Research Article

The Existence of Positive Solutions for Third-Order p-Laplacian m-Point Boundary Value Problems with Sign Changing Nonlinearity on Time Scales

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We study the following third-order p-Laplacian m-point boundary value problems on time scales $(\phi_p(u^{\Delta\nabla}))^{\nabla} + a(t)f(t, u(t)) = 0$, $t \in [0, T]_{\mathbf{T}^\kappa}$, $u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, $u^{\Delta}(T) = 0$, $\phi_p(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} c_i \phi_p(u^{\Delta\nabla}(\xi_i))$, where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$. We obtain the existence of positive solutions by using fixed-point theorem in cones. In particular, the nonlinear term $f(t, u)$ is allowed to change sign. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

The theory of time scales was initiated by Hilger [1] as a mean of unifying and extending theories from differential and difference equations. The study of time scales has lead to several important applications in the study of insect population models, neural networks, heat transfer, and epidemic models, see, for example [2–6]. Recently, the boundary value problems with p-Laplacian operator have also been discussed extensively in literature; for example, see [7–18]. However, to the best of our knowledge, there are not many results concerning the higher-order p-Laplacian mutilpoint boundary value problem on time scales.

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that $0, T$ are points in $\mathbf{T}$. By an interval $(0, T)_{\mathbf{T}}$, we always mean the intersection of the real interval $(0, T)$ with the given time scale; that is $(0, T) \cap \mathbf{T}$.

In [19], Anderson considered the following third-order nonlinear boundary value problem (BVP):

$$x'''(t) = f(t, x(t)), \quad t_1 \leq t \leq t_3,$$
$$x(t_1) = x'(t_2) = 0, \qquad \gamma x(t_3) + \delta x''(t_3) = 0. \tag{1.1}$$

author studied the existence of solutions for the nonlinear boundary value problem by using Krasnoselskii's fixed point theorem and Leggett and Williams fixed point theorem, respectively.

In [9, 10], He considered the existence of positive solutions of the p-Laplacian dynamic equations on time scales

$$\left(\phi_p(u^\Delta)\right)^\nabla + a(t)f(u(t)) = 0, \quad t \in [0, T]_{\mathbf{T}}, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) - B_0\left(u^\Delta(\eta)\right) = 0, \quad u^\Delta(T) = 0, \tag{1.3}$$

or

$$u^\Delta(0) = 0, \quad u(T) - B_1\left(u^\Delta(\eta)\right) = 0, \tag{1.4}$$

where $\eta \in (0, \rho(T))$. He obtained the existence of at least double and triple positive solutions of the problems by using a new double fixed point theorem and triple fixed point theorem, respectively.

In [18], Zhou and Ma firstly studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with p-Laplacian operator

$$(\phi_p(u''))'(t) = q(t)f(t, u(t)), \quad 0 \le t \le 1,$$
$$u(0) = \sum_{i=1}^m \alpha_i u(\xi_i), \qquad u'(n) = 0, \qquad u''(1) = \sum_{i=1}^n \beta_i u''(\theta_i). \tag{1.5}$$

They established a corresponding iterative scheme for the problem by using the monotone iterative technique.

All the above works were done under the assumption that the nonlinear term is nonnegative. The key conditions used in the above papers ensure that positive solution is concave down. If the nonlinearity is negative somewhere, then the solution needs no longer to be concave down. As a result, it is difficult to find positive solutions of the p-Laplacian equation when the nonlinearity changes sign. In particular, little work has been done on the existence of positive solutions for higher order p-Laplacian m-point boundary value problems with nonlinearity f being nonnegative on time scales. Therefore, it is a natural problem to consider the existence of positive solution for higher order p-Laplacian equations with sign changing nonlinearity on time scales. This paper attempts to fill this gap in literature.

