Research Article

Multiple Positive Solutions for a Class of m-Point Boundary Value Problems on Time Scales

Meiqiang Feng,[1] Xuemei Zhang,[2, 3] and Weigao Ge[3]

[1] *School of Science, Beijing Information Science & Technology University,*
  Beijing 100192, China
[2] *Department of Mathematics and Physics, North China Electric Power University,*
  Beijing 102206, China
[3] *Department of Applied Mathematics, Beijing Institute of Technology,*
  Beijing 100081, China

Correspondence should be addressed to Xuemei Zhang, zxm74@sina.com

By constructing an available integral operator and combining Krasnosel'skii-Zabreiko fixed point theorem with properties of Green's function, this paper shows the existence of multiple positive solutions for a class of m-point second-order Sturm-Liouville-like boundary value problems on time scales with polynomial nonlinearity. The results significantly extend and improve many known results for both the continuous case and more general time scales. We illustrate our results by one example, which cannot be handled using the existing results.

1. Introduction

Recently, there have been many papers working on the existence of positive solutions to boundary value problems for differential equations on time scales; see, for example, [1–20]. This has been mainly due to its unification of the theory of differential and difference equations. An introduction to this unification is given in [11, 12, 18, 19]. Now, this study is still a new area of fairly theoretical exploration in mathematics. However, it has led to several important applications, for example, in the study of insect population models, neural networks, heat transfer, and epidemic models; see, for example, [10, 11]. For some other excellent results and applications of the case that boundary value problems on time scales to a variety of problems from Khan et al. [21], Agarose et al. [22], Wang [23], Sun [24], Feng et al. [25], Feng et al. [26] and Feng et al. [27].

Motivated by the works mentioned above, we intend in this paper to study the existence of multiple positive solutions for the second-order m-point nonlinear dynamic

equation on time scales with polynomial nonlinearity:

$$-\left[p(t)x^\nabla\right]^\Delta (t) + q(t) x(t) = f(t, x(t)), \quad t_1 < t < t_m,$$

$$\alpha x(t_1) - \beta p(t_1) x^\nabla(t_1) = \sum_{i=2}^{m-1} a_i x(t_i),$$

$$\gamma x(t_m) + \delta p(t_m) x^\nabla(t_m) = \sum_{i=2}^{m-1} b_i x(t_i),$$

(1.1)

where $\mathbb{T}$ is a time scale,

$$p, q : [t_1, t_m] \longrightarrow (0, \infty), \quad p \in C^\Delta[t_1, t_m), \quad q \in C[t_1, t_m];$$

(1.2)

the points $t_i \in \mathbb{T}_k^k$ for $i \in \{1, 2, \ldots, m\}$ with $t_1 < t_2 < \cdots < t_m;$

$$\alpha, \gamma, \beta, \delta \in [0, \infty) \quad \alpha\gamma + \alpha\delta + \beta\gamma > 0, \qquad a_i, b_i \in [0, \infty), \ i \in \{2, 3, \ldots, m-1\};$$

(1.3)

$$f(t, x) = \sum_{j=1}^{n} c_j(t) x^{v_j}, \quad c_j \in C([t_1, t_m], [0, +\infty)), \ v_j \in [0, \infty), \ j = 1, 2, \ldots, n.$$

(1.4)

Recently, Xu [28] considered the following second-order two-point impulsive singular differential equations boundary value problem:

$$y'' + \sum_{j=1}^{n} a_j(t) x^{\alpha_j} = 0, \quad 0 < t < 1, \ t \neq t_1,$$

$$\Delta y|_{t=t_1} = I(y(t_1)),$$

$$y(0) = y(1) = 0.$$

(1.5)

By means of fixed point index theory in a cone, the author established the existence of two nonnegative solutions for problem (1.5).

More recently, by applying Guo-Krasnosel'skii fixed point theorem in a cone, Anderson and Ma [6] established the existence of at least one positive solution to the multipoint time-scale eigenvalue problem:

$$\left[py^\nabla\right]^\Delta (t) - q(t) y(t) + \lambda h(t) f(y) = 0, \quad t_1 < t < t_n,$$

$$\alpha y(t_1) - \beta p(t_1) y^\nabla(t_1) = \sum_{i=2}^{n-1} a_i y(t_i),$$

$$\gamma x(t_n) + \delta p(t_n) x^\nabla(t_n) = \sum_{i=2}^{n-1} b_i y(t_i),$$

(1.6)

where $f : [0, \infty) \to [0, \infty)$ is continuous.

