Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 143175, 14 pages
doi:10.1155/2009/143175

Research Article

On the Spectrum of Almost Periodic Solution of Second-Order Neutral Delay Differential Equations with Piecewise Constant of Argument

Li Wang and Chuanyi Zhang

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Correspondence should be addressed to Li Wang, wanglimath@yahoo.com.cn

Received 16 December 2008; Accepted 10 April 2009

Recommended by Ondrej Dosly

The spectrum containment of almost periodic solution of second-order neutral delay differential equations with piecewise constant of argument (EPCA, for short) of the form $(x(t)+px(t-1))'' = qx(2[(t+1)/2])+f(t)$ is considered. The main result obtained in this paper is different from that given by some authors for ordinary differential equations (ODE, for short) and clearly shows the differences between ODE and EPCA. Moreover, it is also different from that given for equation $(x(t)+px(t-1))'' = qx([t])+f(t)$ because of the difference between $[t]$ and $2[(t+1)/2]$.

1. Introduction and Some Preliminaries

Differential equations with piecewise constant argument, which were firstly considered by Cooke and Wiener [1] and Shah and Wiener [2], combine properties of both differential and difference equations and usually describe hybrid dynamical systems and have applications in certain biomedical models in the work of Busenberg and Cooke [3]. Over the years, more attention has been paid to the existence, uniqueness, and spectrum containment of almost periodic solutions of this type of equations (see, e.g., [4–12] and reference there in).

If $g_1(t)$ and $g_2(t)$ are almost periodic, then the module containment property $\mathrm{mod}(g_1) \subset \mathrm{mod}(g_2)$ can be characterized in several ways (see [13–16]). For periodic function this inclusion just means that the minimal period of $g_1(t)$ is a multiple of the minimal period of $g_2(t)$. Some properties of basic frequencies (the base of spectrum) were discussed for almost periodic functions by Cartwright. In [17], Cartwright compared basic frequencies (the base of spectrum) of almost periodic differential equations (ODE) $\dot{x} = \psi(x,t)$, $x \in R^n$, with those of its unique almost periodic solution. For scalar equation, $n = 1$, Cartwright's results in [17] implied that the number of basic frequencies of $\dot{x} = \psi(x,t)$, $x \in R$, is the same as that of basic frequencies of its unique solution.

The spectrum containment of almost periodic solution of equation $(x(t)+px(t-1))''=qx([t])+f(t)$ was studied in [9, 10]. Up to now, there have been no papers concerning the spectrum containment of almost periodic solution of equation

$$(x(t)+px(t-1))''=qx\left(2\left[\frac{t+1}{2}\right]\right)+f(t), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, p, q are nonzero real constants, $|p|\neq 1$, $q\neq -2(p^2+1)$, and $f(t)$ is almost periodic. In this paper, we investigate the existence, uniqueness, and spectrum containment of almost periodic solutions of (1.1). The main result obtained in this paper is different from that given in [17] for ordinary differential equations (ODE, for short). This clearly shows differences between ODE and EPCA. Moreover, it is also different from that given in [9, 10] for equation $(x(t)+px(t-1))''=qx([t])+f(t)$. This is due to the difference between $[t]$ and $2[(t+1)/2]$. As well known, both solutions of (1.1) and equation $(x(t)+px(t-1))''=qx([t])+f(t)$ can be constructed by the solutions of corresponding difference equations. However, noticing the difference between $[t]$ and $2[(t+1)/2]$, the solution of difference equation corresponding to the latter can be obtained directly (see [4]), while the solution $\{x_n\}$ of difference equation corresponding to the former (i.e., (1.1)) cannot be obtained directly. In fact, $\{x_n\}$ consists of two parts: $\{x_{2n}\}$ and $\{x_{2n+1}\}$. We will first obtain $\{x_{2n}\}$ by solving a difference equation and then obtain $\{x_{2n+1}\}$ from $\{x_{2n}\}$. (Similar technology can be seen in [8].) A detailed account will be given in Section 2.

Now, We give some preliminary notions, definitions, and theorem. Throughout this paper Z, R, and C denote the sets of integers, real, and complex numbers, respectively. The following preliminaries can be found in the books, for example, [13–16].

Definition 1.1. (1) A subset P of R is said to be relatively dense in R if there exists a number $p>0$ such that $P\cap[t,t+p]\neq\emptyset$ for all $t\in R$.

(2) A continuous function $f:R\to R$ is called almost periodic (abbreviated as $\mathcal{AP}(R)$) if the ϵ-translation set of f

$$T(f,\epsilon)=\{\tau\in R:|f(t+\tau)-f(t)|<\epsilon,\ \forall t\in R\} \tag{1.2}$$

is relatively dense for each $\epsilon>0$.

Definition 1.2. Let f be a bounded continuous function. If the limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)dt \tag{1.3}$$

exists, then we call the limit mean of f and denote it by $M(f)$.

If $f\in\mathcal{AP}(R)$, then the limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T+s}^{T+s}f(t)dt \tag{1.4}$$

exists uniformly with respect to $s\in R$. Furthermore, the limit is independent of s.

For any $\lambda \in R$ and $f \in \mathcal{AP}(R)$ since the function $fe^{-i\lambda\cdot}$ is in $\mathcal{AP}(R)$, the mean exists for this function. We write

$$a(\lambda; f) = M\left(fe^{-i\lambda\cdot}\right), \tag{1.5}$$

then there exists at most a countable set of λ's for which $a(\lambda; f) \neq 0$. The set

$$\Lambda_f = \{\lambda : a(\lambda; f) \neq 0\} \tag{1.6}$$

is called the frequency set (or spectrum) of f. It is clear that if $f(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$, then $a(\lambda; f) = c_k$ if $\lambda = \lambda_k$, for some $k = 1, \ldots, n$; and $a(\lambda; f) = 0$ if $\lambda \neq \lambda_k$, for any $k = 1, \ldots, n$. Thus, $\Lambda_f = \{\lambda_k, k = 1, \ldots, n\}$.

Members of Λ_f are called the Fourier exponents of f, and $a(\lambda; f)$'s are called the Fourier coefficients of f. Obviously, Λ_f is countable. Let $\Lambda_f = \{\lambda_k\}$ and $A_k = a(\lambda_k; f)$. Thus f can associate a Fourier series:

$$f(t) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k t}. \tag{1.7}$$

The Approximation Theorem

Let $f \in \mathcal{AP}(R)$ and $\Lambda_f = \{\lambda_k\}$. Then for any $\epsilon > 0$ there exists a sequence $\{\sigma_\epsilon\}$ of trigonometric polynomials

$$\sigma_\epsilon(t) = \sum_{k=1}^{n(\epsilon)} b_{k,\epsilon} e^{i\lambda_k t} \tag{1.8}$$

such that

$$\|\sigma_\epsilon - f\| \leq \epsilon, \tag{1.9}$$

where $b_{k,\epsilon}$ is the product of $a(\lambda_k; f)$ and certain positive number (depending on ϵ and λ_k) and $\lim_{\epsilon \to 0} b_{k,\epsilon} = a(\lambda_k; f)$.

