ and $qr/(p-q) < p/q$

In this section we restrict our attention to the equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \; x_{-1}, x_0 \in (0, \infty), \tag{4.1}$$

where

$$f(x, y) := \frac{r + px + y}{qx + y}. \tag{4.2}$$

For $p > 0$, $q > 0$, and $r \geq 0$, (1.13) has a unique *positive* equilibrium

$$\overline{y} = \frac{p + 1 + \sqrt{(p+1)^2 + 4r(q+1)}}{2(q+1)}. \tag{4.3}$$

We note that if $I \subset (0, \infty)$ is an invariant compact interval, then necessarily $\overline{y} \in I$.

The goal in this section is to prove the following proposition, which will provide an important part of the proofs of Theorems 1.2 and 1.5.

Proposition 4.1. *Let p, q, and r be real numbers such that*

$$p > q > 0, \qquad r \geq 0, \qquad \frac{qr}{p-q} < \frac{p}{q}, \tag{4.4}$$

and let $[\widetilde{m}, \widetilde{M}] \subset (qr/(p-q), p/q)$ be a compact invariant interval for (1.13)*. Then every solution to* (1.13) *with $x_{-1}, x_0 \in [\widetilde{m}, \widetilde{M}]$ converges to the equilibrium.*

Proposition 4.1 follows from Lemmas 4.2, 4.3, and 4.5, which are stated and proved next.

Lemma 4.2. *Assume the hypotheses to Proposition 4.1. If either $q \geq 1$ or $p \leq 1$, then every solution to* (1.13) *with $x_{-1}, x_0 \in [\widetilde{m}, \widetilde{M}]$ converges to the equilibrium.*

Proof. We verify that hypothesis (iv) of Theorem 2.1 is true. Since $x > rq/(p-q)$ for $x \in [\tilde{m}, \widetilde{M}]$, the function $f(x,y)$ is increasing in x and decreasing in y for $(x,y) \in [\tilde{m}, \widetilde{M}]^2$ by Lemma 3.4. Let $m, M \in [\tilde{m}, \widetilde{M}]$ be such that $m \neq M$ and

$$\begin{aligned} f(M,m) - M &= 0, \\ f(m,M) - m &= 0. \end{aligned} \tag{4.5}$$

We show first that system (4.5) has no solutions if either $q \geq 1$ or $p \leq 1$. By eliminating denominators in both equations in (4.5),

$$\begin{aligned} m - mM + Mp - M^2q + r &= 0, \\ M - mM + mp - m^2q + r &= 0, \end{aligned} \tag{4.6}$$

and by subtracting terms in (4.6) one obtains

$$(M-m)\bigl(1 - p + q(m+M)\bigr) = 0. \tag{4.7}$$

Since $m \neq M$, we have $q(m+M) = p-1$, which implies that for $p \leq 1$ there are no solutions to system (4.5) which have both coordinates positive. Now assume $p > 1$; from (4.7), $m = (p-1-Mq)/q$, and substitute the latter into (4.5) to see that $x = M$ is a solution to the quadratic equation

$$(1-q)x^2 + \frac{(-1+p)(-1+q)}{q}x + \frac{-1+p+qr}{q} = 0. \tag{4.8}$$

By a symmetry argument, one has that $x = m$ is also a solution to (4.8). By inspection of the coefficients of the polynomial in the left-hand side of (4.8) one sees that two positive solutions are possible only when $q < 1$. To get the conclusion of the lemma, note that the fact that (4.5) has no solutions with $m \neq M$ is just hypothesis (iv) of Theorem 2.1. □

Lemma 4.3. *Assume the hypotheses to Proposition 4.1. If*

$$r \leq p^2q - p, \tag{4.9}$$

then every solution to (1.13) *with* $x_{-1}, x_0 \in [\tilde{m}, \widetilde{M}]$ *converges to the equilibrium.*

Proof. By substituting $x_n = f(x_{n-1}, x_{n-2})$ into $x_{n+1} = f(x_n, x_{n-1})$ we obtain

$$x_{n+1} = \frac{r + px_n + x_{n-1}}{qx_n + x_{n-1}} = \frac{r + p\bigl((r + px_{n-1} + x_{n-2})/(qx_{n-1} + x_{n-2})\bigr) + x_{n-1}}{q\bigl((r + px_{n-1} + x_{n-2})/(qx_{n-1} + x_{n-2})\bigr) + x_{n-1}}, \quad n = 0, 1, \ldots, \tag{4.10}$$

that is,

$$x_{n+1} = \widehat{f}(x_n, x_{n-1}, x_{n-2}), \quad \text{where } \widehat{f}(x, y, z) = \frac{pr + p^2y + qry + qy^2 + pz + rz + yz}{qr + pqy + qy^2 + qz + yz}, \tag{4.11}$$

where the x has been kept in $\widehat{f}(x, y, z)$ for bookkeeping purposes. Thus $\widehat{f}(x, y, z)$ is constant in x. We claim that $\widehat{f}(x, y, z)$ is decreasing in both y and z. To see that the partial derivative

$$D_3\widehat{f}(x, y, z) = -\frac{(r + (p-q)y)(-qr + (p-q)y)}{(qr + pqy + qy^2 + qz + yz)^2} \tag{4.12}$$

is negative, just use $p > q$ and the inequality $(p-q)y - qr > 0$, which is true by the hypotheses of Proposition 4.1. The remaining partial derivative is

$$D_2\widehat{f}(x, y, z) = -\frac{h(y, z)}{(qr + pqy + qy^2 + qz + yz)^2}, \tag{4.13}$$

where

$$\begin{aligned} h(y, z) := &-q^2r^2 + 2pqry - 2q^2ry + p^2qy^2 - pq^2y^2 + q^2ry^2 + prz \\ &- qrz + pqrz - q^2rz + 2pqyz - 2q^2yz + 2qryz + pz^2 - qz^2 + rz^2. \end{aligned} \tag{4.14}$$