As far as we know, there is no paper to study the existence of multiple positive solutions to problem (1.1) on time scales with polynomial nonlinearity. The objective of the present paper is to fill this gap. On the other hand, many difficulties occur when we study BVPs on time scales. For example, basic tools from calculus such as Fermat's theorem, Rolle's theorem and the intermediate value theorem may not necessarily hold. So it is interesting and important to discuss the problem (1.1). The purpose of this paper is to prove that the problem (1.1) possesses at least two positive solutions. Moreover, the methods used in this paper are different from [6, 28] and the results obtained in this paper generalize some results in [6, 28] to some degree.

The time scale related notations adopted in this paper can be found, if not explained specifically, in almost all literature related to time scales. The readers who are unfamiliar with this area can consult for example [11, 12, 18, 19] for details.

For convenience, we list the following well-known definitions.

Definition 1.1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

Definition 1.2. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ ($\rho(t)$ at t for $t > \inf \mathbb{T}$) by $\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}$ ($\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}$) for all $t \in \mathbb{T}$.

We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$ and say t is right-scattered, left-scattered, right-dense and left-dense if $\sigma(t) > t, \rho(t) < t, \sigma(t) = t$ and $\rho(t) = t$, respectively. Finally, we introduce the sets $\mathbb{T}^k$ and $\mathbb{T}_k$. which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1^*, then $\mathbb{T}^k = \mathbb{T} - t_1^*$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2^*, then $\mathbb{T}_k = \mathbb{T} - t_2^*$, otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 1.3. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t with

$$\left| [y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \tag{1.7}$$

for all $s \in U$, where y^Δ denotes the (delta) derivative of y with respect to the first variable, then

$$g(t) := \int_a^t \omega(t, \tau) \, \Delta\tau \tag{1.8}$$

implies

$$g^\Delta(t) = \int_a^t \omega^\Delta(t, \tau) \, \Delta\tau + \omega(\sigma(t), \tau). \tag{1.9}$$

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$$\left| [y(\rho(t)) - y(s)] - y^\nabla(t)[\rho(t) - s] \right| < \varepsilon |\rho(t) - s| \tag{1.10}$$

for all $s \in U$. Call $y^\nabla(t)$ the nabla derivative of $y(t)$ at the point t.

If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = f(t+1) - f(t)$ is the forward difference operator while $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

Definition 1.5. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at all right dense points of $\mathbb{T}$ and its left sided limit exists (finite) at left dense *points of* $\mathbb{T}$. We let $C_{\mathrm{rd}}^{0}(\mathbb{T})$ denote the set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$.

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Definition 1.7. A function $F : \mathbb{T}^{k} \rightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T}^{k} \rightarrow \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{k}$. In this case we define the delta integral of f by

$$\int_{a}^{t} f(s)\, \Delta s = F(t) - F(a), \tag{1.11}$$

for all $a, t \in \mathbb{T}$.

Definition 1.8. A function $\Phi : \mathbb{T}_{k} \rightarrow \mathbb{R}$ is called a nabla-antiderivative of $f : \mathbb{T}_{k} \rightarrow \mathbb{R}$ provided $\Phi^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_{k}$. In this case we define the nabla integral of f by

$$\int_{a}^{t} f(s)\, \nabla s = \Phi(t) - \Phi(a), \tag{1.12}$$

for all $a, t \in \mathbb{T}$.

2. Preliminaries

In this section, we provide some necessary background. In particular, we state some properties of Green's function associated with problem (1.1), and we then state a fixed-point theorem which is crucial to prove our main results.

The basic space used in this paper is $E = C[\rho(t_1), t_m]$. It is well known that E is a Banach space with the norm $\| \cdot \|$ defined by $\|x\| = \sup_{t \in [\rho(t_1), t_m]} |x(t)|$. Let P be a cone of E, $P_r = \{x \in P : \|x\| \leq r\}$, $\partial P_r = \{x \in P : \|x\| = r\}$, where $r > 0$.