Definition 1.3. (1) For a sequence $\{g(n) : n \in Z\}$, define $[g(n),\ g(n+p)] = \{g(n), \ldots, g(n+p)\}$ and call it sequence interval with length $p \in Z$. A subset P of Z is said to be relatively dense in Z if there exists a positive integer p such that $P \cap [n, n+p] \neq \emptyset$ for all $n \in Z$.

(2) A bounded sequence $g : Z \to R$ is called an almost periodic sequence (abbreviated as $\mathcal{APS}(R)$) if the ϵ-translation set of g

$$T(g, \epsilon) = \{\tau \in Z : |g(n+\tau) - g(n)| < \epsilon,\ \forall n \in Z\} \tag{1.10}$$

is relatively dense for each $\epsilon > 0$.

For an almost periodic sequence $\{g(n)\}$, it follows from the lemma in [13] that

$$a(z;g) = \lim_{N\to\infty}\frac{1}{2N}\sum_{k=-N}^{N} z^{-k}g(k), \quad \forall z \in S^1 = \{\overline{z} \in C : |\overline{z}| = 1\} \tag{1.11}$$

exists. The set

$$\sigma_b(g) = \left\{z : a(z;g) \neq 0,\ z \in S^1\right\} \tag{1.12}$$

is called the Bohr spectrum of $\{g(n)\}$. Obviously, for almost periodic sequence $g(n) = \sum_{k=1}^{m} r_k z_k^n$, $a(z;g) = r_k$ if $z = z_k$, for some $k = 1,\ldots,m$; $a(z;g) = 0$ if $z \neq z_k$, for any $k = 1,\ldots,m$. So, $\sigma_b(g) = \{z_k, k = 1,\ldots,m\}$.

2. The Statement of Main Theorem

We begin this section with a definition of the solution of (1.1).

Definition 2.1. A continuous function $x : R \to R$ is called a solution of (1.1) if the following conditions are satisfied:

(i) $x(t)$ satisfies (1.1) for $t \in R$, $t \neq n \in Z$;

(ii) the one-sided second-order derivatives $(x(t) + px(t-1))''$ exist at n, $n \in Z$.

In [8], the authors pointed out that if $x(t)$ is a solution of (1.1), then $(x(t) + px(t-1))'$ are continuous at $t \in R$, which guarantees the uniqueness of solution of (1.1) and cannot be omitted.

To study the spectrum of almost periodic solution of (1.1), we firstly study the solution of (1.1). Let

$$f_n^{(1)} = \int_n^{n+1}\int_n^s f(\sigma)d\sigma\, ds, \qquad f_n^{(2)} = \int_n^{n-1}\int_n^s f(\sigma)d\sigma\, ds, \qquad h_n = f_n^{(1)} + f_n^{(2)}. \tag{2.1}$$

Suppose that $x(t)$ is a solution of (1.1), then $(x(t) + px(t-1))'$ exist and are continuous everywhere on R. By a process of integrating (1.1) two times in $t \in [2n-1, 2n+1)$ or $t \in [2n, 2n+2)$ as in [7, 8, 18], we can easily get

$$\begin{gathered} x(2n+1) + (p-2-q)x(2n) + (1-2p)x(2n-1) + px(2n-2) = h_{2n}, \\ \left(1-\frac{q}{2}\right)x(2n+2) + (p-2)x(2n+1) + \left(1-2p-\frac{q}{2}\right)x(2n) + px(2n-1) = h_{2n+1}. \end{gathered} \tag{2.2}$$

These lead to the difference equations

$$px_{2n-2} + (1-2p)x_{2n-1} + (p-2-q)x_{2n} + x_{2n+1} = h_{2n}, \tag{2.3}$$

$$px_{2n-1} + \left(1-2p-\frac{q}{2}\right)x_{2n} + (p-2)x_{2n+1} + \left(1-\frac{q}{2}\right)x_{2n+2} = h_{2n+1}. \tag{2.4}$$

Suppose that $|p| \neq 1$. First, multiply the two sides of (2.3) and (2.4) by p and $(2p-1)$, respectively, then add the resulting equations to get

$$\begin{aligned} x_{2n+1} = {} & \frac{1}{2(p-1)^2}\left(ph_{2n} - p(p-2-q)x_{2n} - p^2 x_{2n-2} + (2p-1)h_{2n+1}\right) \\ & - \frac{1}{2(p-1)^2}\left((2p-1)\left(1-\frac{q}{2}\right)x_{2n+2} + (2p-1)\left(1-2p-\frac{q}{2}\right)x_{2n}\right). \end{aligned} \tag{2.5}$$

Similarly, one gets

$$\begin{aligned} x_{2n-1} = {} & \frac{1}{2(p-1)^2}\left((2-p)h_{2n} - (2-p)(p-2-q)x_{2n} - (2-p)p x_{2n-2}\right) \\ & + \frac{1}{2(p-1)^2}\left(h_{2n+1} - \left(1-\frac{q}{2}\right)x_{2n+2} - \left(1-2p-\frac{q}{2}\right)x_{2n}\right). \end{aligned} \tag{2.6}$$

Replacing $2n$ by $(2n+2)$ in (2.6) and comparing with (2.5), one gets

$$\begin{aligned} & \left(1-\frac{q}{2}\right)x_{2n+4} - \left(p^2 - 2pq + 3q + 2\right)x_{2n+2} + \left(2p^2 + 2pq - \frac{q}{2} + 1\right)x_{2n} - p^2 x_{2n-2} \\ & \quad = h_{2n+3} + (2-p)h_{2n+2} + (1-2p)h_{2n+1} - ph_{2n}. \end{aligned} \tag{2.7}$$

The corresponding homogeneous equation is

$$\left(1-\frac{q}{2}\right)x_{2n+4} - \left(p^2 - 2pq + 3q + 2\right)x_{2n+2} + \left(2p^2 + 2pq - \frac{q}{2} + 1\right)x_{2n} - p^2 x_{2n-2} = 0. \tag{2.8}$$

We can seek the particular solution as $x_{2n} = \xi^n$ for this homogeneous difference equation. At this time, ξ will satisfy the following equation:

$$p_1(\xi) = \left(1-\frac{q}{2}\right)\xi^3 - \left(p^2 - 2pq + 3q + 2\right)\xi^2 + \left(2p^2 + 2pq - \frac{q}{2} + 1\right)\xi - p^2 = 0. \tag{2.9}$$

From the analysis above one sees that if $x(t)$ is a solution of (1.1) and $|p| \neq 1$, then one gets (2.3) and (2.4). In fact, a solution of (1.1) is constructed by the common solution $\{x_n\}$ of (2.3) and (2.4). Moreover, it is clear that $\{x_n\}$ consists of two parts: $\{x_{2n}\}$ and $\{x_{2n+1}\}$. $\{x_{2n}\}$ can be obtained by solving (2.7), and $\{x_{2n+1}\}$ can be obtained by substituting $\{x_{2n}\}$ into (2.5) or (2.6). Without loss of generality, we consider (2.5) only. These will be shown in Lemmas 2.5 and 2.6.