We have,

$$\begin{aligned} D_1h(y, z) &= 2(p-q)qr + 2q\left(p^2 - pq + qr\right)y + 2q(p-q+r)z > 0, \\ D_2h(y, z) &= (p-q)(1+q)r + 2q(p-q+r)y + 2(p-q+r)z > 0. \end{aligned} \tag{4.15}$$

Since $qr/(p-q) \le \tilde{m}$,

$$h(y, z) \ge h\left(\frac{qr}{p-q}, \frac{qr}{p-q}\right) = \frac{q(1+q)^2r^2(p^2 - pq + qr)}{(p-q)^2} > 0, \quad y, z \in \left[\tilde{m}, \widetilde{M}\right], \tag{4.16}$$

thus we conclude that $D_2\widehat{f}(x, y, z) < 0$ for $x, y, z \in [\tilde{m}, \widetilde{M}]$.

To complete the proof we verify the hypotheses of Theorem 2.2. We claim that the system of equations

$$\begin{aligned} \widehat{f}(m, m, m) - M = 0, \\ \widehat{f}(M, M, M) - m = 0 \end{aligned} \tag{4.17}$$

has no solutions (m, M) with $m \neq M$ whenever hypothesis (4.9) holds.

By eliminating denominators in both equations in (4.17) one obtains

$$-m^2 + m^2M - mp - mp^2 - m^2q + mMq + m^2Mq + mMpq - mr - pr - mqr + Mqr = 0,$$
$$-M^2 + mM^2 - Mp - Mp^2 + mMq - M^2q + mM^2q + mMpq - Mr - pr + mqr - Mqr = 0, \tag{4.18}$$

and by subtracting terms in (4.18) one obtains

$$(m - M)\left(-m - M + mM - p - p^2 - mq - Mq + mMq - r - 2qr\right) = 0. \tag{4.19}$$

Since $m \neq M$, we may use the second factor in the left-hand side term of (4.19) to solve for M in terms of m, which upon substitution into $\hat{f}(m, m, m) = M$ and simplification yields the equation

$$\frac{a_2m^2 + a_1m + a_0}{(-1 + m)(1 + q)(m^2 + mq + m^2q + mpq + qr)} = 0, \tag{4.20}$$

where

$$a_0 = r\left(p + 2pq + p^2q + qr + 2q^2r\right),$$
$$a_1 = p + p^2 + 2pq + 3p^2q + p^3q + r - pr + 4qr + 4q^2r + 2pq^2r, \tag{4.21}$$
$$a_2 = (1 + q)(1 + 2q + pq + qr).$$

By hypothesis (4.9) we have $rp \leq p^3q - p^2 < p^3q$, hence $p^3q - rp > 0$, which implies $a_1 \geq 0$. By direct inspection one can see that $a_0 > 0$ and $a_2 > 0$. Thus (4.20) has no positive solutions, and we conclude that (4.17) has no solutions $(m, M) \in [\widetilde{m}, \widetilde{M}]$ with $m \neq M$. We have verified the hypotheses of Theorem 2.2, and the conclusion of the lemma follows. □

Lemma 4.4. *Let $p > 0$, $q > 0$ and $r \geq 0$. If the positive equilibrium $\overline{y}$ of* (1.13) *satisfies $\overline{y} < p/q$, then $\overline{y}$ is locally asymptotically stable (L.A.S.).*

Proof. Solving for r in

$$\overline{y} = \frac{r + (p + 1)\overline{y}}{(q + 1)\overline{y}} \tag{4.22}$$

gives

$$r = (q + 1)\overline{y}^2 - (p + 1)\overline{y}. \tag{4.23}$$

Then a calculation shows

$$D_1 f(\overline{y},\overline{y}) = \frac{p - q\overline{y}}{\overline{y}(q+1)},$$
$$D_2 f(\overline{y},\overline{y}) = -\frac{\overline{y} - 1}{\overline{y}(q+1)}. \tag{4.24}$$

Set $t_1 := D_1 f(\overline{y},\overline{y})$ and $t_2 := D_2 f(\overline{y},\overline{y})$. The equilibrium $\overline{y}$ is locally asymptotically stable if the roots of the characteristic polynomial

$$\rho(x) = x^2 - t_1 x - t_2 \tag{4.25}$$

have modulus less than one [1]. By the Schur-Cohn Theorem, $\overline{y}$ is L.A.S. if and only if $|t_1| < 1 - t_2 < 2$. It can be easily verified that $1 - t_2 < 2$ if and only if $0 < q\overline{y} + 1$ which is true regardless of the allowable parameter values. Since $p - q\overline{y} > 0$ by the hypothesis, we have $|t_1| = |(p - q\ \overline{y})/\overline{y}(q+1)| = (p - q\ \overline{y})/\overline{y}(q+1)$; hence some algebra gives $|t_1| < 1 - t_2$ if and only if

$$\frac{1}{2}\frac{p+1}{q+1} < \overline{y}. \tag{4.26}$$

But (4.26) is a true statement by formula (4.3). We conclude that $\overline{y}$ is L.A.S. □

Lemma 4.5. *Assume the hypotheses to Proposition 4.1. If*

$$p > 1, \qquad q < 1, \qquad r > p^2 q - p, \tag{4.27}$$

then every solution to (1.13) *with* $x_{-1}, x_0 \in [\tilde{m}, \widetilde{M}]$ *converges to the equilibrium.*

Proof. The proof begins with a change of variable in (1.13) to produce a transformed equation with normalized coefficients analogous to those in the standard *normalized Lyness' Equation* [2–4]

$$x_{n+1} = \frac{\tilde{\alpha} + x_n}{x_{n-1}}. \tag{4.28}$$

We seek to use an argument of proof similar to the one used in [9], in which one takes advantage of the existence of *invariant curves* of Lyness' Equation to produce a Lyapunov-like function for (1.13).