In this paper, the Green's function of the corresponding homogeneous BVP is defined by

$$G(t, s) = \frac{1}{d} \begin{cases} \psi(t)\phi(s), & \text{if } \rho(t_1) \leq t \leq s \leq t_m, \\ \psi(s)\phi(t), & \text{if } \rho(t_1) \leq s \leq t \leq t_m, \end{cases} \tag{2.1}$$

where

$$d := \alpha\phi(t_1) - \beta p(t_1)\phi^{\nabla}(t_1) = \gamma\psi(t_m) + \delta p(t_m)\psi^{\nabla}(t_m), \tag{2.2}$$

and ϕ and ψ satisfy

$$-\left(p\psi^\nabla\right)^\Delta(t) + q(t)\,\psi(t) = 0, \qquad \psi(t_1) = \beta, \qquad p(t_1)\,\psi^\nabla(t_1) = \alpha,$$

$$-\left(p\phi^\nabla\right)^\Delta(t) + q(t)\,\phi(t) = 0, \qquad \phi(t_m) = \delta, \qquad p(t_m)\,\phi^\nabla(t_m) = -\gamma, \tag{2.3}$$

respectively.

Lemma 2.1 (see [6]). *Assume that (1.2) and (1.3) hold. Then $d > 0$ and the functions ψ and ϕ satisfy*

$$\psi(t) \geq 0, \quad t \in [\rho(t_1), t_m], \qquad \psi(t) > 0, \quad t \in (\rho(t_1), t_m],$$

$$p(t)\,\psi^\nabla(t) \geq 0, \quad t \in [\rho(t_1), t_m], \qquad \phi(t) \geq 0, \quad t \in [\rho(t_1), t_m], \tag{2.4}$$

$$\phi(t) > 0, \quad t \in [\rho(t_1), t_m), \qquad p(t)\,\phi^\nabla(t) \leq 0, \quad t \in [\rho(t_1), t_m].$$

From Lemma 2.1 and the definition of $G(t,s)$, we can prove that $G(t,s)$ has the following properties.

Proposition 2.2. *For $t, s \in [\xi_1, \xi_2]$, one has*

$$G(t,s) > 0, \tag{2.5}$$

where ξ_1, $\xi_2 \in \mathbb{T}_k^k$, $\rho(t_1) < \xi_1 < \xi_2 < t_m$.

In fact, from Lemma 2.1, we have $\psi(t) > 0, \quad \phi(t) > 0$ for $t \in [\xi_1, \xi_2]$. Therefore (2.5) holds.

Proposition 2.3. *If (1.2) holds, then for $t, s \in [\rho(t_1), t_m] \times [\rho(t_1), t_m]$, one has*

$$0 \leq G(t,s) \leq G(s,s). \tag{2.6}$$

Proof. In fact, from Lemma 2.1, we obtain $\psi(t) \geq 0, \quad \phi(t) \geq 0$ for $t \in [\rho(t_1), t_m]$. So $G(t,s) \geq 0$.

On the other hand, from Lemma 2.1, we know that $p(t)\psi^\nabla(t) \geq 0, \quad p(t)\phi^\nabla(t) \leq 0$ for $t \in [\rho(t_1), t_m]$. This together with $p(t) > 0$ implies that $\psi^\nabla(t) \geq 0, \quad \phi^\nabla(t) \leq 0$ for $t \in [\rho(t_1), t_m]$. Hence $\psi(t)$ is nondecreasing on $[\rho(t_1), t_m]$, ϕ is nonincreasing on $[\rho(t_1), t_m]$. So (2.6) holds. $\square$

Proposition 2.4. *For all $t \in [\xi_1, \xi_2]$, $s \in [\rho(t_1), t_m]$ one has*

$$G(t,s) \geq \sigma(t)\,G(s,s), \tag{2.7}$$

where

$$\sigma(t) := \min \left\{ \frac{\psi(t)}{\psi(t_m)}, \frac{\phi(t)}{\phi(\rho(t_1))} \right\}. \tag{2.8}$$

Proof. In fact, for $t \in [\xi_1, \xi_2]$, we have

$$\frac{G(t,s)}{G(s,s)} \geq \min \left\{ \frac{\psi(t)}{\psi(s)}, \frac{\phi(t)}{\phi(s)} \right\} \geq \min \left\{ \frac{\psi(t)}{\psi(t_m)}, \frac{\phi(t)}{\phi(\rho(t_1))} \right\} =: \sigma(t). \tag{2.9}$$