Lemma 2.2. *If $f \in \mathcal{AP}(R)$, then $\{f_n^{(i)}\}$, $\{h_n\} \in \mathcal{APS}(R)$, $i = 1, 2$.*

Lemma 2.3. *Suppose that $|p| \neq 1$ and $q \neq -2(p^2+1)$, then the roots of polynomial $p_1(\xi)$ are of moduli different from 1.*

Lemma 2.4. *Suppose that X is a Banach space, $\mathcal{L}(X)$ denotes the set of bounded linear operators from X to X, $A \in \mathcal{L}(X)$, and $\|A\| < 1$, then $Id - A$ is bounded invertible and*

$$(Id - A)^{-1} = \sum_{n=0}^{\infty} A^n, \qquad \left\|(I - A)^{-1}\right\| \le \frac{1}{(1 - \|A\|)}, \tag{2.10}$$

where $A^0 = Id$, and Id is an identical operator.

The proofs of Lemmas 2.2, 2.3, and 2.4 are elementary, and we omit the details.

Lemma 2.5. *Suppose that $|p| \neq 1$ and $q \neq -2(p^2 + 1)$, then* (2.7) *has a unique solution $\{x_{2n}\} \in \mathcal{APS}(R)$.*

Proof. As the proof of Theorem 9 in [8], define $A : X \to X$ by $A\{x_{2n}\} = \{x_{2n+2}\}$, where X is the Banach space consisting of all bounded sequences $\{x_n\}$ in C with $\|\{x_n\}\| = \sup_{n\in Z}|x_n|$. It follows from Lemmas 2.2–2.4 that (2.7) has a unique solution $\{x_{2n}\} = P(A)^{-1}\{h_{2n+5} + (2 - p)h_{2n+4} + (1 - 2p)h_{2n+3} - ph_{2n+2}\} \in \mathcal{APS}(R)$.

Substituting x_{2n} into (2.5), we obtain x_{2n+1}. Easily, we can get $\{x_{2n+1}\} \in \mathcal{APS}(R)$. Consequently, the common solution $\{x_n\}$ of (2.3) and (2.4) can be obtained. Furthermore, we have that $\{x_n\} \in \mathcal{APS}(R)$ is unique. □

Lemma 2.6. *Suppose that $|p| \neq 1$ and $q \neq -2(p^2 + 1)$, $f \in \mathcal{AP}(R)$. Let $\{x_n\} \in \mathcal{APS}(R)$ be the common solution of* (2.3) *and* (2.4)*. Then* (1.1) *has a unique solution $x(t) \in \mathcal{AP}(R)$ such that $x(n) = x_n$, $n \in Z$. In this case the solution $x(t)$ is given for $t \in R$ by*

$$x(t) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} (-p)^k \omega(t - k), & |p| < 1, \\ \displaystyle -\sum_{k=1}^{\infty} (-p)^{-k} \omega(t + k), & |p| > 1, \end{cases} \tag{2.11}$$

where

$$\omega(t) = x_{2n} + px_{2n-1} + y_{2n}(t - 2n) + \frac{qx_{2n}(t - 2n)^2}{2} + \int_{2n}^{t}\int_{2n}^{s} f(\sigma)d\sigma\, ds, \qquad y_{2n} = x_{2n+1} + \left(p - 1 - \frac{q}{2}\right)x_{2n} - px_{2n-1} - f_{2n}^{(1)}, \tag{2.12}$$

for $t \in [2n - 1, 2n + 1)$, $n \in Z$; $\{y_{2n}\} \in \mathcal{APS}(R)$, $\omega(t) \in \mathcal{AP}(R)$.

The proof is easy, we omit the details. Since the almost periodic solution $x(t)$ of (1.1) is constructed by the common almost periodic solution of (2.3) and (2.4), easily, we have that $(x(t) + px(t - 1))'$ are continuous at $t \in R$. It must be pointed out that in many works only one of (2.3) and (2.4) is considered while seeking the unique almost periodic solution of (1.1), and it is not true for the continuity of $(x(t) + px(t - 1))'$ on R, consequently, it is not true for the uniqueness (see [8]).

The expressions of x_{2n}, x_{2n+1}, y_{2n}, $\omega(t)$, and $x(t)$ are important in the process of studying the spectrum containment of the almost periodic solution of (1.1). Before giving the main theorem, we list the following assumptions which will be used later.

(H_1) $|p| \neq 1$, $q \neq -2(p^2+1)$.

(H_2) $k\pi \notin \Lambda_f$, for all $k \in Z$.

(H_3) If $\lambda \in \Lambda_f$, then $\lambda + k\pi \notin \Lambda_f$, $0 \neq k \in Z$.

Our result can be formulated as follows.

Main Theorem

Let $f \in \mathcal{AP}(R)$ and (H_1) be satisfied. Then (1.1) has a unique almost periodic solution $x(t)$ and $\Lambda_x \subset \Lambda_f + \{k\pi : k \in Z\}$. Additionally, if ($H_2$) and ($H_3$) are also satisfied, then $\Lambda_f + \{k\pi : k \in Z\} \subset \Lambda_x$, that is, the following spectrum relation $\Lambda_x = \Lambda_f + \{k\pi : k \in Z\}$ holds, where the sum of sets A and B is defined as $A + B = \{a + b : a \in A,\ b \in B\}$.

We postpone the proof of this theorem to the next section.

3. The Proof of Main Theorem

To show the Main Theorem, we need some more lemmas.

Lemma 3.1. *Let $f \in \mathcal{AP}(R)$, then $\sigma_b(f_{2n}^{(i)}), \sigma_b(f_{2n+1}^{(i)}), \sigma_b(h_{2n}), \sigma_b(h_{2n+1}) \subset e^{i2\Lambda_f}$, $i = 1, 2$. If (H_3) is satisfied, then $\sigma_b(f_{2n}^{(i)}) = \sigma_b(f_{2n+1}^{(i)}) = e^{i2\Lambda_f}$, $i = 1, 2$. Furthermore, if (H_3) and (H_2) are both satisfied, then $\sigma_b(h_{2n}) = \sigma_b(h_{2n+1}) = e^{i2\Lambda_f}$.*

Proof. Since $f \in \mathcal{AP}(R)$, by Lemma 2.2 we know that $\{f_{2n}^{(i)}\}, \{f_{2n+1}^{(i)}\}, \{h_{2n}\}, \{h_{2n+1}\} \in \mathcal{APS}(R)$, $i = 1, 2$. It follows from The Approximation Theorem that, for any $m > 0, m \in Z$, there exists $P_m(t) = \sum_{k=1}^{n(m)} b_{k,m} e^{i\lambda_k t}$, $\lambda_k \in \Lambda_f$ such that $\|P_m - f\| \leq 1/m$, where $\lim_{m\to\infty} b_{k,m} = a(\lambda_k; f)$, and we can assume that $b_{k,m} e^{i\lambda_k t}$ and $\overline{b}_{k,m} e^{-i\lambda_k t}$ appear together in the trigonometric polynomial $P_m(t)$. Define