Set $y_n = p z_n$ in (1.13) to obain the equation

$$z_{n+1} = \frac{a + z_n + g z_{n-1}}{b z_n + z_{n-1}}, \quad n = 0, 1, \ldots, \ z_{-1}, z_0 \in (0, \infty), \tag{4.29}$$

where

$$a = \frac{r}{p^2}, \qquad g = \frac{1}{p}, \qquad b = q. \tag{4.30}$$

We will denote with $\overline{z}$ the unique equilibrium of (4.29). Note that

$$\overline{y} = p\overline{z}. \tag{4.31}$$

It is convenient to parametrize (4.29) in terms of the equilibrium. We will use the symbol u to represent the equilibrium $\overline{z}$ of (4.29). By direct substitution of the equilibrium $u = \overline{z}$ into (4.29) we obtain

$$a = (b+1)u^2 - (g+1)u. \tag{4.32}$$

By (4.32), $a \geq 0$ if and only if $u \geq (g+1)/(b+1)$. Using (4.32) to eliminate a from (4.29) gives the following equation for $b > 0$, $g > 0$, and $u \geq (g+1)/(b+1)$, equivalent to (4.29):

$$z_{n+1} = \frac{(b+1)u^2 - (g+1)u + z_n + gz_{n-1}}{bz_n + z_{n-1}}, \quad n = 0, 1, 2, \ldots, \ y_{-1}, y_0 \in (0, \infty). \tag{4.33}$$

Therefore it suffices to prove that all solutions of (4.33) converge to the equilibrium u.

The following statement is crucial for the proof of the proposition.

Claim 1. $u > 1$ if and only if $r > p^2q - p$.

Proof. Since $\overline{y} = p\overline{z} = pu$, we have $u > 1$ if and only if $\overline{y} > p$, which holds if and only if

$$\frac{p + 1 + \sqrt{(p+1)^2 + 4r(q+1)}}{2(q+1)} > p. \tag{4.34}$$

After an elementary simplification, the latter inequality can be rewritten as $r > p^2q - p$. □

By the hypotheses of the lemma, by Claim 1, and by (4.30) and (4.32) we have

$$b < 1, \qquad g < 1, \qquad 1 < u < \frac{1}{b}, \qquad \frac{g+1}{b+1} \leq u. \tag{4.35}$$

We now introduce a function which is the invariant function for (4.28) with constant $\tilde{\alpha} = u^2 - u$ (in this case the the equilibrium of (4.28) is u):

$$g(x, y) = \frac{(1+x)(1+y)(u^2 - u + x + y)}{xy}. \tag{4.36}$$

Note that $g(x,y) > 0$ for all $x,y \in (0,\infty)$ whenever $u > 1$. By using elementary calculus, one can show that the function $g(x,y)$ has a strict global minimum at (u,u) [3, 4], that is,

$$g(u,u) < g(x,y), \quad (x,y) \in (0,\infty)^2. \tag{4.37}$$

We need some elementary properties of the sublevel sets

$$S(c) := \{(s,t) \in (0,\infty) : g(s,t) \le c\}, \quad c > 0. \tag{4.38}$$

We denote with $Q_\ell(u,u)$, $\ell = 1,2,3,4$ the four regions

$$\begin{aligned}
Q_1(u,u) &:= \{(x,y) \in (0,\infty) \times (0,\infty) : u \le x, u \le y\},\\
Q_2(u,u) &:= \{(x,y) \in (0,\infty) \times (0,\infty) : x \le u, u \le y\},\\
Q_3(u,u) &:= \{(x,y) \in (0,\infty) \times (0,\infty) : x \le u, y \le u\},\\
Q_4(u,u) &:= \{(x,y) \in (0,\infty) \times (0,\infty) : u \le x, y \le u\}.
\end{aligned} \tag{4.39}$$

Let

$$T(x,y) := \left(y, \frac{(g+1)u^2 - (b+1)u + y + gx}{by + x}\right), \quad (x,y) \in (0,\infty) \times (0,\infty), \tag{4.40}$$

be the map associated to (4.33) (see [16]).

Claim 2. If $(x,y) \in Q_2(u,u) \cup Q_4(u,u) \setminus \{(u,u)\}$, then $g(T(x,y)) < g(x,y)$.

Proof. Set

$$\Delta_1(x,y) := g(x,y) - g(T(x,y)). \tag{4.41}$$

A calculation yields

$$\Delta_1(x,y) = -\frac{(1+x)F_1(x,y)F_2(x,y)}{xy(bx+y)F_3(x,y)}, \tag{4.42}$$

where

$$\begin{aligned}
F_1(x,y) &:= b(x-u)\left(y - \frac{1}{b}\right) + (y-u)(bu + y + u - g),\\
F_2(x,y) &:= b(x-u)^2 + b(x-u)u + b(x-u)u^2 + (u-y)\left(bu^2 + yg\right),\\
F_3(x,y) &:= (b+1)u^2 - (1+g)u + x + gy.
\end{aligned} \tag{4.43}$$

By (4.35), for $(x, y) \in Q_4(u, u) \setminus \{(u, u)\}$ we have $u \leq x$ and $y \leq u < 1/b$ with $(x, y) \neq (u, u)$, therefore $F_1(x, y) < 0$, $F_2(x, y) > 0$ and $F_3(x, y) > 0$. Consequently $\Delta_1(x, y) > 0$ for $(x, y) \in Q_4(u, u) \setminus \{(u, u)\}$. To see that $\Delta_1(x, y) > 0$ for $(x, y) \in Q_2(u, u) \setminus \{(u, u)\}$ as well, rewrite $F_1(x, y)$ and $F_2(x, y)$ as follows:

$$\begin{aligned} F_1(x, y) &= b(u - x)\left(\frac{1}{b} - u\right) + (y - u)^2 + (y - u)u + (y - u)(u - g) + b(y - u)x, \\ F_2(x, y) &= b(x - u)\left(x + u^2\right) - b(y - u)u^2 - (y - u)^2 g - (y - u)ug. \end{aligned} \tag{4.44}$$