Therefore (2.7) holds. □

It is easy to see that $0 < \sigma(t) < 1$, for $t \in [\xi_1, \xi_2]$. Thus, there exists $\overline{\gamma} > 0$ such that $G(t,s) \geq \overline{\gamma}G(s,s)$ for $t \in [\xi_1, \xi_2]$, where

$$\overline{\gamma} = \min \{\sigma(t) : t \in [\xi_1, \xi_2]\}. \tag{2.10}$$

We remark that Proposition 2.2 implies that there exists $\tau > 0$ such that for $t, s \in [\xi_1, \xi_2]$

$$G(t,s) \geq \tau. \tag{2.11}$$

Set

$$D := \begin{vmatrix} -\displaystyle\sum_{i=1}^{m-2} a_i \psi(t_i) & d - \displaystyle\sum_{i=1}^{m-2} a_i \phi(t_i) \\ d - \displaystyle\sum_{i=1}^{m-2} b_i \psi(t_i) & -\displaystyle\sum_{i=1}^{m-2} b_i \phi(t_i) \end{vmatrix}. \tag{2.12}$$

Lemma 2.5 (see [6]). *Assume that (1.2) and (1.3) hold. If $D \neq 0$ and $u \in C_{rd}[t_1, t_m]$, then the nonhomogeneous boundary value problem*

$$-\left[p(t) x^{\nabla}\right]^{\Delta}(t) + q(t) x(t) = u(t), \quad t_1 < t < t_m,$$

$$\alpha x(t_1) - \beta p(t_1) x^{\nabla}(t_1) = \sum_{i=2}^{m-1} a_i x(t_i), \tag{2.13}$$

$$\gamma x(t_m) + \delta p(t_m) x^{\nabla}(t_m) = \sum_{i=2}^{m-1} b_i x(t_i)$$

has a unique solution x for which the formula

$$x(t) = \int_{t_1}^{t_m} G(t,s) u(s) \Delta s + \Gamma(u(t)) \psi(t) + \Upsilon(u(t)) \phi(t) \tag{2.14}$$

holds, where

$$\Gamma\left(u\left(s\right)\right):=\frac{1}{D}\begin{vmatrix} \sum\limits_{i=2}^{m-1}a_i\int_{t_1}^{t_m}G\left(t_i,s\right)u\left(s\right)\Delta s & d-\sum\limits_{i=2}^{m-1}a_i\phi\left(t_i\right) \\ \sum\limits_{i=2}^{m-1}b_i\int_{t_1}^{t_m}G\left(t_i,s\right)u\left(s\right)\Delta s & -\sum\limits_{i=2}^{m-1}b_i\phi\left(t_i\right) \end{vmatrix}, \tag{2.15}$$

$$\Upsilon\left(u\left(s\right)\right):=\frac{1}{D}\begin{vmatrix} -\sum\limits_{i=2}^{m-1}a_i\psi\left(t_i\right) & \sum\limits_{i=2}^{m-1}a_i\int_{t_1}^{t_m}G\left(t_i,s\right)u\left(s\right)\Delta s \\ d-\sum\limits_{i=2}^{m-1}b_i\psi\left(\xi_i\right) & \sum\limits_{i=2}^{m-1}b_i\int_{t_1}^{t_m}G\left(t_i,s\right)u\left(s\right)\Delta s \end{vmatrix}. \tag{2.16}$$

By similar method, one can define

$$\begin{aligned}
\Gamma_0\left(f\left(t,x_0\left(t\right)\right)\right), & \quad \Gamma_1\left(f\left(t,x_1\left(t\right)\right)\right), & \quad \Gamma_2\left(f\left(t,x_2\left(t\right)\right)\right), & \quad \Gamma_*\left(f\left(t,x_*\left(t\right)\right)\right), \\
\Upsilon_0\left(f\left(t,x_0\left(t\right)\right)\right), & \quad \Upsilon_1\left(f\left(t,x_1\left(t\right)\right)\right), & \quad \Upsilon_2\left(f\left(t,x_2\left(t\right)\right)\right), & \quad \Upsilon_*\left(f\left(t,x_*\left(t\right)\right)\right).
\end{aligned} \tag{2.17}$$

The following lemma is crucial to prove our main results.