$$
\begin{aligned}
Q_{m,2n}^{(1)} &= \int_{2n}^{2n+1} \int_{2n}^{s} P_m(\sigma) d\sigma\, ds = \sum_{k=1}^{n(m)} c_{k,m}^{(1)} e^{i2\lambda_k n}, \\
Q_{m,2n}^{(2)} &= \int_{2n}^{2n-1} \int_{2n}^{s} P_m(\sigma) d\sigma\, ds = \sum_{k=1}^{n(m)} c_{k,m}^{(2)} e^{i2\lambda_k n}, \\
Q_{m,2n+1}^{(1)} &= \int_{2n+1}^{2n+2} \int_{2n+1}^{s} P_m(\sigma) d\sigma\, ds = \sum_{k=1}^{n(m)} c_{k,m}^{(1)} e^{i\lambda_k} e^{i2\lambda_k n}, \\
Q_{m,2n+1}^{(2)} &= \int_{2n+1}^{2n} \int_{2n+1}^{s} P_m(\sigma) d\sigma\, ds = \sum_{k=1}^{n(m)} c_{k,m}^{(2)} e^{i\lambda_k} e^{i2\lambda_k n},
\end{aligned}
\tag{3.1}
$$

where

$$c_{k,m}^{(1)}=\begin{cases}\dfrac{b_{k,m}}{2}, & \lambda_k=0,\\ \dfrac{-b_{k,m}\left(e^{i\lambda_k}-1-i\lambda_k\right)}{\lambda_k^2}, & \lambda_k\neq 0,\end{cases}\qquad c_{k,m}^{(2)}=\begin{cases}\dfrac{b_{k,m}}{2}, & \lambda_k=0,\\ \dfrac{-b_{k,m}\left(e^{-i\lambda_k}-1+i\lambda_k\right)}{\lambda_k^2}, & \lambda_k\neq 0.\end{cases}\tag{3.2}$$

Obviously, $\sigma_b(Q_{m,2n}^{(i)}),\sigma_b(Q_{m,2n+1}^{(i)})\subset e^{i2\Lambda_f}$, $i=1,2$, for all $m\in Z$. For any $z\in S^1$, $a(z;f_{2n}^{(i)})=\lim_{m\to\infty}a(z;Q_{m,2n}^{(i)})$, $a(z;f_{2n+1}^{(i)})=\lim_{m\to\infty}a(z;Q_{m,2n+1}^{(i)})$, thus, we have $\sigma_b(f_{2n}^{(i)}),\sigma_b(f_{2n+1}^{(i)})\subset e^{i2\Lambda_f}$, $i=1,2$.

Since $h_{2n}=f_{2n}^{(1)}+f_{2n}^{(2)}$ and $h_{2n+1}=f_{2n+1}^{(1)}+f_{2n+1}^{(2)}$, for all $n\in Z$. For all $z\in S^1$, we have

$$a(z;h_{2n})=a\left(z;f_{2n}^{(1)}\right)+a\left(z;f_{2n}^{(2)}\right),\tag{3.3}$$

$$a(z;h_{2n+1})=a\left(z;f_{2n+1}^{(1)}\right)+a\left(z;f_{2n+1}^{(2)}\right).\tag{3.4}$$

Thus, $\sigma_b(f_{2n}^{(i)})\subset e^{i2\Lambda_f}$ and $\sigma_b(f_{2n+1}^{(i)})\subset e^{i2\Lambda_f}$ imply $\sigma_b(h_{2n})\subset e^{i2\Lambda_f}$ and $\sigma_b(h_{2n+1})\subset e^{i2\Lambda_f}$, respectively, $i=1,2$.

If (H_3) is satisfied, then for any $\lambda_j\in\Lambda_f$, we have

$$\begin{aligned}
a\left(e^{i2\lambda_j};f_{2n}^{(1)}\right)&=\lim_{m\to\infty}a\left(e^{i2\lambda_j};Q_{m,2n}^{(1)}\right)=\lim_{m\to\infty}c_{j,m}^{(1)}\\
&=\begin{cases}\dfrac{a(\lambda_j;f)}{2}, & \lambda_j=0,\\ \dfrac{-a(\lambda_j;f)\left(e^{i\lambda_j}-1-i\lambda_j\right)}{\lambda_j^2}, & \lambda_j\neq 0,\end{cases}\\
a\left(e^{i2\lambda_j};f_{2n}^{(2)}\right)&=\lim_{m\to\infty}a\left(e^{i2\lambda_j};Q_{m,2n}^{(2)}\right)=\lim_{m\to\infty}c_{j,m}^{(2)}\\
&=\begin{cases}\dfrac{a(\lambda_j;f)}{2}, & \lambda_j=0,\\ \dfrac{-a(\lambda_j;f)\left(e^{-i\lambda_j}-1+i\lambda_j\right)}{\lambda_j^2}, & \lambda_j\neq 0,\end{cases}\\
a\left(e^{i2\lambda_j};f_{2n+1}^{(1)}\right)&=\lim_{m\to\infty}a\left(e^{i2\lambda_j};Q_{m,2n+1}^{(1)}\right)=\lim_{m\to\infty}e^{i\lambda_j}c_{j,m}^{(1)},\\
a\left(e^{i2\lambda_j};f_{2n+1}^{(2)}\right)&=\lim_{m\to\infty}a\left(e^{i2\lambda_j};Q_{m,2n+1}^{(2)}\right)=\lim_{m\to\infty}e^{i\lambda_j}c_{j,m}^{(2)}.
\end{aligned}\tag{3.5}$$

Easily, we have $a(e^{i2\lambda_j};f_{2n}^{(i)})\neq 0$ and $a(e^{i2\lambda_j};f_{2n+1}^{(i)})\neq 0$, that is, $e^{i2\lambda_j}\subset\sigma_b(f_{2n}^{(i)})$, $e^{i2\lambda_j}\subset\sigma_b(f_{2n+1}^{(i)})$, $i=1,2$. By the arbitrariness of λ_j, we get $e^{i2\Lambda_f}\subset\sigma_b(f_{2n}^{(i)})$ and $e^{i2\Lambda_f}\subset\sigma_b(f_{2n+1}^{(i)})$. So, $e^{i2\Lambda_f}=\sigma_b(f_{2n}^{(i)})=\sigma_b(f_{2n+1}^{(i)})$, $i=1,2$.

If (H_3) and (H_2) are both satisfied, suppose that there exists $z_0 = e^{i2\lambda_j} \in e^{i2\Lambda_f}$ such that $a(z_0; h_{2n}) = 0$. (H_2) implies $e^{i\lambda_j} \neq \pm 1$. Moreover, since ($H_3$) holds, we have $a(z_0; f_{2n}^{(i)}) \neq 0$, $i = 1, 2$. $a(z_0; h_{2n}) = a(z_0; f_{2n}^{(1)}) + a(z_0; f_{2n}^{(2)})$ leads to $e^{i\lambda_j} = 1$, which contradicts with $e^{i\lambda_j} \neq \pm 1$. So, $e^{i2\Lambda_f} \subset \sigma_b(h_{2n})$. Noticing that $\sigma_b(h_{2n}) \subset e^{i2\Lambda_f}$, we have $e^{i2\Lambda_f} = \sigma_b(h_{2n})$. Similarly, we can get $e^{i2\Lambda_f} = \sigma_b(h_{2n+1})$. The proof is completed. □

Lemma 3.2. *Suppose that (H_1) is satisfied, then $\sigma_b(x_{2n}) \subset e^{i2\Lambda_f}$. If ($H_1$), ($H_2$), and ($H_3$) are all satisfied, then $\sigma_b(x_{2n}) = e^{i2\Lambda_f}$, where $\{x_{2n}\}$ is the unique almost periodic sequence solution of* (2.7).