In this paper, by using different method, we are concerned with the existence of positive solutions for the following third-order p-Laplacian m-point boundary value problems on time scales:

$$\left(\phi_p(u^{\Delta\nabla})\right)^{\nabla} + a(t)f(t, u(t)) = 0, \quad t \in [0, T]_{\mathbf{T}^\kappa},$$

$$u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \quad u^{\Delta}(T) = 0, \quad \phi_p\left(u^{\Delta\nabla}(0)\right) = \sum_{i=1}^{m-2} c_i \phi_p\left(u^{\Delta\nabla}(\xi_i)\right), \tag{1.6}$$

where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, and b_i, c_i, a, f satisfy

(H_1) $b_i, c_i \in [0, +\infty)$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $0 < \sum_{i=1}^{m-2} c_i < 1$;

(H_2) $f : [0, T]_{\mathbf{T}^\kappa} \times [0, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, $a \in C_{ld}([0, T]_{\mathbf{T}^\kappa}, [0, +\infty))$, and there exists $t_0 \in [0, T)_{\mathbf{T}^\kappa}$ such that $a(t_0) > 0$.

2. Preliminaries and Lemmas

For convenience, we list the following definitions which can be found in [1–5].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$

$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T} \tag{2.1}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided that it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s| \tag{2.2}$$

for all $s \in U$.

For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t, denoted by $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \leq \epsilon |\rho(t) - s| \tag{2.3}$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e., *ld*-continuous), if f is continuous at each left-dense point in **T** and its right-sided limit exists at each right-dense point in **T**.

Definition 2.4. If $\phi^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = \phi(b) - \phi(a). \tag{2.4}$$

If $F^\nabla(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.5}$$

Lemma 2.5. *If condition (H_1) holds, then for $h \in C_{ld}[0,T]_{\mathbf{T}^\kappa}$, the boundary value problem (BVP)*

$$u^{\Delta\nabla} + h(t) = 0, \quad t \in [0,T]_{\mathbf{T}^\kappa},$$

$$u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \quad u^\Delta(T) = 0 \tag{2.6}$$

has the unique solution

$$u(t) = \int_0^t (T-s)h(s)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)h(s)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}. \tag{2.7}$$

Proof. By caculating, we can easily get (2.7). So we omit it.   □

Lemma 2.6. *If condition (H_1) holds, then for $h \in C_{ld}[0,T]_{\mathbf{T}^\kappa}$, the boundary value problem (BVP)*

$$\left(\phi_p\left(u^{\Delta\nabla}\right)\right)^\nabla + h(t) = 0, \quad t \in [0,T]_{\mathbf{T}^\kappa},$$

$$u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i), \quad u^\Delta(T) = 0, \quad \phi_p\left(u^{\Delta\nabla}(0)\right) = \sum_{i=1}^{m-2} c_i \phi_p\left(u^{\Delta\nabla}(\xi_i)\right) \tag{2.8}$$

has the unique solution

$$u(t) = \int_0^t (T-s)\phi_q\left(\int_0^s h(r)\nabla r + C\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i}(T-s)\phi_q\left(\int_0^s h(r)\nabla r + C\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}, \tag{2.9}$$

where $C = \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r)\nabla r / (1 - \sum_{i=1}^{m-2} c_i)$.

Proof. Integrating both sides of equation in (2.8) on $[0, t]$, we have

$$\phi_p\left(u^{\Delta\nabla}(t)\right) = \phi_p\left(u^{\Delta\nabla}(0)\right) - \int_0^t h(r)\nabla r. \tag{2.10}$$

So,

$$\phi_p\left(u^{\Delta\nabla}(\xi_i)\right) = \phi_p\left(u^{\Delta\nabla}(0)\right) - \int_0^{\xi_i} h(r)\nabla r. \tag{2.11}$$

By boundary value condition $\phi_p(u^{\Delta\nabla}(0)) = \sum_{i=1}^{m-2} c_i \phi_p(u^{\Delta\nabla}(\xi_i))$, we have

$$\phi_p\left(u^{\Delta\nabla}(0)\right) = -\frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i}. \tag{2.12}$$