For $(x, y) \in Q_2(u, u) \setminus \{(u, u)\}$ we have $x \leq u \leq y$ and $(x, y) \neq (u, u)$. Thus $F_1(x, y) > 0$, $F_2(x, y) < 0$, and $F_3(x, y) > 0$, which imply $\Delta_1(x, y) > 0$. □

Claim 3. Suppose $g > b$. If $(x, y) \in Q_1(u, u) \cup Q_3(u, u) \setminus \{(u, u)\}$, then $g(T^2(x, y)) < g(x, y)$.

Proof. This proof requires extensive use of a computer algebra system to verify certain inequalities involving rational expressions. Here we give an outline of the steps, and refer the reader to Tables 1 and 2 for the details.

Since $b < g < 1 < u < 1/b$ and $(g + 1)/(b + 1) < u$, we may write

$$\begin{aligned} u &= \frac{g + 1}{b + 1} + t, \quad t > 0, \\ g &= \frac{b + s/b}{1 + s}, \quad s > 0. \end{aligned} \tag{4.45}$$

The expression $\Delta_2 := g(x, y) - g(T^2(x, y))$ may be written as a single ratio of polynomials, $\Delta_2 = N/D$ with $D > 0$. The next step is to show $N > 0$ for $(x, y) \neq (u, u)$.

Points (x, y) in $Q_1(u, u)$ may be written in the form $x = u + v$, $y = u + w$, where $v, w \in [0, \infty)$. Substituting x, y, u, and g in terms of v, w, s, and t into the expression for N one obtains a rational expression $\widetilde{N}/\widetilde{D}$ with positive denominator. The numerator $\widetilde{N}$ has some negative coefficients. At this points two cases are considered, $w \geq v$, and $w \leq v$. These can be written as $w = v + k$ and $v = w + k$ for nonnegative k. Substitution of each one of the latter expressions in $\widetilde{N}$ gives a polynomial with positive coefficients. This proves $\Delta_2(x, y) > 0$ for $(x, y) \in Q_2(u, u)$.

If now we assume $(x, y) \in Q_3(u, u)$ with $(x, y) \neq (u, u)$, we may write

$$\begin{aligned} x &= \frac{u}{v + 1}, \quad v \in [0, \infty), \\ y &= \frac{u}{w + 1}, \quad w \in [0, \infty). \end{aligned} \tag{4.46}$$

The rest of the proof is as in the first case already discussed. Details can be found in Tables 1 and 2. □

Claim 4. Suppose $g < b$ and $u > 1$. If $(x, y) \in Q_1(u, u) \cup Q_3(u, u) \setminus \{(u, u)\}$, then $g(T^3(x, y)) < g(x, y)$.

Table 1: Mathematica code needed to do the calculations in Claim 3. Here we define the functions g, f and T, as well as the expression DELTA2. The reparametrizations indicated in the proof of Claim 3 for the case $g > b$ are defined as substitution rules. To verify the positive sign of a polynomial of nonnegative variables $z, s, \ldots$, we form a list with the terms of the polynomial and then substitute the number 1 for the variables in order to extract the smallest coefficient. This input was tested on Mathematica Version 5.0 [18].

$g[\{x\_, y\_\}] = \dfrac{(1+x)(1+y)(u^2 - u + x + y)}{xy};$

$f[x\_, y\_] = \dfrac{(b+1)u^2 - (g+1)u + x + gy}{bx + y};$

$T[\{x\_, y\_\}] = \{f[x, y], x\};$

DELTA2 $= g[\{x, y\}] - g[T[T[\{x, y\}]]];$

rule$b = b \to \dfrac{z}{z+1};$

rule$g = g \to$ Factor$[\dfrac{(1/b)s + b}{s+1}$/.rule $b];$

rule$u = u \to$ Factor$[\dfrac{g+1}{b+1} + t$/.rule$g$/.rule $b];$

num$D2$ = Numerator[Together[DELTA2]];

num1$D2vk$ = Numerator[Together[num $D2$/.$\{x \to u + v + k,\ y \to u + v\}$]];

list1$D2vk$ = Numerator[Together[num1$D2vk$/.{rule u, rule g, rule b}]]/.Plus $\to$ List;

min 21 = Min[list1$D2vk$/.$\{z \to 1,\ s \to 1,\ t \to 1,\ v \to 1,\ k \to 1\}$];

num2$D2wk$ = Numerator[Together[num $D2$/.$\{x \to u + v,\ y \to u + v + k\}$]];

list2$D2wk$ = Numerator[Together[num2$D2wk$/.{rule u, rule g, rule b}]]/.Plus $\to$ List;

min 22 = Min[list2$D2wk$/.$\{z \to 1,\ s \to 1,\ t \to 1,\ v \to 1,\ k \to 1\}$];

num22 = Numerator[Together[num $D2$/.$\{x \to u/(w+1),\ y \to u/(v+1)\}$]];

num2$D2vk$ = Expand[num22/.$\{v \to w + k\}$];

list2$D2vk$ = Numerator[Together[num2$D2vk$/.{rule u, rule g, rule b}]]/.Plus $\to$ List;

min 23 = Min[list2$D2vk$/.$\{z \to 1,\ s \to 1,\ t \to 1,\ w \to 1,\ k \to 1\}$];

num2$D2wk$ = Expand[num22/.$\{w \to v + k\}$];

list2$D2wk$ = Numerator[Together[num2$D2wk$/.{rule$u$, rule$g$, rule$b$}]]/.Plus $\to$ List;

min 24 = Min[list2$D2wk$/.$\{z \to 1,\ s \to 1,\ t \to 1,\ v \to 1,\ k \to 1\}$];

Print["Minimal coefficients:", {min 21, min 22, min 23, min 24}]

Proof. The proof is analogous to the proof of Claim 3. We provide an outline. More details can be found in Tables 1 and 2.