Proof. Since (H_1) holds, from Lemma 2.5 we know $\{x_{2n}\} = p_1(A)^{-1}\{g_{n+1}\} \in \mathcal{APS}(R)$, where $g_n = h_{2n+3} + (2-p)h_{2n+2} + (1-2p)h_{2n+1} - ph_{2n}$, for all $n \in Z$. For any $z \in S^1$, it follows from Lemma 2.3 that $p_1(z) \neq 0$. Noticing the expressions of $\{x_{2n}\}$ and g_n, we obtain

$$za(z; g_n) = p_1(z)a(z; x_{2n}), \tag{3.6}$$

$$a(z; g_n) = (z + 1 - 2p)a(z; h_{2n+1}) + (2z - pz - p)a(z; h_{2n}). \tag{3.7}$$

Those equalities and Lemma 3.1 imply that $\sigma_b(x_{2n}) = \sigma_b(g_n)$ and $\sigma_b(x_{2n}) \subset e^{i2\Lambda_f}$, when ($H_1$) is satisfied. If ($H_1$), ($H_2$), and ($H_3$) are all satisfied, we only need to prove $e^{i2\Lambda_f} \subset \sigma_b(g_n)$. Suppose that there exists $z_0 = e^{i2\lambda_j} \in e^{i2\Lambda_f}$, obviously, $e^{i\lambda_j} \neq \pm 1$, such that $a(z_0; g_n) = 0$. From Lemma 3.1, $a(z_0; h_{2n}) \neq 0$, $a(z_0; h_{2n+1}) \neq 0$. Thus, $0 = (z_0 + 1 - 2p)a(z_0; h_{2n+1}) + (2z_0 - pz_0 - p)a(z_0; h_{2n})$, that is, $(e^{i2\lambda_j} + 1 - 2p)e^{i\lambda_j} = pe^{i2\lambda_j} - 2e^{i2\lambda_j} + p$, which leads to $e^{i\lambda_j} = p$. This contradicts with (H_1). Thus, $e^{i2\Lambda_f} \subset \sigma_b(g_n)$, that is, $e^{i2\Lambda_f} \subset \sigma_b(x_{2n})$. Noticing that $\sigma_b(x_{2n}) \subset e^{i2\Lambda_f}$, so, $e^{i2\Lambda_f} = \sigma_b(x_{2n})$. The proof is completed. □

As mentioned above, the common almost periodic sequence solution $\{x_n\}$ of (2.3) and (2.4) consists of two parts: $\{x_{2n}\}$ and $\{x_{2n+1}\}$, where $\{x_{2n}\} \in \mathcal{APS}(R)$ is the unique solution of (2.7), and $\{x_{2n+1}\}$ is obtained by substituting $\{x_{2n}\}$ into (2.5). Obviously, $\{x_{2n+1}\} \in \mathcal{APS}(R)$. In the following, we give the spectrum containment of $\{x_{2n+1}\}$.

Lemma 3.3. *Suppose that (H_1) is satisfied, then $\sigma_b(x_{2n+1}) \subset e^{i2\Lambda_f}$. If ($H_1$), ($H_2$), and ($H_3$) are all satisfied, then $\sigma_b(x_{2n+1}) = e^{i2\Lambda_f}$.*

Proof. Since $\{x_{2n}\}, \{h_{2n}\}, \{h_{2n+1}\} \in \mathcal{APS}(R)$, $\{x_{2n+1}\} \in \mathcal{APS}(R)$. Noticing the expression of x_{2n+1}, for any $z \in S^1$, we have

$$2(p-1)^2 a(z, x_{2n+1}) = pa(z, h_{2n}) + (2p-1)a(z, h_{2n+1}) - z^{-1}p_2(z)a(z, x_{2n}), \tag{3.8}$$

where $p_2(z) = (2p-1)(1-q/2)z^2 + (-3p^2 + 2p - 1 - 2pq + q/2)z + p^2$. If ($H_1$) is satisfied, it follows from Lemmas 3.1 and 3.2 that $\sigma_b(x_{2n+1}) \subset e^{i2\Lambda_f}$.

If (H_1), (H_2), and (H_3) are all satisfied, supposing there exists $z_0 = e^{i2\lambda_j} \in e^{i2\Lambda_f}$, obviously, $e^{i\lambda_j} \neq \pm 1$, such that $a(z_0; x_{2n+1}) = 0$, that is, $z_0^{-1}p_2(z_0)a(z_0, x_{2n}) = pa(z_0, h_{2n}) + (2p-1)a(z_0, h_{2n+1})$. Noticing (3.3)–(3.7), this equality is equivalent to $(p_2(e^{i2\lambda_j})(e^{i2\lambda_j} + 1 - 2p) - p_1(e^{i2\lambda_j})(2p-1))e^{i\lambda_j} + p_2(e^{i2\lambda_j})(2e^{i2\lambda_j} - pe^{i2\lambda_j} - p) - pp_1(e^{i2\lambda_j}) = 0$, that is, $(q-2)e^{i3\lambda_j} + (2p - 4 - 2q)e^{i2\lambda_j} + (4p + q - 2)e^{i\lambda_j} + 2p = 0$. Considering equation $(q-2)x^3 + (2p - 4 - 2q)x^2 + (4p + q - 2)x + 2p = 0$, its roots are x_1, x_3, and x_2, obviously, $x_i \neq \pm 1$, $i = 1, 2, 3$. We claim that $|x_i| \neq 1$, $i = 1, 2, 3$, that is, this equation has no imaginary root. Otherwise, suppose that $|x_1| = 1$ and $x_3 = \overline{x}_1$, then by the relationship between roots and coefficient of three-order equation, we know $q = 0$, which leads to a contradiction. Thus $(q-2)e^{i3\lambda_j}+(2p-4-2q)e^{i2\lambda_j}+(4p+q-2)e^{i\lambda_j}+2p\neq 0$; this contradiction shows $e^{i2\Lambda_f}\subset\sigma_b(x_{2n+1})$. Noticing that $\sigma_b(x_{2n+1})\subset e^{i2\Lambda_f}$, thus, $\sigma_b(x_{2n+1})=e^{i2\Lambda_f}$. The proof is completed. □

Lemma 3.4. *Suppose that* (H_1) *is satisfied, then* $\sigma_b(y_{2n})\subset e^{i2\Lambda_f}$. *If* ($H_1$), ($H_2$), *and* ($H_3$) *are all satisfied, then* $\sigma_b(y_{2n})=e^{i2\Lambda_f}$, *where* $\{y_{2n}\}$ *is defined in Lemma 2.6.*

Proof. From Lemma 2.6, we have $y_{2n}=x_{2n+1}+(p-1-q/2)x_{2n}-px_{2n-1}-f_{2n}^{(1)}$, for all $n\in Z$. For any $z\in S^1$

$$a(z,y_{2n})=\left(1-pz^{-1}\right)a(z,x_{2n+1})+\left(p-1-\frac{q}{2}\right)a(z,x_{2n})-a\left(z,f_{2n}^{(1)}\right). \tag{3.9}$$

Since (H_1) holds, it follows from Lemmas 3.1–3.3 that we have $\sigma_b(y_{2n})\subset e^{i2\Lambda_f}$.