By (2.10) and (2.12) we know

$$u^{\Delta\nabla}(t) = -\phi_q\left(\frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i} + \int_0^t h(r)\nabla r\right). \tag{2.13}$$

This together with Lemma 2.5 implies that

$$u(t) = \int_0^t (T-s)\phi_q\left(\int_0^s h(r)\nabla r + C\right)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\phi_q\left(\int_0^s h(r)\nabla r + C\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}, \tag{2.14}$$

where $C = \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r)\nabla r / (1 - \sum_{i=1}^{m-2} c_i)$. The proof is complete. $\qquad\square$

Lemma 2.7. *Let condition (H_1) holds If $h \in C_{ld}[0, T]_{\mathbf{T}^\kappa}$ and $h(t) \geq 0$, then the unique solution $u(t)$ of (2.8) satisfies*

$$u(t) \geq 0, \quad t \in [0, T]_{\mathbf{T}^\kappa}. \tag{2.15}$$

Proof. By $u^{\Delta\nabla}(t) = -\phi_q(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r)\nabla r / (1 - \sum_{i=1}^{m-2} c_i) + \int_0^t h(r)\nabla r) \leq 0$, we can know that the graph of $u(t)$ is concave down on $(0, T)_{\mathbf{T}^\kappa}$, and $u^\Delta(t)$ is nonincreasing on $[0, T]_{\mathbf{T}^\kappa}$. This together with the assumption that the boundary condition $u^\Delta(T) = 0$ implies that $u^\Delta(t) \geq 0$ for $t \in [0, T]_{\mathbf{T}^\kappa}$. This implies that

$$\min_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) = u(0). \tag{2.16}$$

So we only prove $u(0) \geq 0$. By condition (H_1) we have

$$u(0) = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s) \phi_q \left(\int_0^s h(r) \nabla r + C \right) \nabla s}{1 - \sum_{i=1}^{m-2} b_i} \geq 0. \tag{2.17}$$

The proof is completed. □

Lemma 2.8. *Let condition (H_1) hold. If $h \in C_{ld}[0,T]_{\mathbf{T}^\kappa}$ and $h(t) \geq 0$, then the unique positive solution $u(t)$ of (BVP) (2.8) satisfies*

$$\inf_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) \geq \sigma_1 \|u\|, \tag{2.18}$$

where $\sigma_1 = \sum_{i=1}^{m-2} b_i \xi_i / (T - \sum_{i=1}^{m-2} b_i (T - \xi_i))$, $\|u\| = \sup_{t \in [0,T]_{\mathbf{T}^\kappa}} |u(t)|$.

Proof. By $u^{\Delta \nabla}(t) = -\phi_q \left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} h(r) \nabla r / (1 - \sum_{i=1}^{m-2} c_i) + \int_0^t h(r) \nabla r \right) \leq 0$, we can know that the graph of $u(t)$ is concave down on $(0,T)_{\mathbf{T}^\kappa}$, and $u^\Delta(t)$ is nonincreasing on $[0,T]_{\mathbf{T}^\kappa}$. This together with the assumption that the boundary condition $u^\Delta(T) = 0$ implies that $u^\Delta(t) \geq 0$ for $t \in [0,T]_{\mathbf{T}^\kappa}$. This implies that

$$\|u\| = u(T), \quad \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) = u(0). \tag{2.19}$$

For all $i \in \{1,2,\ldots,m-2\}$, we have from the concavity of u that

$$\frac{u(\xi_i) - u(0)}{\xi_i} \geq \frac{u(T) - u(0)}{T}, \tag{2.20}$$

that is,

$$u(\xi_i) - u(0) + \frac{\xi_i}{T} u(0) \geq \frac{\xi_i}{T} u(T). \tag{2.21}$$

This together with the boundary condition $u(0) = \sum_{i=1}^{m-2} b_i u(\xi_i)$ implies that

$$\min_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) \geq \frac{\sum_{i=1}^{m-2} b_i \xi_i}{T - \sum_{i=1}^{m-2} b_i (T - \xi_i)} u(T). \tag{2.22}$$

This completes the proof. □

Let $E = C_{ld}[0,T]_{\mathbf{T}^\kappa}$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,T]_{\mathbf{T}^\kappa}$, and $\|u\| = \max_{t \in [0,T]_{\mathbf{T}^\kappa}} |u(t)|$ is defined as usual by maximum norm. Clearly, it follows that $(E, \|u\|)$ is a Banach space.