Since $1 < u$, we may write $u = 1 + t$ with $t > 0$. Also, $u < 1/b$ implies $b < 1/u$, and $b = 1/(1 + t + s)$ for $s > 0$. Since $g < b$, we may write $g = 1/(1 + t + s + \ell)$ for $\ell > 0$.

The expression $\Delta_3 := g(x, y) - g(T^3(x, y))$ may be written as a single ratio of polynomials, $\Delta_2 = N/D$ with $D > 0$. The next step is to show $N > 0$ for $(x, y) \neq (u, u)$. This is done in a way similar to the procedure described in Claim 3. $\square$

Table 2: Mathematica code needed to do the calculations in Claim 4 when $g \le b$. The functions g, f and T are defined as before (not shown). This input was tested on Mathematica Version 5.0 [18].

```
Tz[{x_, y_}] = Together[T[{x, y}]/.{B → 1/(1+s+t), g → 1/(1+s+t+r), u → 1 + s }];
DELTA3 = Together[g[{x, y}] − g[Tz[Tz[Tz[{x, y}]]]]];
numDELTA3 = Numerator [step3];
numDELTA32 = Numerator[Together[numDELTA3/.{x → 1 + s + v, y → 1 + s + w}]];
numDELTA33 = Expand[numDELTA32];
numDELTA33vk = Expand[numDELTA33/.w → v + k];
min 31 = Min[numDELTA33vk/.Plus → List/.{s → 1, t → 1, r → 1, v → 1, k → 1}];
numDELTA33wk = Expand[numDELTA33/.v → w + k];
min 32 = Min[numDELTA33wk/.Plus → List/.{s → 1, t → 1, r → 1, w → 1, k → 1}];
numDELTA32 = Numerator[Together[numDELTA3/.{x → (1+s)/(1+v), y → (1+s)/(1+w)}]];
numDELTA33 = Expand[numDELTA32];
numDELTA33vk = Expand[numDELTA33/.w → v + k];
min 33 = Min[numDELTA33vk/.Plus → List/.{s → 1, t → 1, r → 1, v → 1, k → 1}];
numDELTA33wk = Expand[numDELTA33/.v → w + k];
min 34 = Min[numDELTA33wk/.Plus → List/.{s → 1, t → 1, r → 1, w → 1, k → 1}];
Print["Minimal coefficients:", {min 31, min 32, min 33, min 34}];
```

To complete the proof of the lemma, let $(\phi, \psi) \in (0, \infty) \times (0, \infty)$. Let $\{y_n\}_{n \ge -1}$ be the solution to (4.33) with initial condition $(y_{-1}, y_0) = (\phi, \psi)$, and let $\{T^n(\phi, \psi)\}_{n \ge 0}$ be the corresponding orbit of T. The following argument is essentially the same as the one found in [9]; we provided here for convenience. Define

$$\hat{c} := \liminf_n g(T^n(\phi, \psi)). \tag{4.47}$$

Note that $\hat{c} < \infty$, which can be shown by applying Claims 2, 3, and 4 repeatedly as needed to obtain a nonincreasing subsequence of $\{g(T^n(\phi, \psi))\}_{n \ge 0}$ that is bounded below by $g(u, u)$. Let $\{g(T^{n_k}(\phi, \psi))\}_{k \ge 0}$ be a subsequence convergent to $\hat{c}$. Therefore there exists $c > 0$ such that

$$g(T^{n_k}(\phi, \psi)) \le c, \quad \forall k \ge 0, \tag{4.48}$$

that is,

$$T^{n_k}(\phi, \psi) \in S(c) := \{(s, t) : g(s, t) \le c\}, \quad \text{for } k \ge 0. \tag{4.49}$$

The set $S(c)$ is closed by continuity of $g(x,y)$. Boundedness of $S(c)$ follows from

$$0<x,\quad y<\frac{(1+x)(1+y)(u^2-u+x+y)}{xy}=g(x,y)\le c,\quad \text{for } (x,y)\in S(c). \tag{4.50}$$

Thus $S(c)$ is compact, and there exists a convergent subsequence $\{T^{n_{k_\ell}}(\phi,\varphi)\}_\ell$ with limit $(\hat{x},\hat{y})$. Note that

$$\hat{c}=\lim_{\ell\to\infty} g(T^{n_{k_\ell}}(\phi,\varphi))=g(\hat{x},\hat{y}). \tag{4.51}$$

We claim that $(\hat{x},\hat{y})=(u,u)$. If not, then by Claims 2, 3, and 4,

$$\min\left\{g(T(\hat{x},\hat{y})),g\left(T^2(\hat{x},\hat{y})\right),g\left(T^3(\hat{x},\hat{y})\right)\right\}<\hat{c}. \tag{4.52}$$

Let $\|\cdot\|$ denotes the Euclidean norm. By (4.52) and continuity, there exists $\delta>0$ such that

$$\|(s,t)-(\hat{x},\hat{y})\|<\delta \Longrightarrow \min\left\{g(T(s,t)),g\left(T^2(s,t)\right),g\left(T^3(s,t)\right)\right\}<\hat{c}. \tag{4.53}$$