If (H_1), (H_2), and (H_3) are all satisfied, supposing there exists $z_0=e^{i2\lambda_j}\in e^{i2\Lambda_f}$ such that $a(z_0;\ y_{2n})=0$, it follows from (H_2) that $e^{i\lambda_j}\neq\pm 1$. Notice that (3.3)–(3.8), $a(z_0;y_{2n})=0$ is equivalent to $p(z_0-p)a(z_0,h_{2n})+(z_0-p)(2p-1)a(z_0,h_{2n+1})-2(p-1)^2z_0a(z_0,f_{2n}^{(1)})+((p-1-q/2)2(p-1)^2z_0^2-(z_0-p)p_2(z_0))p_1(z_0)^{-1}((z_0+1-2p)a(z_0,h_{2n+1})+(2z_0-pz_0-p)a(z_0,h_{2n}))=0$. This equality is equivalent to $e^{i\lambda_j}-1-i\lambda_j=(e^{i\lambda_j}+e^{-i\lambda_j}-2)p_1(e^{i2\lambda_j})^{-1}((1-q/2)e^{i6\lambda_j}+(1+q/2)e^{i5\lambda_j}+(pq-p^2-1-3q/2)e^{i4\lambda_j}-(p^2+1+q/2)e^{i3\lambda_j}+(p^2+pq)e^{i2\lambda_j}+p^2e^{i\lambda_j})$. Since $\lambda_j\in R$, that is, $\lambda_j=\overline{\lambda}_j$, this leads to $e^{-i5\lambda_j}(e^{i\lambda_j}-1)^2(e^{i\lambda_j}+1)^2(e^{i2\lambda_j}+1)(-pe^{i4\lambda_j}+(p^2+1-2p-q/2)e^{i3\lambda_j}+(2p^2-2p+2+q)e^{i2\lambda_j}+(p^2+1-2p-q/2)e^{i\lambda_j}-p)=0$. We firstly claim that the equation $-px^4+(p^2+1-2p-q/2)x^3+(2p^2-2p+2+q)x^2+(p^2+1-2p-q/2)x-p=0$ has no imaginary root, that is, equations $x^2+(a/2-\sqrt{a^2/4-b+2})x+1-\sqrt{1-a}=0$ and $x^2+(a/2+\sqrt{a^2/4-b+2})x+1+\sqrt{1-a}=0$ both have no imaginary roots, where $a=(q/2-1-p^2+2p)/p$, $b=(2p-q-2-2p^2)/p$. If these two equations have imaginary roots, then $a=1$, $b=4-4(p+1/p)$. Since $p\neq 0$, $|p|\neq 1$, then $b<-4$ or $b>12$. If the first equation has imaginary roots, then $-4<b\leq 9/4$, which contradicts with $b<-4$ or $b>12$. If the second equation has imaginary roots, then $0<b\leq 9/4$, which also contradicts with $b<-4$ or b >12. The claim follows. Thus $-pe^{i4\lambda_j}+(p^2+1-2p-q/2)e^{i3\lambda_j}+(2p^2-2p+2+q)e^{i2\lambda_j}+(p^2+1-2p-q/2)e^{i\lambda_j}-p\neq 0$, and $e^{i\lambda_j}=\pm i$. Substituting $e^{i\lambda_j}=\pm i$ into $e^{i\lambda_j}-1-i\lambda_j=(e^{i\lambda_j}+e^{-i\lambda_j}-2)p_1(e^{i2\lambda_j})^{-1}((1-q/2)e^{i6\lambda_j}+(1+q/2)e^{i5\lambda_j}+(pq-p^2-1-3q/2)e^{i4\lambda_j}-(p^2+1+q/2)e^{i3\lambda_j}+(p^2+pq)e^{i2\lambda_j}+p^2e^{i\lambda_j})$, we get $\lambda_j=0$. This is impossible. Thus, for any $z_0=e^{i2\lambda_j}\in e^{i2\Lambda_f}$, we have $a(z_0;y_{2n})\neq 0$, that is, $e^{i2\Lambda_f}\subset\sigma_b(y_{2n})$. Noticing that $\sigma_b(y_{2n})\subset e^{i2\Lambda_f}$, we have $\sigma_b(y_{2n})=e^{i2\Lambda_f}$. The proof has finished. □

In Lemma 2.6, we have given the expression of the almost periodic solution of (1.1) explicitly by a known function ω. This brings more convenience to study the spectrum containment of almost periodic solution of (1.1). Now, we are in the position to show the Main Theorem.

The proof of Main Theorem

Since (H_1) is satisfied, by Lemma 2.6, (1.1) has a unique almost periodic solution $x(t)$ satisfying $x(t)+px(t-1)=\omega(t)$. Thus, for any $\lambda\in R$, we have $a(\lambda;\omega(t))=(1+pe^{-i\lambda})a(\lambda;x(t))$. Since ($H_1$) holds, then $\Lambda_x=\Lambda_\omega$. We only need to prove $\Lambda_\omega\subset\Lambda_f+\{k\pi,\ \ k\in Z\}$ when (H_1) is satisfied, and $\Lambda_f+\{k\pi,\ \ k\in Z\}=\Lambda_\omega$ when (H_1)–(H_3) are all satisfied.

When (H_1) is satisfied, we prove $\Lambda_\omega \subset \Lambda_f + \{k\pi,\ k \in Z\}$ firstly. For any $\lambda \notin \Lambda_f + \{k\pi, k \in Z\}$, it follows from Lemmas 3.1–3.4 that $e^{i2\lambda} \notin \sigma_b(x_{2n})$, $e^{i2\lambda} \notin \sigma_b(x_{2n+1})$ and $e^{i2\lambda} \notin \sigma_b(y_{2n})$, that is, $a(e^{i2\lambda}; x_{2n}) = a(e^{i2\lambda}; y_{2n}) = a(e^{i2\lambda}; x_{2n+1}) = 0$. From the expression of $\omega(t)$ given in Lemma 2.6, we know

$$a(\lambda;\omega(t)) = \lim_{N\to\infty}\frac{1}{4N}\sum_{j=-N}^{N}\int_{2j-1}^{2j+1}\int_{2j}^{t}\int_{2j}^{s} f(\sigma)e^{-i\lambda t}d\sigma\, ds\, dt. \tag{3.10}$$

As mentioned in Lemma 3.1, for any $m>0$, $m\in Z$, there exists $P_m(t) = \sum_{k=1}^{n(m)} b_{k,m}e^{i\lambda_k t}$, $\lambda_k \in \Lambda_f$ such that $P_m(t) \to f(t)$, as $m\to\infty$. By simple calculation, we have

$$0 = \lim_{N\to\infty}\frac{1}{4N}\sum_{j=-N}^{N}\int_{2j-1}^{2j+1}\int_{2j}^{t}\int_{2j}^{s} P_m(\sigma)e^{-i\lambda t}d\sigma\, ds\, dt, \quad \forall m>0,\ m\in Z. \tag{3.11}$$

Therefore, $a(\lambda;\omega(t)) = 0$, that is, $\lambda\notin\Lambda_\omega$, which implies $\Lambda_\omega \subset \Lambda_f + \{k\pi, k\in Z\}$.