Lemma 2.6 (see[29, 30]). *Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let the operator $A : P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to P$ be completely continuous, where P is a cone in E. Suppose that one of the two conditions*

$$(i) \quad Ax \not\geq x, \quad \forall x \in P \cap \partial\Omega_1; \qquad Ax \not\leq x, \quad \forall x \in P \cap \partial\Omega_2, \tag{2.18}$$

or

$$(ii) \quad Ax \not\leq x, \quad \forall x \in P \cap \partial\Omega_1; \qquad Ax \not\geq x, \quad \forall x \in P \cap \partial\Omega_2, \tag{2.19}$$

is satisfied. Then A has at least one fixed point in $P \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right)$.

3. Main Results

In this section, we apply Lemma 2.6 to establish the existence of at least two positive solutions for BVP (1.1).

The following assumptions will stand throughout this paper.

(H_1) There exist $v_{j1} < 1$, $v_{j2} > 1$ such that

$$\inf_{t\in[\xi_1,\xi_2]} c_{j1}\left(t\right) = \tau_1 > 0, \qquad \inf_{t\in[\xi_1,\xi_2]} c_{j2}\left(t\right) = \tau_2 > 0, \quad j = 1,2,\ldots,n, \tag{3.1}$$

where v_{j1}, v_{j2}, $c_{j1}(t)$ and $c_{j2}(t)$ are defined in (1.4), respectively.

(H_2) We have

$$D < 0, \qquad d - \sum_{i=2}^{m-1} a_i \phi(t_i) > 0, \qquad d - \sum_{i=2}^{m-1} b_i \psi(t_i) > 0 \qquad (3.2)$$

for d and D given in (2.2) and (2.12), respectively.

If (H_2) holds, then we can show that $\Gamma(f(t,x)), \Upsilon(f(t,x))$ have the following properties.

Proposition 3.1. *If* (1.2)–(1.4) *and* (H_2) *hold, then from* (2.15), *for* $x \in C[\rho(t_1), t_m]$, *one has*

$$\left| \Gamma(f(t,x)) \right| \le \frac{1}{D} \begin{vmatrix} \sum_{i=2}^{m-1} a_i & d - \sum_{i=2}^{m-1} a_i \phi(t_i) \\ \sum_{i=2}^{m-1} b_i & -\sum_{i=2}^{m-1} b_i \phi(t_i) \end{vmatrix} M \sum_{j=1}^{n} \|c_j\|_L \|x\|^{v_j} =: \tilde{\Gamma} M \sum_{j=1}^{n} \|c_j\|_L \|x\|^{v_j}, \qquad (3.3)$$

where $\|c_j\|_L := \int_{t_1}^{t_m} |c_j(s)| \Delta s$, $M = \max_{(t,s) \in [\rho(t_1), t_m] \times [\rho(t_1), t_m]} G(t,s)$.

Proof. Let

$$G = \sum_{i=2}^{m-1} a_i \int_{t_1}^{t_m} G(t_i, s) f(s, x(s)) \Delta s, \qquad H = d - \sum_{i=2}^{m-1} a_i \phi(t_i),$$

$$F = \sum_{i=2}^{m-1} b_i \int_{t_1}^{t_m} G(t_i, s) f(s, x(s)) \Delta s, \qquad Q = -\sum_{i=2}^{m-1} b_i \phi(t_i). \qquad (3.4)$$

Then from (1.2)–(1.4) and (H_2), we obtain $G \ge 0$, $F \ge 0$, $H > 0$, $Q \le 0$. Therefore, $GQ \le 0$, $-FH \le 0$.
    On the other hand, since

$$\int_{t_1}^{t_m} G(t_i, s) f(s, x(s)) \Delta s \le M \sum_{j=1}^{n} \|c_j\|_L \|x\|^{v_j} =: \Lambda, \qquad (3.5)$$

we have $G \le \sum_{i=2}^{m-1} a_i \Lambda$, $F \le \sum_{i=2}^{m-1} b_i \Lambda$. So one has

$$\sum_{i=2}^{m-1} a_i \Lambda Q - H \sum_{i=2}^{m-1} b_i \Lambda \le GQ - FH \le 0. \qquad (3.6)$$