For the convenience, let

$$\psi(s) = \phi_q\left(\int_0^s a(r)\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i}\right). \tag{2.23}$$

We define two cones by

$$P = \{u : u \in E, u(t) \geq 0, t \in (0,T)_{\mathbb{T}^\kappa}\},$$

$$K = \left\{u : u \in E, u(t) \text{ is concave, nonincreasing and nonnegative on } (0,T)_{\mathbb{T}^\kappa} \right.$$

$$\left. \min_{t \in [0,T]_{\mathbb{T}^\kappa}} u(t) \geq \sigma\|u\|\right\}, \tag{2.24}$$

where $\sigma = \sigma_1\sigma_2$, σ_1 is defined in Lemma 2.8 and

$$\sigma_2 = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(s)\nabla s}{\left(1 - \sum_{i=1}^{m-2} b_i\right)\left(\int_0^T (T-s)\psi(T)\nabla s + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(T)\nabla s \big/ \left(1 - \sum_{i=1}^{m-2} b_i\right)\right)}. \tag{2.25}$$

Define the operators $F : P \to E$ and $S : K \to E$ by setting

$$(Fu)(t) = \int_0^t (T-s)\phi_q\left(\int_0^s a(r)f(r,u(r))\nabla r + A\right)\nabla s$$

$$+ \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (T-s)\phi_q\left(\int_0^s a(r)f(r,u(r))\nabla r + A\right)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}, \tag{2.26}$$

where $A = \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)f(r,u(r))\nabla r \big/ (1 - \sum_{i=1}^{m-2} c_i)$,

$$(Su)(t) = \int_0^t (T-s)\varphi(s)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}, \tag{2.27}$$

where $\varphi(s) = \phi_q(\int_0^s a(r)f^+(r,u(r))\nabla r + \tilde{A})$, $\tilde{A} = \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)f^+(r,u(r))\nabla r \big/ (1 - \sum_{i=1}^{m-2} c_i)$, and $f^+(t,u(t)) = \max\{f(t,u(t)),0\}$. Obviously, u is a solution of the BVP(1.6) if and only if u is a fixed point of operator F.

Lemma 2.9. *$S : K \to K$ is completely continuous.*

Proof. It is easy to see that $SK \subset K$ by $f^+ \geq 0$ and Lemma 2.8. By Arzela-Ascoli theorem and Lebesgue dominated convergence theorem, we can easily prove that operator S is completely continuous. $\qquad\square$

Lemma 2.10 (see [20, 21]). *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $A : \overline{D}_K \rightarrow K$ is a compact map such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold.*

(1) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i_K(A, D_K) = 1$.*

(2) *If there exists $x_0 \in K \setminus \{0\}$ such that $x \neq Ax + \lambda x_0$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(A, D_K) = 0$.*

(3) *Let U be open in X such that $\overline{U} \subset D_K$. If $i_K(A, D_K) = 1$ and $i_K(A, U_K) = 0$, then A has a fixed point in $D_K \setminus \overline{U}_K$. The same result holds if $i_K(A, D_K) = 0$ and $i_K(A, U_K) = 1$, where $i_K(A, D_K)$ denotes fixed point index.*

We define

$$K_\rho = \{u(t) \in K : \|u\| < \rho\}, \qquad \Omega_\rho = \left\{ u(t) \in K : \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) < \sigma\rho \right\}. \tag{2.28}$$