Choose $L\in\mathbb{N}$ large enough so that

$$\|T^{n_{k_L}}(\phi,\varphi)-(\hat{x},\hat{y})\|<\delta. \tag{4.54}$$

But then (4.53) and (4.54) imply

$$\min\left\{g\left(T^{n_{k_L}+1}(\phi,\varphi)\right),g\left(T^{n_{k_L}+2}(s,t)\right),g\left(T^{n_{k_L}+3}(s,t)\right)\right\}<\hat{c}, \tag{4.55}$$

which contradicts the definition (4.47) of $\hat{c}$. We conclude $(\hat{x},\hat{y})=(u,u)$. From this and the definition of convergence of sequences we have that for every $\epsilon>0$ there exists $L\in\mathbb{N}$ such that $\|T^{n_{k_L}}(\phi,\varphi)-(u,u)\|<\epsilon$. Finally, since

$$\max\left\{\left|y_{n_{k_L}-1}-u\right|,\left|y_{n_{k_L}}-u\right|\right\}\le\left\|\left(y_{n_{k_L}-1}-u,y_{n_{k_L}}-u\right)\right\|=\|T^{n_{k_L}}(\phi,\varphi)-(u,u)\|, \tag{4.56}$$

we have that for every $\epsilon>0$ there exists $L\in\mathbb{N}$ such that $|y_{n_{k_L}}-u|<\epsilon$ and $|y_{n_{k_L}-1}-u|<\epsilon$. Since u is a locally asymptotically stable equilibrium for (4.33) by Lemma 4.4, it follows that $y_n\to u$. This completes the proof of the lemma. □

5. Proof of Theorem 1.4

To prove Theorem 1.4 it is enough to assume statement (B) of Proposition 3.1. Also by Lemma 3.3 we may assume $p\neq q$ without loss of generality. Thus we make the following standing assumption valid throughout the rest of this section for (1.15).

Standing Assumption (SA)

Assume $p \neq q$ and that there exist m^*, M^* with $\mathcal{L} \le m^* < M^* \le \mathcal{U}$ such that for (1.15) and its associated function $f(x,y)$,

(i) $[m^*, M^*]$ is an invariant interval;

(ii) every solution eventually enters $[m^*, M^*]$;

(iii) $f(x,y)$ is coordinatewise strictly monotonic on $[m^*, M^*]^2$.

The function $f(x,y)$ is assumed to be coordinatewise monotonic on $[m^*, M^*]$, and there are four possible cases in which this can happen: (a) $f(x,y)$ is increasing in both variables, (b) $f(x,y)$ is decreasing in both variables, (c) $f(x,y)$ is decreasing in x and increasing in y, and (d) $f(x,y)$ is increasing in x and decreasing in y.

We present several lemmas before completing the proof of Theorem 1.4.

By considering the restriction of the map T of (1.15) to $[m^*, M^*]^2$, an application of the Schauder Fixed Point Theorem [17] gives that $[m^*, M^*]^2$ contains the fixed point of T, namely, $(\overline{y}, \overline{y})$. Thus we have the following result.

Lemma 5.1. *One has* $\overline{y} \in [m^*, M^*]$.

Lemma 5.2. *Neither one of the systems of equations*

$$\begin{aligned} M &= f(M, M), \\ m &= f(m, m), \end{aligned} \tag{S1}$$

and

$$\begin{aligned} M &= f(m, m), \\ m &= f(M, M), \end{aligned} \tag{S2}$$

has solutions $(m, M) \in [m^*, M^*]^2$ *with* $m < M$.

Proof. Since $x = \overline{y}$ is the only solution to $f(x,x) = x$, it is clear that only $(\overline{y}, \overline{y})$ satisfies (S1). Now let (m, M) be a solution to (S2). From straightforward algebra applied to $M - m = f(m,m) - f(M,M)$ one arrives at $(p+1)(M-m) = 0$, which implies $m = M$. □

Lemma 5.3. *Suppose that $f(x,y)$ is increasing in x and decreasing in y for $(x,y) \in [m^*, M^*]$. Then $p - q > 0$.*

Proof. By the standing assumption (SA), $p \neq q$. By Lemma 3.4, the coordinatewise monotonicity hypothesis, and the fact $\overline{y} \in [m^*, M^*]$ from Lemma 5.1, we have

$$(p-q)\overline{y} > qr, \qquad (q-p)\overline{y} < r. \tag{5.1}$$

The inequalities in (5.1) cannot hold simultaneously unless $p - q > 0$. □

Lemma 5.4. *If $f(x,y)$ is increasing in x and decreasing in y for $(x,y) \in [m^*,M^*]$, then $f([m^*,M^*]^2) \subset (1,p/q)$.*

Proof. For $(x,y) \in [m^*,M^*]$, the function f is well defined and is componentwise strictly monotonic on the set $[x,\infty) \times [y,\infty)$. Then,

$$
\begin{aligned}
f(x,y) &< \lim_{s\to\infty} f(s,y) = \lim_{s\to\infty} \frac{r+ps+y}{qs+y} = \frac{p}{q}, \\
f(x,y) &> \lim_{t\to\infty} f(x,t) = \lim_{t\to\infty} \frac{r+px+t}{qx+t} = 1.
\end{aligned}
\tag{5.2}
$$

□

Lemma 5.5. *Let $p > 0$, $q > 0$ and $r \geq 0$. If $f(x,y)$ is increasing in x and decreasing in y on $[m^*,M^*]$, then*

$$
\frac{qr}{p-q} < \frac{p}{q}. \tag{5.3}
$$

Proof. Since $\overline{y} \in [m^*,M^*]$ by Lemma 5.1, we have $D_1(\overline{y},\overline{y}) > 0$ and $D_2(\overline{y},\overline{y}) < 0$. By Lemma 5.3, $p > q$, and by Lemma 3.4,

$$
(p-q)\overline{y} > qr, \qquad (q-p)\overline{y} < r. \tag{5.4}
$$

Then,

$$
\overline{y} > \frac{qr}{p-q}. \tag{5.5}
$$

In addition, by Lemma 3.4,

$$
\overline{y} = f(\overline{y},\overline{y}) < \lim_{s\to\infty} f(s,\overline{y}) = \lim_{s\to\infty} \frac{r+ps+\overline{y}}{qs+\overline{y}} = \frac{p}{q}. \tag{5.6}
$$