This and $D < 0$ imply (3.3) holds. $\qquad \qquad \square$

Proposition 3.2. *If (1.2)–(1.4) and (H_2) hold, then from (2.16), $x \in C[\rho(t_1), t_m]$, one has*

$$|\Upsilon(f(t,x))| \le \frac{1}{D} \begin{vmatrix} -\sum_{i=2}^{m-1} a_i \psi(t_i) & \sum_{i=2}^{m-1} a_i \\ d - \sum_{i=2}^{m-1} b_i \psi(\xi_i) & \sum_{i=2}^{m-1} b_i \end{vmatrix} M \sum_{j=1}^{n} \|c_j\|_L \|x\|^{\upsilon_j} =: \tilde{\Upsilon} M \sum_{j=1}^{n} \|c_j\|_L \|x\|^{\upsilon_j}. \tag{3.7}$$

Proof. The proof is similar to that of Proposition 3.1. So we omit it. □

For the sake of applying fixed point theorem on cone, we construct a cone in $E = C[\rho(t_1), t_m]$ by

$$P = \left\{ x \in E : x(t) \ge 0, \ t \in [\rho(t_1), t_m], \min_{t \in [\xi_1, \xi_2]} x(t) \ge \overline{\gamma} \|x\| \right\}, \tag{3.8}$$

where $\overline{\gamma}$ is defined in (2.10).

Define $A : P \to P$ by

$$(Ax)(t) = \int_{t_1}^{t_m} G(t,s) f(s, x(s)) \Delta s + \Gamma(f(t, x(t))) \psi(t) + \Upsilon(f(t, x(t))) \phi(t). \tag{3.9}$$

By (2.14), it is well known that the problem (1.1) has a positive solution x if and only if $x \in P$ is a fixed point of A.

Lemma 3.3. *Suppose that (1.2)–(1.4) and (H_1)-(H_2) hold. Then $A(P) \subset P$ and $A : P \to P$ is completely continuous.*

Proof. For $x \in P$, by (2.14), we have $Ax(t) \ge 0$ and

$$\|Ax\| \le \int_{t_1}^{t_m} G(s,s) f(s, x(s)) \Delta s + \Gamma(f(t, x(t))) \psi(t_m) + \Upsilon(f(t, x(t))) \phi(\rho(t_1)). \tag{3.10}$$

On the other hand, for $t \in [\xi_1, \xi_2]$, by (3.9),(3.10) and (2.7), we obtain

$$\min_{t \in [\xi_1, \xi_2]} Ax(t) = \min_{t \in [\xi_1, \xi_2]} \left[\int_{t_1}^{t_m} G(t,s) f(s, x(s)) \Delta s + \Gamma(f(t, x(t))) \psi(t) + \Upsilon(f(t, x(t))) \phi(t) \right]$$

$$\ge \sigma(t) \left[\int_{t_1}^{t_m} G(s,s) f(s, x(s)) \Delta s + \Gamma(f(t, x(t))) \psi(t_m) + \Upsilon(f(t, x(t))) \phi(\rho(t_1)) \right]$$

$$\ge \sigma(t) \|Ax\| \ge \overline{\gamma} \|Ax\|. \tag{3.11}$$

Therefore $Ax \in P$, that is, $A(P) \subset P$.

Next by standard methods and the Ascoli-Arzela theorem one can prove that $A : P \to P$ is completely continuous. So it is omitted. □

Theorem 3.4. *Suppose that* (1.2)–(1.4) *and* (H_1)-(H_2) *hold. Then problem* (1.1) *has at least two positive solutions provided*

$$\sum_{j=1}^{n} \|c_j\|_L \left[1 + \widetilde{\Gamma}\psi(t_n) + \widetilde{\Upsilon}\phi(t_1)\right] < M^{-1}, \tag{3.12}$$

where $\widetilde{\Gamma}$, $\widetilde{\Upsilon}$ *and* M *are defined in* (3.3), (3.7) *and in Proposition 3.1, respectively.*

Proof. Let A be the cone preserving, completely continuous operator that was defined by (3.9).