Additionally, if (H_2) and (H_3) are also satisfied, to show the equality $\Lambda_\omega = \Lambda_f + \{k\pi, k\in Z\}$, we only need to show the inverse inclusion, that is, $\Lambda_f + \{k\pi, k\in Z\} \subset \Lambda_\omega$. For any $m>0$, $m,n\in Z$, define $\widehat{h}_{m,2n} = Q^{(1)}_{m,2n} + Q^{(2)}_{m,2n}$, $\widehat{h}_{m,2n+1} = Q^{(1)}_{m,2n+1} + Q^{(2)}_{m,2n+1}$, $\{\widehat{x}_{m,2n}\} = p_1(A)^{-1}\{\widehat{g}_{m,n+1}\}$, where $\widehat{g}_{m,n} = \widehat{h}_{m,2n+3} + (2-p)\widehat{h}_{m,2n+2} + (1-2p)\widehat{h}_{m,2n+1} - p\widehat{h}_{m,2n}$, and define

$$\begin{aligned}
\widehat{y}_{m,2n} &= \widehat{x}_{m,2n+1} + \left(p-1-\frac{q}{2}\right)\widehat{x}_{m,2n} - p\widehat{x}_{m,2n-1} - Q^{(1)}_{m,2n},\\
2(p-1)^2\widehat{x}_{m,2n+1} &= p\widehat{h}_{m,2n} - p(p-2-q)\widehat{x}_{m,2n} - p^2\widehat{x}_{m,2n-2} + (2p-1)\widehat{h}_{m,2n+1}\\
&\quad -(2p-1)\left(1-\frac{q}{2}\right)\widehat{x}_{m,2n+2} - (2p-1)\left(1-2p-\frac{q}{2}\right)\widehat{x}_{m,2n},\\
\widehat{\omega}_m(t) &= \widehat{x}_{m,2n} + p\widehat{x}_{m,2n-1} + \widehat{y}_{m,2n}(t-2\mathrm{n}) + \frac{q\widehat{x}_{m,2n}(t-2n)^2}{2} + \int_{2n}^{t}\int_{2n}^{s}P_m(\sigma)d\sigma\, ds,
\end{aligned} \tag{3.12}$$

$t \in [2n-1, 2n+1)$, then, $\{\widehat{h}_{m,\ 2n}\}, \{\widehat{h}_{m,2n+1}\}, \{\widehat{g}_{m,n}\}, \{\widehat{x}_{m,2n}\}, \{\widehat{x}_{m,2n+1}\}, \{\widehat{y}_{m,2n}\} \in \mathcal{APS}(R)$, $\widehat{\omega}_m(t) \in \mathcal{AP}(R)$, and as $m\to\infty$, $\{\widehat{h}_{m,2n}\} \to \{h_{2n}\}$, $\{\widehat{h}_{m,2n+1}\} \to \{h_{2n+1}\}$, $\{\widehat{g}_{m,n}\} \to \{g_n\}$, $\{\widehat{x}_{m,2n}\} \to \{x_{2n}\}$, $\{\widehat{x}_{m,2n+1}\} \to \{x_{2n+1}\}$, $\{\widehat{y}_{m,\ 2n}\} \to \{y_{2n}\}$ in $\mathcal{APS}(R)$, $\omega_m(t) \to \omega(t)$ in $\mathcal{AP}(R)$, which implies for any $\lambda\in R$, $a(\lambda;\omega_m(t)) \to a(\lambda;\omega(t))$ as $m\to\infty$, where

$Q^{(i)}_{m,2n}, Q^{(i)}_{m,2n+1}$, and $P_m(t)$ are as in Lemma 3.1, $i = 1,2$. Similarly as above, for all $z \in S^1$, we can get

$$a\left(z;\hat{h}_{m,2n}\right) = a\left(z;Q^{(1)}_{m,2n}\right) + a\left(z;Q^{(2)}_{m,2n}\right),$$
$$a\left(z;\hat{h}_{m,2n+1}\right) = a\left(z;Q^{(1)}_{m,2n+1}\right) + a\left(z;Q^{(2)}_{m,2n+1}\right),$$
$$a(z,\hat{y}_{m,2n}) = \left(1 - pz^{-1}\right)a(z,\hat{x}_{m,2n+1}) + \left(p - 1 - \frac{q}{2}\right)a(z,\hat{x}_{m,2n}) - a\left(z,Q^{(1)}_{m,2n}\right),$$
$$2(p-1)^2 a(z,\hat{x}_{m,2n+1}) = pa\left(z,\hat{h}_{m,2n}\right) + (2p-1)a\left(z,\hat{h}_{m,2n+1}\right) - z^{-1}p_2(z)a(z,\hat{x}_{m,2n}),$$
$$a(z;\hat{g}_{m,n}) = z^{-1}p_1(z)a(z;\hat{x}_{m,2\mathrm{n}}) = (z+1-2p)a\left(z;\hat{h}_{m,2n+1}\right) + (2z - pz - p)a\left(z;\hat{h}_{m,2n}\right). \tag{3.13}$$

We claim: $\Lambda_f \subset \Lambda_\omega$. Suppose that the claim is false, then there would exist $\lambda_j \in \Lambda_f$ such that $\lambda_j \notin \Lambda_\omega$. Noticing ($H_2$) and ($H_3$), an elementary calculation leads to

$$\begin{aligned} 2a(\lambda_j;\omega_m) = {} & \frac{\left(2\lambda_j^2 + q\lambda_j^2 - 2q\right)\left(e^{i\lambda_j} - e^{-i\lambda_j}\right) + i2q\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right)}{2i\lambda_j^3} a\left(e^{i2\lambda_j};\hat{x}_{m,2n}\right) \\ & + \frac{pe^{-2i\lambda_j}\left(e^{i\lambda_j} - e^{-i\lambda_j}\right)}{i\lambda_j} a\left(e^{i2\lambda_j};\hat{x}_{m,2n+1}\right) + \frac{i\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right)}{\lambda_j^2} a\left(e^{i2\lambda_j};\hat{y}_{m,2n}\right) \\ & + \frac{e^{-i\lambda_j} - e^{i\lambda_j}}{\lambda_j^2} a\left(e^{i2\lambda_j};\hat{y}_{m,2n}\right) + \frac{-2i\lambda_j + 2\left(e^{i\lambda_j} - e^{-i\lambda_j}\right) - i\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right)}{i\lambda_j^3} b_{j,m}. \end{aligned} \tag{3.14}$$