Lemma 2.11 (see [20]). *Ω_ρ defined above has the following properties:*

(a) *$K_{\sigma\rho} \subset \Omega_\rho \subset K_\rho$;*

(b) *Ω_ρ is open relative to K;*

(c) *$u \in \partial\Omega_\rho$ if and only if $\min_{t \in [0,T]_{\mathbf{T}^\kappa}} u(t) = \sigma\rho$;*

(d) *if $u \in \partial\Omega_\rho$, then $\sigma\rho \leq u(t) \leq \rho$ for $t \in [0,T]_{\mathbf{T}^\kappa}$.*

For the convenience, we introduce the following notations:

$$\frac{1}{m} = \int_0^T (T-s)\psi(T)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(T)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}, \qquad \frac{1}{M} = \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(s)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}. \tag{2.29}$$

Remark 2.12. By (H_1) we can know that $0 < m, M < +\infty$, $M\sigma = M\sigma_1\sigma_2 = m\sigma_1 < m$.

Lemma 2.13. *If f satisfies the following condition :*

$$f(t, u) \leq \phi_p(m\rho), \quad (t, u) \in [0,T]_{\mathbf{T}^\kappa} \times [0,\rho], \ u \neq Su, \ u \in \partial K_\rho, \tag{2.30}$$

then

$$i_K(S, K_\rho) = 1. \tag{2.31}$$

Proof. For $u \in \partial K_\rho$, then from (2.30) we have

$$
\begin{aligned}
\int_0^s a(r)f^+(r, u(r))\nabla r + \widetilde{A} &= \int_0^s a(r)f^+(r, u(r))\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)f^+(r, u(r))\nabla r}{1 - \sum_{i=1}^{m-2} c_i} \\
&\leq \phi_p(m\rho)\left(\int_0^T a(r)\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i} \right).
\end{aligned}
\tag{2.32}
$$

So that

$$\varphi(s) = \phi_q\left(\int_0^s a(r)f^+(r,u(r))\nabla r + \tilde{A}\right) \le m\rho\psi(T). \tag{2.33}$$

Therefore,

$$\begin{aligned} Su(t) &\le \int_0^T (T-s)\varphi(s)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\nabla s}{1 - \sum_{i=1}^{m-2} b_i} \\ &\le m\rho\left(\int_0^T (T-s)\psi(T)\nabla s + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(T)\nabla s}{1 - \sum_{i=1}^{m-2} b_i}\right) = \rho. \end{aligned} \tag{2.34}$$

This implies that $\|Su\| \le \|u\|$ for $u \in \partial K_\rho$. Hence by Lemma 2.10(1) it follows that $i_K(S,K_\rho) = 1$. $\qquad\square$

Lemma 2.14. *If f satisfies the following condition:*

$$f(t,u) \ge \phi_p(M\sigma\rho), \quad (t,u) \in [0,T]_{\mathbf{T}^\kappa} \times [\sigma\rho,\rho], \ u \ne Su, \ u \in \partial\Omega_\rho, \tag{2.35}$$

then

$$i_K(S,\Omega_\rho) = 0. \tag{2.36}$$

Proof. Let $e(t) \equiv 1$ for $t \in [0,T]_{\mathbf{T}^\kappa}$. Then $e \in \partial K_1$. We claim that

$$u \ne Su + \lambda e, \quad u \in \partial\Omega_\rho, \ \lambda > 0. \tag{2.37}$$

In fact, if not, there exist $u_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $u_0 = Su_0 + \lambda_0 e$. By $f(t,u_0) \ge \phi_p(M\sigma\rho)$, we have

$$\begin{aligned} \int_0^s a(r)f^+(r,u_0(r))\nabla r + \tilde{A} &= \int_0^s a(r)f^+(r,u_0(r))\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)f^+(r,u_0(r))\nabla r}{1 - \sum_{i=1}^{m-2} c_i} \\ &\ge \phi_p(M\sigma\rho)\left(\int_0^s a(r)\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i}\right). \end{aligned} \tag{2.38}$$