Let $S_l = \{x \in E : \|x\| < l\}$, where $l > 0$. Choosing r and $\bar{r}$ satisfy

$$0 < r < \min\left\{1, (\tau\tau_1(\xi_2 - \xi_1))^{1/(1-v_{j1})} \overline{\gamma}^{v_{j1}/(1-v_{j1})}\right\},$$
$$\bar{r} > \max\left\{1, (\tau\tau_2(\xi_2 - \xi_1))^{-1/(v_{j2}-1)} \overline{\gamma}^{(-v_{j2})/(v_{j2}-1)}\right\}. \tag{3.13}$$

Now we prove that

$$Ax \nleq x, \quad \forall x \in P \cap \partial S_r, \tag{3.14}$$

$$Ax \nleq x, \quad \forall x \in P \cap \partial S_{\bar{r}}. \tag{3.15}$$

In fact, if there exists $x_1 \in P \cap \partial S_r$ such that $Ax_1 \leq x_1$, then for $t \in [\xi_1, \xi_2]$, we have

$$
\begin{aligned}
x_1(t) &\geq Ax_1(t) \\
&= \int_{t_1}^{t_m} G(t,s) f(s, x_1(s)) \Delta s + \Gamma_1\left(f(t, x_1(t))\right) \psi(t) + \Upsilon_1\left(f(t, x_1(t))\right) \phi(t) \\
&\geq \int_{t_1}^{t_m} G(t,s) f(s, x_1(s)) \Delta s \\
&\geq \int_{\xi_1}^{\xi_2} G(t,s) c_{j1}(s) [x_1(s)]^{v_{j1}} \Delta s \\
&\geq \tau\tau_1(\xi_2 - \xi_1) \overline{\gamma}^{v_{j1}} \|x_1\|^{v_{j1}},
\end{aligned} \tag{3.16}
$$

where $\Gamma_1(f(t, x(t)))$, $\Upsilon_1(f(t, x(t)))$ defined by (2.17).

Therefore $r \geq \tau\tau_1(\xi_2 - \xi_1)\overline{\gamma}^{v_{j1}} r^{v_{j1}}$, that is, $r \geq (\tau\tau_1(\xi_2 - \xi_1))^{1/(1-v_{j1})} \overline{\gamma}^{v_{j1}/(1-v_{j1})}$, which is a contradiction. Hence (3.14) holds.

Next, turning to (3.15). If there exists $x_2 \in P \cap \partial S_{\bar{r}}$ such that $Ax_2 \leq x_2$, then for $t \in [\xi_1, \xi_2]$, we have

$$
\begin{aligned}
x_2(t) &\geq Ax_2(t) \\
&= \int_{t_1}^{t_m} G(t,s) f(s, x_2(s)) \Delta s + \Gamma_2\left(f(t, x_2(t))\right) \psi(t) + \Upsilon_2\left(f(t, x_2(t))\right) \phi(t) \\
&\geq \int_{t_1}^{t_m} G(t,s) f(s, x_2(s)) \Delta s \\
&\geq \int_{\xi_1}^{\xi_2} G(t,s) c_{j2}(s) [x_2(s)]^{v_{j2}} \Delta s \\
&\geq \tau \tau_2 (\xi_2 - \xi_1) \bar{\gamma}^{v_{j2}} \|x_2\|^{v_{j2}},
\end{aligned}
$$

(3.17)

where $\Gamma_2(f(t, x(t)))$, $\Upsilon_2(f(t, x(t)))$ are defined by (2.17).

Therefore $\bar{r} \geq \tau\tau_2(\xi_2 - \xi_1)\bar{\gamma}^{v_{j2}}\bar{r}^{v_{j2}}$, that is, $\bar{r} \leq (\tau\tau_2(\xi_2 - \xi_1))^{-1/(v_{j2}-1)}\bar{\gamma}^{-v_{j2}/(v_{j2}-1)}$, which is a contradiction. Hence (3.15) holds.

It remains to prove

$$
Ax \not\geq x, \quad \forall x \in P \cap \partial S_1. \tag{3.18}
$$

In fact, if there exists $x_0 \in P \cap \partial S_1$ such that $Ax_0 \geq x_0$, then for $t \in (t_1, t_m) \cap \mathbb{T}$, we have

$$
1 = \|x_0\| \leq \|Ax_0\| \leq M \sum_{j=1}^{n} \|c_j\|_L \|x_0\|^{v_j} \left[1 + \tilde{\Gamma}\psi(t_m) + \tilde{\Upsilon}\phi(t_1)\right] \tag{3.19}
$$

that is,

$$
\sum_{j=1}^{n} \|c_j\|_L \left[1 + \tilde{\Gamma}\psi(t_m) + \tilde{\Upsilon}\phi(t_1)\right] \geq M^{-1}, \tag{3.20}
$$

which is a contradiction, where $\Gamma_0(f(t, x(t)))$, $\Upsilon_0(f(t, x(t)))$ are defined by (2.17). Hence (3.18) holds. From Lemma 2.6, (3.14), (3.15) and (3.18) yield that the problem (1.1) has at least two solutions x_*, x_{**} and $x_* \in P \cap (S_{\bar{r}} \setminus \overline{S}_1)$, $x_{**} \in P \cap (S_1 \setminus \overline{S}_r)$. The proof is complete. $\square$

4. Example

Example 4.1. To illustrate how our main results can be used in practice we present an example.