□

Lemma 5.6. *Suppose that $f(x,y)$ is increasing in x and decreasing in y for $(x,y) \in [m^*,M^*]$. If $r < 0$, then $p - q + r > 0$.*

Proof. Since $D_1 f(x,y) > 0$ for $(x,y) \in [m^*,M^*]$, and by Lemmas 3.4, 5.1 and 5.3, we have $\overline{y} > -r/(p-q)$, that is,

$$
\sqrt{(p+1)^2 + 4r(q+1)} > \frac{-2r(q+1)}{p-q} - p - 1. \tag{5.7}
$$

If the right-hand side of inequality (5.7) is nonnegative, then, after squaring both sides of (5.7), we have

$$
(p+1)^2 + 4r(q+1) > \left(\frac{-2r(q+1)}{p-q}\right)^2 + \frac{4r(q+1)(p+1)}{p-q} + (p+1)^2. \tag{5.8}
$$

Further simplification of (5.8) and the hypothesis $r < 0$ yield

$$1 < \frac{r(q+1)}{(p-q)^2} + \frac{p+1}{p-q}, \tag{5.9}$$

which, after some elementary algebra, implies $p - q + r > 0$. Now assume that the right-hand side of inequality (5.7) is negative relation that we may rewrite as

$$\frac{-r}{p-q} < \frac{1}{2}\,\frac{p+1}{q+1}. \tag{5.10}$$

If $(1/2)((p+1)/(q+1)) \le 1$, then $-r/(p-q) < 1$, which gives the conclusion $p - q + r > 0$. If $(1/2)((p+1)/(q+1)) > 1$, that is, $p > 2q + 1$, then

$$p - q + r > q + r + 1. \tag{5.11}$$

Therefore if $q + r + 1 \ge 0$, the conclusion of the lemma follows from this and from (5.11). Assume

$$q + r + 1 < 0. \tag{5.12}$$

From relations (1.16) we have

$$q + r + 1 = \frac{(b+c)\left(a^2c^2 + bc^2\alpha + c^3\alpha - ac^2\beta + 2abc\gamma + ac^2\gamma + b^2\gamma^2 + 2bc\gamma^2 + c^2\gamma^2\right)}{c(ac + b\gamma + c\gamma)^2}, \tag{5.13}$$

hence assumption (5.12) and relation (5.13) imply

$$R := a^2c^2 + bc^2\alpha + c^3\alpha - ac^2\beta + 2abc\gamma + ac^2\gamma + b^2\gamma^2 + 2bc\gamma^2 + c^2\gamma^2 < 0. \tag{5.14}$$

Further algebra gives

$$\begin{aligned} &\frac{\gamma}{ac}R - \left(-c^2\alpha + ac\gamma - c\beta\gamma + b\gamma^2\right) \\ &\quad = c^2\alpha + \frac{bc\alpha\gamma}{a} + \frac{c^2\alpha\gamma}{a} + b\gamma^2 + c\gamma^2 + \frac{2b\gamma^3}{a} + \frac{b^2\gamma^3}{ac} + \frac{c\gamma^3}{a} > 0. \end{aligned} \tag{5.15}$$

Since $R < 0$ by (5.14), from inequality (5.15) we have

$$-c^2\alpha + ac\gamma - c\beta\gamma + b\gamma^2 < 0. \tag{5.16}$$

Finally, from (1.16) we have

$$p-q+r=-\frac{(b+c)^2\left(-c^2\alpha+ac\gamma-c\beta\gamma+b\gamma^2\right)}{c\left(ac+b\gamma+c\gamma\right)^2}. \tag{5.17}$$

Combining (5.16) with (5.17) we obtain $p-q+r>0$. □

Lemma 5.7. *If $r<0$ and $f(x,y)$ is increasing in x and decreasing in y for $(x,y)\in[m^*,M^*]$, then every solution converges to the equilibrium.*

Proof. Since $p-q+r>0$ by Lemma 5.6, we have $K_2=-r/(p-q)<1$, which together with Lemma 5.4 implies that $[1,p/q]$ is an invariant, attracting compact interval such that $f(x,y)$ is increasing in x and decreasing in y on $[1,p/q]^2$. Since $f([1,p/q]^2)\subset(1,p/q)$, we see that every solution to (1.15) eventually enters the invariant interval $(1,p/q)$. The change of variables

$$y_n=\frac{1+(p/q)z_n}{1+z_n},\quad \text{or}\quad z_n=\frac{y_n-1}{p/q-y_n} \tag{5.18}$$

transforms the equation

$$y_{n+1}=\frac{r+py_n+y_{n-1}}{qy_n+y_{n-1}},\quad n=0,1,\ldots,\ y_{-1},y_0\in\left(1,\frac{p}{q}\right) \tag{5.19}$$

into the equivalent equation

$$z_{n+1}=g(z_n,z_{n-1}),\quad n=0,1,\ldots,\ z_{-1},z_0\in(0,\infty), \tag{5.20}$$

where

$$g(w,v):=\frac{q(1+v)\left(-(q(p-q+r))+\left(-p^2+pq-qr\right)w\right)}{(1+w)\left(-q(p-q-qr)+\left(-p^2+pq+q^2r\right)v\right)}. \tag{5.21}$$