So that

$$
\begin{aligned}
\varphi(s) &= \phi_q \left(\int_0^s a(r) f^+(r, u_0(r)) \nabla r + \tilde{A} \right) \\
&\geq M\sigma\rho\phi_q \left(\int_0^s a(r)\nabla r + \frac{\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} a(r)\nabla r}{1 - \sum_{i=1}^{m-2} c_i} \right) \\
&= M\sigma\rho\psi(s).
\end{aligned}
\tag{2.39}
$$

For $t \in [0, T]_{\mathbf{T}^\kappa}$, then

$$
\begin{aligned}
u_0(t) &= Su_0(t) + \lambda_0 e(t) \\
&\geq Su_0(0) + \lambda_0 \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\nabla s}{1 - \sum_{i=1}^{m-2} b_i} + \lambda_0 \\
&\geq \frac{M\sigma\rho}{1 - \sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \psi(s)\nabla s + \lambda_0 \\
&= \sigma\rho + \lambda_0.
\end{aligned}
\tag{2.40}
$$

This together with Lemma 2.11(c) implies that

$$
\sigma\rho \geq \sigma\rho + \lambda_0,
\tag{2.41}
$$

a contradiction. Hence by Lemma 2.10(2) it follows that $i_K(S, \Omega_\rho) = 0$.   □

3. Main Results

We now give our results on the existence of positive solutions of BVP (1.6).

Theorem 3.1. *Suppose that conditions (H_1) and (H_2) hold, and assume that one of the following conditions holds.*

   (H_3) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \sigma\rho_2$ such that*

   (i) $f(t, u) \leq \phi_p(m\rho_1)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [0, \rho_1]$;
   (ii) $f(t, u) \geq 0$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_1, \rho_2]$, *moreover* $f(t, u) \geq \phi_p(M\sigma\rho_2)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_2, \rho_2]$.

   (H_4) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that*

   (i) $f(t, u) \leq \phi_p(m\rho_2)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [0, \rho_2]$;
   (ii) $f(t, u) \geq \phi_p(M\sigma\rho_1)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma^2\rho_1, \rho_2]$.

Then, the BVP (1.6) has at least one positive solution.

Proof. Assume that (H_3) holds, we show that S has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. By $f(t,u) \leq \phi_p(m\rho_1)$ and Lemma 2.13, we have that

$$i_K(S, K_{\rho_1}) = 1. \tag{3.1}$$

By $f(t,u) \geq \phi_p(M\sigma\rho_2)$ and Lemma 2.14, we have that

$$i_K(S, \Omega_{\rho_2}) = 0. \tag{3.2}$$

By Lemma 2.11(a) and $\rho_1 < \sigma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\sigma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.10(3) that S has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Clearly,

$$\|u_1\| > \rho_1, \quad \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u_1(t) \geq \sigma\|u_1\| > \sigma\rho_1, \tag{3.3}$$

which implies that $\sigma\rho_1 \leq u_1(t) \leq \rho_2, t \in [0,T]_{\mathbf{T}^\kappa}$. By condition (H_3)(ii), we have $f(t, u_1(t)) \geq 0$, $t \in [0,T]_{\mathbf{T}^\kappa}$, that is, $f^+(t, u_1(t)) = f(t, u_1(t))$. Hence,

$$Fu_1 = Su_1. \tag{3.4}$$

This means that u_1 is a fixed point of operator F.

When condition (H_4) holds, by $f(t,u) \leq \phi_p(m\rho_2)$ and Lemma 2.13, we have that

$$i_K(S, K_{\rho_2}) = 1. \tag{3.5}$$

By $f(t,u) \geq \phi_p(M\sigma\rho_1)$ and Lemma 2.14, we have that

$$i_K(S, \Omega_{\rho_1}) = 0. \tag{3.6}$$

By Lemma 2.11(a) and $\rho_1 < \rho_2$, we have $\overline{K}_{\sigma\rho_1} \subset \Omega_{\rho_1} \subset K_{\rho_2}$. It follows from Lemma 2.10(3) that S has a fixed point u_2 in $K_{\rho_2} \setminus \overline{\Omega}_{\rho_1}$. Obviously,