Let $\mathbb{T} = \{0, 1/2, 1/4, \ldots, 1/2^n, \ldots, 1\}$. Take $p(t) \equiv 1$, $q(t) \equiv 0$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, $t_1 = 0$, $t_2 = 1/2$, $t_m = 1$, $a_2 = 1/2$, $b_2 = 1$ in (1.1). Now we consider the following three point boundary value problem

$$x^{\nabla\Delta}(t) = f(t, x(t)), \quad 0 < t < 1,$$

$$x(0) = \frac{1}{2}x\left(\frac{1}{2}\right) \qquad x(1) = x\left(\frac{1}{2}\right), \tag{4.1}$$

where

$$f(t, x) = \frac{1}{10}tx^{1/2} + \frac{1}{20}t^2x + \left(t - t^2\right)x^2. \tag{4.2}$$

It is not difficult to see that

$$c_1(t) = \frac{1}{10}t, \qquad c_2(t) = \frac{1}{20}t^2, \qquad c_3(t) = t - t^2, \qquad v_1 = \frac{1}{2}, \qquad v_2 = 1, \qquad v_3 = 2. \tag{4.3}$$

On the other hand, by calculating we have $\psi(t) = t$, $\phi(t) = 1 - t$, $d = 1$, $d - \sum_{i=2}^{m-1} a_i\phi(t_i) = 1 - 1/2 \times 1/2 = 3/4 > 0$, $d - \sum_{i=2}^{m-1} b_i\psi(t_i) = 1 - 1 \times 1/2 = 1/2 > 0$,

$$D = \begin{vmatrix} -\dfrac{1}{4} & \dfrac{3}{4} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} < 0, \qquad G(t, s) = \begin{cases} s(1 - t), \\ t(1 - s), \end{cases} \tag{4.4}$$

and $M = \max_{t,s \in [0,1]} G(t, s) = 1/4, \tilde{\Gamma} = 7/2, \tilde{\Upsilon} = 2$.

Let $v_{j1} = 1/2$, $v_{j2} = 2$, $c_{j1}(t) = (1/10)t$, $c_{j2}(t) = t - t^2$. Then $v_{j1} < 1$, $v_{j2} > 1$ and

$$\inf_{t \in [\xi_1, \xi_2]} c_{j1}(t) = \frac{1}{10}\xi_1 > 0, \qquad \inf_{t \in [\xi_1, \xi_2]} c_{j2}(t) = \min\{\xi_1(1 - \xi_1), \xi_2(1 - \xi_2)\} > 0, \quad j = 1, 2, 3. \tag{4.5}$$

It follows that (H_1) and (H_2) hold.

Finally, we prove that

$$\sum_{j=1}^{n} \|c_j\|_L \left[1 + \tilde{\Gamma}\psi(t_n) + \tilde{\Upsilon}\phi(t_1)\right] < M^{-1}. \tag{4.6}$$

In fact, from $\tilde{\Gamma} = 7/2$, $\tilde{\Upsilon} = 2$, $\psi(1) = 1$, $\phi(0) = 1$, we have $1 + \tilde{\Gamma}\psi(t_n) + \tilde{\Upsilon}\phi(t_1) = 13/2$, and

$$\sum_{j=1}^{n} \|c_j\|_L \left[1 + \tilde{\Gamma}\psi(t_n) + \tilde{\Upsilon}\phi(t_1)\right] = \frac{2}{5} \times \frac{13}{2} = \frac{13}{5} < 4 = M^{-1}. \tag{4.7}$$

Therefore, the conditions of **Theorem 3.4** hold. Hence problem (4.1) has at least two positive solutions.

Remark 4.2. **Example** 4.1 implies that there is a large number of functions that satisfy the conditions of **Theorem 3.4**. In addition, the conditions of **Theorem 3.4** are also easy to check.

Acknowledgments

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