We claim that for $w,v\in(0,\infty)$, (a) $g(w,w)$ is increasing in w, (b) $g(w,v)/w$ is decreasing in w, and (c) $g(w,v)/w$ is decreasing in v. Indeed, since $p>q$, $r<0$, $p-q+r>0$, and $-r/(p-q)<p/q$, we have

$$\begin{aligned}
\frac{d}{dw}(g(w,w))&=\frac{-rq^2(1+q)(-p+q)^2}{\left(-pq+q^2+q^2r-p^2w+pqw+q^2rw\right)^2}>0,\\
\frac{\partial}{\partial v}\left(\frac{g(w,v)}{w}\right)&=-\frac{q(-p+q)^2\left(q(p-q+r)+\left(p^2-pq+qr\right)w\right)}{\left(-pq+q^2+q^2r-p^2v+pqv+q^2rv\right)^2w(1+w)}<0,\\
\frac{\partial}{\partial w}\left(\frac{g(w,v)}{w}\right)&=-\frac{q(1+v)\left(q(p-q+r)+2q(p-q+r)w+\left(p^2-pq+qr\right)w^2\right)}{\left(q(p-q-qr)+\left(p^2-pq-q^2r\right)v\right)\ w^2(1+w)^2}<0.
\end{aligned} \tag{5.22}$$

Also, note that (5.20) has a unique equilibrium $\overline{z}$. Therefore hypotheses (i)–(iv) of Theorem 2.5 are satisfied; so every solution $\{z_n\}$ to (5.20) converges to $\overline{z}$. By reversing the change of variables, one can conclude that every solution to (5.19) converges to the equilibrium. □

Proof of Theorem 1.4. The four parts of the proof are as follows.

(a) *$f(x,y)$ is increasing in both x and y on $[m^*,M^*]^2$.* By Lemma 5.2 the hypotheses of Theorem 2.1 part (i). are satisfied; hence every solution converges to the equilibrium $\overline{y}$.

(b) *$f(x,y)$ is decreasing in both x and y on $[m^*,M^*]^2$.* By Lemma 5.2 the hypotheses of Theorem 2.1 part (ii). are satisfied; hence every solution converges to the equilibrium $\overline{y}$.

(c) *$f(x,y)$ is decreasing in x and increasing in y on $[m^*,M^*]^2$.* By the corollary to Theorem 2.3 we conclude that every solution converges to the unique equilibrium or to a prime period-two solution.

(d) *$f(x,y)$ is increasing in x and decreasing in y on $[m^*,M^*]^2$.* By Lemmas 3.4, 5.3, and 5.4, there is no loss of generality in assuming $[m^*,M^*] \subset (K,p/q)$, where $K := \max\{-r/(p-q), qr/(p-q)\}$, which we do. We consider two subcases. If $r \geq 0$, then Lemmas 5.3, 5.5, and Proposition 4.1 imply that every solution converges to the unique equilibrium. If $r < 0$, then Lemma 5.7 implies that every solution converges to the unique equilibrium.

This completes the proof of Theorem 1.4. Since Theorem 1.4 is just a version of Theorem 1.2 obtained by an affine change of coordinates, we have also proved Theorem 1.2 as well. □

6. Proof of Theorem 1.5

The first lemma guarantees solutions to (1.13) to be bounded.

Lemma 6.1. *Let $p > 0$, $q > 0$ and $r \geq 0$. There exist positive constants $\mathcal{L}$ and $\mathcal{U}$ such that every solution $\{x_n\}_{n=-1}^{\infty}$ to (1.13) satisfies $x_n \in [\mathcal{L},\mathcal{U}]$ for $n \geq 2$, and the function*

$$f(x,y) = \frac{r+px+y}{qx+y}, \quad (x,y) \in (0,\infty)^2 \tag{6.1}$$

satisfies

$$f([\mathcal{L},\mathcal{U}] \times [\mathcal{L},\mathcal{U}]) \subset [\mathcal{L},\mathcal{U}]. \tag{6.2}$$

Proof. Set

$$\mathcal{L} := \min\left\{\frac{p}{q},1\right\}, \qquad \mathcal{U} := \max\left\{\frac{p}{q},1,\frac{r+(p+1)\mathcal{L}}{(q+1)\mathcal{L}}\right\}. \tag{6.3}$$

Then

$$f(x,y) \geq \frac{px+y}{qx+y} \geq \min\left\{\frac{p}{q}, 1\right\} = \mathcal{L}. \tag{6.4}$$

Let $x = \mathcal{L} + u$ and $y = \mathcal{L} + v$ with $u, v \geq 0$. Thus

$$f(x,y) = \frac{r + \mathcal{L}(p+1) + pu + v}{\mathcal{L}(q+1) + qu + v} \leq \max\left\{\frac{p}{q}, 1, \frac{r + (p+1)\mathcal{L}}{(q+1)\mathcal{L}}\right\} = \mathcal{U}. \tag{6.5}$$

Hence $f([\mathcal{L},\mathcal{U}] \times [\mathcal{L},\mathcal{U}]) \subset [\mathcal{L},\mathcal{U}]$. □

Inspection of the proof of Proposition 3.1 reveals that, given that we have Lemma 6.1, the conclusion of the proposition is true concerning (1.13). The statement is given next.

Proposition 6.2. *At least one of the following statements is true.*

(A) *Every solution to* (1.13) *converges to the equilibrium.*

(B) *There exist* m^*, M^* *with* $L \leq m^* < M^*$ *s.t. the following.*

(i) $[m^*, M^*]$ *is an invariant interval for* (1.13), *that is,* $f([m^*, M^*] \times [m^*, M^*]) \subset [m^*, M^*]$.

(ii) *Every solution to* (1.13) *eventually enters* $[m^*, M^*]$.

(iii) $f(x,y)$ *is coordinatewise strictly monotonic on* $[m^*, M^*]^2$.

The proof of Theorem 1.4 may be reproduced here in its entirety with the only change being the elimination of the case $r < 0$, which presently does not apply. Everything else in the proof applies to (1.13). The proof of Theorem 1.5 is complete.

Acknowledgment

The authors wish to thank an anonymous referee for his/her careful review of the paper and valuable suggestions.

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