$$\|u_2\| > \sigma\rho_1, \quad \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u_2(t) \geq \sigma\|u_2\| > \sigma^2\rho_1, \tag{3.7}$$

which implies that $\sigma^2\rho_1 \leq u_2(t) \leq \rho_2, t \in [0,T]_{\mathbf{T}^\kappa}$. By condition (H_4)(ii), we have $f(t, u_2(t)) \geq 0, t \in [0,T]_{\mathbf{T}^\kappa}$, that is, $f^+(t, u_2(t)) = f(t, u_2(t))$. Hence,

$$Fu_2 = Su_2. \tag{3.8}$$

This means that u_2 is a fixed point of operator F. Therefore, the BVP (1.6) has at least one positive solution. $\square$

Theorem 3.2. *Assume that conditions (H_1) and (H_2) hold, and suppose that one of the following conditions holds.*

   (H_5) *There exist ρ_1, ρ_2, and $\rho_3 \in (0, +\infty)$ with $\rho_1 < \sigma \rho_2$, and $\rho_2 < \rho_3$ such that*

      (i) $f(t, u) \leq \phi_p(m\rho_1)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [0, \rho_1]$;

      (ii) $f(t, u) \geq 0$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_1, \rho_3]$, *moreover* $f(t, u) \geq \phi_p(M\sigma\rho_2)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_2, \rho_2]$, $u \neq Su, \forall u \in \partial\Omega_{\rho_2}$;

      (iii) $f(t, u) \leq \phi_p(m\rho_3)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [0, \rho_3]$.

   (H_6) *There exist ρ_1, ρ_2, and $\rho_3 \in (0, +\infty)$ with $\rho_1 < \rho_2 < \sigma\rho_3$ such that*

      (i) $f(t, u) \geq \phi_p(M\sigma\rho_1)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma^2\rho_1, \rho_2]$;

      (ii) $f(t, u) \leq \phi_p(m_1\rho_2)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [0, \rho_2]$, $u \neq Su$, $\forall u \in \partial K_{\rho_2}$;

      (iii) $f(t, u) \geq 0$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_2, \rho_3]$, *moreover,* $f(t, u) \geq \phi_p(M\sigma\rho_3)$, $(t, u) \in [0, T]_{\mathbf{T}^\kappa} \times [\sigma\rho_3, \rho_3]$.

Then, the BVP (1.6) *has at least two positive solutions.*

Proof. Assume that condition (H_5) holds, we show that S has a fixed point u_1 either in ∂K_{ρ_1} or in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. If $u \neq Su$ for $u \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$, by Lemmas 2.13 and 2.14, we have that

$$
\begin{aligned}
i_K(S, K_{\rho_1}) &= 1, \\
i_K(S, K_{\rho_3}) &= 1, \\
i_K(S, \Omega_{\rho_2}) &= 0.
\end{aligned}
\tag{3.9}
$$

By Lemma 2.11(a) and $\rho_1 < \sigma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\sigma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.10(3) that S has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Similarly, S has a fixed point u_2 in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. Clearly,

$$
\|u_1\| > \rho_1, \quad \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u_1(t) \geq \sigma\|u_1\| > \sigma\rho_1,
\tag{3.10}
$$

which implies that $\sigma\rho_1 \leq u_1(t) \leq \rho_2$, $t \in [0, T]_{\mathbf{T}^\kappa}$. By condition (H_5)(ii), we have $f(t, u_1(t)) \geq 0$, $t \in [0, T]_{\mathbf{T}^\kappa}$, that is, $f^+(t, u_1(t)) = f(t, u_1(t))$. Hence,

$$
Fu_1 = Su_1.
\tag{3.11}
$$

This means that u_1 is a fixed point of operator F. On the other hand, from $u_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$, $\rho_2 < \rho_3$ and Lemma 2.11(a), we have $K_{\sigma\rho_2} \subset \Omega_{\rho_2} \subset K_{\rho_3}$. Clearly,