From the above equality, we have $-\lambda_j^2 p_1(e^{i2\lambda_j})2a(\lambda_j;\omega_m) = b_{j,m}c_1$, where, $c_1 = 2p_1(e^{i2\lambda_j}) - q(e^{i\lambda_j} - 1)^3(e^{i\lambda_j} + 1)^3(e^{i\lambda_j} - p)/i\lambda_j^3$. So, $-\lambda_j^2 p_1(e^{i2\lambda_j})2a(\lambda_j;\omega(t)) = a(\lambda_j;f)c_1$. Since $\lambda_j \notin \Lambda_\omega$, $a(\lambda_j;\omega) = 0$, we have $c_1 = 0$, which is equivalent to $2i\lambda_j^3 = q(e^{i\lambda_j} - 1)^3(e^{i\lambda_j} + 1)^3(e^{i\lambda_j} - p)/p_1(e^{i2\lambda_j})$. Since $\lambda_j \in R$, that is, $\lambda_j = \overline{\lambda}_j$, this leads to $p(e^{i\lambda_j} - 1)^3(e^{i\lambda_j} + 1)(pe^{i4\lambda_j} + e^{i3\lambda_j}(q/2 - 1 - p^2 + 2p) + e^{i2\lambda_j}(-q - 2 - 2p^2 + 2p) + e^{i\lambda_j}(q/2 - 1 - p^2 + 2p) + p) = 0$. Noticing $\lambda_j \in \Lambda_f$, it follows from (H_2) that $e^{i\lambda_j} \neq \pm 1$, that is, $(e^{i\lambda_j} - 1)^3(e^{i\lambda_j} + 1) \neq 0$. From Lemma 3.4, we know that the equation $px^4 + x^3(q/2 - 1 - p^2 + 2p) + x^2(-q - 2 - 2p^2 + 2p) + x(q/2 - 1 - p^2 + 2p) + p = 0$ has no imaginary root. Thus $pe^{i4\lambda_j} + e^{i3\lambda_j}(q/2 - 1 - p^2 + 2p) + e^{i2\lambda_j}(-q - 2 - 2p^2 + 2p) + e^{i\lambda_j}(q/2 - 1 - p^2 + 2p) + p \neq 0$, which leads to a contradiction. The claim follows.

Now we are able to prove $\Lambda_f + \{k\pi, k \in Z\} \subset \Lambda_\omega$. For any $\lambda_{j_0} \in \Lambda_f$ let $\lambda_j = \lambda_{j_0} + k\pi$, $0 \neq k \in Z$. Noticing (H$_2$) and (H$_3$), $e^{i\lambda_{j_0}} \neq \pm 1, e^{i\lambda_{j_0}} \neq p$, and we have

$$2a(\lambda_j;\omega_m) = \frac{\left(2\lambda_j^2 + q\lambda_j^2 - 2q\right)\left(e^{i\lambda_j} - e^{-i\lambda_j}\right) + i2q\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right)}{2i\lambda_j^3} a\left(e^{i2\lambda_j};\hat{x}_{m,2n}\right)$$
$$+ \frac{pe^{-2i\lambda_j}\left(e^{i\lambda_j} - e^{-i\lambda_j}\right)}{i\lambda_j} a\left(e^{i2\lambda_j};\hat{x}_{m,2n+1}\right) + \frac{i\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right)}{\lambda_j^2} a\left(e^{i2\lambda_j};\hat{y}_{m,2n}\right)$$
$$+ \frac{e^{-i\lambda_j} - e^{i\lambda_j}}{\lambda_j^2} a\left(e^{i2\lambda_j};\hat{y}_{m,2n}\right) + \left(\frac{e^{-i\lambda_j} - e^{i\lambda_j}}{-i\lambda_j\lambda_{j_0}^2} - \frac{i\lambda_j\left(e^{i\lambda_j} + e^{-i\lambda_j}\right) + e^{-i\lambda_j} - e^{i\lambda_j}}{i\lambda_j^2\lambda_{j_0}}\right) b_{j_0,m}. \tag{3.15}$$

The above equality is equivalent to $-\lambda_{j_0}^2 p_1(e^{i2\lambda_j}) 2a(\lambda_j;\omega_m(t)) = (-1)^k(-q(e^{i\lambda_{j_0}} - 1)^3 (e^{i\lambda_{j_0}} + 1)^3 (e^{i\lambda_{j_0}} - p)/i\lambda_j^3) b_{j_0,m}$. So, $-\lambda_{j_0}^2 p_1(e^{i2\lambda_j}) 2a(\lambda_j;\omega(t)) = (-1)^k(-q(e^{i\lambda_{j_0}} - 1)^3 (e^{i\lambda_{j_0}} + 1)^3 (e^{i\lambda_{j_0}} - p)/i\lambda_j^3) a(\lambda_{j_0}; f) \neq 0$, which implies that $\lambda_j \in \Lambda_\omega$, that is, $\Lambda_f + \{k\pi, k \neq 0\} \subset \Lambda_\omega$. From the claim above, we get $\Lambda_f + \{k\pi, k \in Z\} \subset \Lambda_\omega$. This completes the proof.

References

[1] K. L. Cooke and J. Wiener, "Retarded differential equations with piecewise constant delays," *Journal of Mathematical Analysis and Applications*, vol. 99, no. 1, pp. 265–297, 1984.

[2] S. M. Shah and J. Wiener, "Advanced differential equations with piecewise constant argument deviations," *International Journal of Mathematics and Mathematical Sciences*, vol. 6, no. 4, pp. 671–703, 1983.

[3] S. Busenberg and K. L. Cooke, "Models of vertically transmitted diseases with sequential-continuous dynamics," in *Nonlinear Phenomena in Mathematical Sciences*, V. Lakshmikantham, Ed., pp. 179–187, Academic Press, New York, NY, USA, 1982.

[4] G. Seifert, "Second-order neutral delay-differential equations with piecewise constant time dependence," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 1–9, 2003.

[5] D. X. Piao, "Almost periodic solutions of neutral differential difference equations with piecewise constant arguments," *Acta Mathematica Sinica*, vol. 18, no. 2, pp. 263–276, 2002.

[6] H.-X. Li, "Almost periodic solutions of second-order neutral delay-differential equations with piecewise constant arguments," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 2, pp. 693–709, 2004.

[7] H.-X. Li, "Almost periodic weak solutions of neutral delay-differential equations with piecewise constant argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 3, pp. 530–545, 2006.

[8] E. A. Dads and L. Lhachimi, "New approach for the existence of pseudo almost periodic solutions for some second order differential equation with piecewise constant argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 6, pp. 1307–1324, 2006.

[9] R. Yuan, "On the spectrum of almost periodic solution of second order scalar functional differential equations with piecewise constant argument," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 1, pp. 103–118, 2005.

[10] L. Wang, R. Yuan, and C. Zhang, "Corrigendum to: "On the spectrum of almost periodic solution of second order scalar functional differential equations with piecewise constant argument"," *Journal of Mathematical Analysis and Applications*, vol. 349, no. 1, p. 299, 2009.

[11] X. Yang and R. Yuan, "On the module containment of the almost periodic solution for a class of differential equations with piecewise constant delays," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 540–555, 2006.

[12] R. Yuan, "On the spectrum of almost periodic solution of second-order differential equations with piecewise constant argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 59, no. 8, pp. 1189–1205, 2004.

[13] C. Corduneanu, *Almost Periodic Functions*, Interscience Tracts in Pure and Applied Mathematics, no. 22, John Wiley & Sons, New York, NY, USA, 1968.

[14] A. M. Fink, *Almost Periodic Differential Equations*, vol. 377 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1974.

[15] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, UK, 1982.

[16] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press, Beijing, China; Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[17] M. L. Cartwright, "Almost periodic differential equations and almost periodic flows," *Journal of Differential Equations*, vol. 5, no. 1, pp. 167–181, 1969.

[18] R. Yuan, "Pseudo-almost periodic solutions of second-order neutral delay differential equations with piecewise constant argument," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 41, no. 7-8, pp. 871–890, 2000.