$$
\|u_2\| > \sigma\rho_2, \quad \min_{t \in [0,T]_{\mathbf{T}^\kappa}} u_2(t) \geq \sigma\|u_2\| > \sigma^2\rho_2,
\tag{3.12}
$$

which implies that $\sigma^2\rho_2 \leq u_2(t) \leq \rho_3$, $t \in [0, T]_{\mathbf{T}^\kappa}$. By $\rho_1 < \sigma\rho_2$ and condition (H_5)(ii), we have $f(t, u_2(t)) \geq 0$, $t \in [0, T]_{\mathbf{T}^\kappa}$, that is, $f^+(t, u_2(t)) = f(t, u_2(t))$. Hence,

$$
Fu_2 = Su_2.
\tag{3.13}
$$

This means that u_2 is a fixed point of operator F. Then, the BVP (1.6) has at least two positive solutions.

When condition (H_6) holds, the proof is similar to the above, and so we omit it here.

$\square$

4. An Example

In the section, we present some simple examples to explain our results.

Example 4.1. Let $\mathbf{T} = [0, 1/2] \bigcup \{1\}$, $T = 1$. Consider the following three-point boundary value problem with p-Laplacian

$$\left(\phi_p(u^{\Delta\nabla})\right)^\nabla + a(t)f(t, u) = 0, \quad 0 < t < 1,$$

$$u(0) = \frac{1}{3}u\left(\frac{1}{2}\right), \quad u^\Delta(1) = 0, \quad \phi_p\left(u^{\Delta\nabla}(0)\right) = \frac{1}{4}\phi_p\left(u^{\Delta\nabla}\left(\frac{1}{2}\right)\right),$$

(4.1)

where $a(t) \equiv 1$, $b_1 = 1/3$, $c_1 = 1/4$, $\xi_1 = 1/2$, $p = q = 2$.

By computing, we can know $\sigma_1 = 1/5$, $\sigma_2 = 1/7$, $M = 48/5$, $m = 48/35$. Obviously, $\sigma = \sigma_1\sigma_2 = 1/35$, $M\sigma = 48/5 \times 1/35 < 48/35 = m$.

Let $\rho_1 = 1$, $\rho_2 = 78$, then $\sigma\rho_1 < \rho_1 < \sigma\rho_2 < \rho_2$. We define a sign changing nonlinearity as follows:

$$f(t, u) = \begin{cases} \dfrac{48t^3}{35}\left(u - \dfrac{1}{35}\right)^3, & 0 < t < 1, \ u \in \left[0, \dfrac{1}{35}\right], \\[3mm] \dfrac{48t^3}{35}\sin\left(\dfrac{35}{34}\dfrac{\pi}{2}u - \dfrac{1}{34}\dfrac{\pi}{2}\right), & 0 < t < 1, \ u \in \left[\dfrac{1}{35}, 1\right], \\[3mm] \dfrac{48t^3}{35}(2 - u) + \dfrac{3744}{175}(u - 1), & 0 < t < 1, \ u \in [1, 2], \\[3mm] \dfrac{3744}{175} + t^3(u - 2)^2, & 0 < t < 1, \ u \in [2, 78], \\[3mm] \dfrac{3744}{175} + t^3(78 - 2)^2[1 + (u - 78)], & 0 < t < 1, \ u \in [78, +\infty]. \end{cases}$$

(4.2)

Then, by the definition of f we have

(i) $f(t, u) \le \phi_p(m\rho_1) = 48/35$, $(t, u) \in [0, 1] \times [0, \rho_1]$;

(ii) $f(t, u) \ge 0$, $(t, u) \in [0, 1] \times [\sigma\rho_1, \rho_2]$, moreover $f(t, u) \ge \phi_p(M\sigma\rho_2) = 3744/175$, $(t, u) \in [0, 1] \times [\sigma\rho_2, \rho_2]$.

So condition (H_3) holds, and by Theorem 3.1, BVP (4.1) has at least one positive solution.

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