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Research Article

Stabilization of Discrete-Time Control Systems with Multiple State Delays

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We give sufficient conditions for the exponential stabilizability of a class of perturbed time-varying difference equations with multiple delays and slowly varying coefficients. Under appropiate growth conditions on the perturbations, combined with the "freezing" technique, we establish explicit conditions for global feedback exponential stabilizability.

1. Introduction

Let us consider a discrete-time control system described by the following equation in C^n:

$$x(k+1) = A(k)x(k) + A_1(k)x(k-r) + B(k)u(k), \tag{1.1}$$

$$x(k) = \varphi(k), \quad k \in \{-r, -r+1, \ldots, 0\}, \tag{1.2}$$

where C^n denotes the n-dimensional space of complex column vectors, $r \geq 1$ is a given integer, $x : Z^+ \to C^n$ is the state, $u : Z^+ \to C^m$ $(m \leq n)$ is the input, Z^+ is the set of nonnegative integers. Hence forward, $\|\cdot\| = \|\cdot\|_{C^n}$ is the Euclidean norm; A and B are variable matrices of compatible dimensions, A_1 is a variable $n \times n$-matrix such that

$$\sup_{k \geq 0} \|A_1(k)\| < \infty, \tag{1.3}$$

and φ is a given vector-valued function, that is, $\varphi(k) \in C^n$.

The stabilizability question consists on finding a feedback control law $u(k) = L(k)x(k)$, for keeping the closed-loop system

$$x(k+1) = [A(k) + B(k)L(k)]x(k) + A_1(k)x(k-r), \tag{1.4}$$

asymptotically stable in the Lyapunov sense.

The stabilization of control systems is one of the most important properties of the systems and has been studied widely by many reseachers in control theory; (see, e.g., [1–11]) and the references therein. It is recognized that the Lyapunov function method serves as a main technique to reduce a given complicated system into a relatively simpler system and provides useful applications to control theory, but finding Lyapunov functions is still a difficult task (see, e.g., [1–3, 12, 13]). By contrast, many methods different from Lyapunov functions have been successfully applied to establish stabilizability results for discrete-time equations. For example, to the linear system

$$x(k+1) = A(k)x(k), \quad k \in Z^+, \tag{1.5}$$

if the evolution operator $\Phi(k,s)$ generated by $A(k)$ is stable, then the delay control system (1.1)-(1.2) is asymptotically stabilizable under appropiate conditions on $A_1(k)$ (see [4, 8, 14]). For infinite-dimensional control systems, the study of stabilizabilization is more complicated and requires sophisticated techniques from semigroup theory.

The concept of stabilizability has been developed and successfully applied in different settings (see, e.g., [9, 15, 16]). For example, finite- and infinite-dimensional discrete-time control systems have been studied extensively (see, e.g., [2, 5, 6, 10, 17–20]).

The stabilizability conditions obtained in this paper are derived by using the "freezing" technique (see, e.g., [21–23]) for perturbed systems of difference equations with slowly varying coefficients and do not involve either Lyapunov functions or stability assumptions on the associated evolution operator $\Phi(k,s)$. With more precision, the freezing technique can be described as follows. If $m \in Z^+$ is any fixed integer, then we can think of the autonomous system

$$x(k+1) = A(m)x(k) + A_1(m)x(k-r) + B(m)u(k) \tag{1.6}$$

as a particular case of the system (1.1), with its time dependence "frozen" at time m. Thus, in this paper it is shown that if each frozen system is exponentially stabilizable and the rate of change of the coefficients of system (1.1) is small enough, then the nonautonomous system (1.1)-(1.2) is indeed exponentially stabilizable.

The purpose of this paper is to establish sufficient conditions for the global exponential feedback stabilizability of perturbed control systems with both time-varying and time-delayed states.

Our main contributions are as follows. By applying the "freezing" technique to the control system (1.1)-(1.2), we derive explicit stabilizability conditions, provided that the coefficients are slowly varying. Applications of the main results to control systems with many delays and nonlinear perturbations will also be established in this paper. This technique will allow us to avoid constructing the Lyapunov functions in some situations. For instance, it is worth noting that Niamsup and Phat [2] established sufficient stabilizability conditions for the zero solution of a discrete-time control system with many delays, under exponential growth assumptions on the corresponding transition matrix. By contrast, our approach does not involve any stability assumption on the transition matrix.

The paper is organized as follows. In Section 2 we introduce notations, definition, and some preliminary results. In Section 3, we give new sufficient conditions for the global exponential stabilizability of discrete-time systems with time-delayed states. Finally, as an application, we consider the global stabilization of the nonlinear control systems.

2. Preliminaries

In this paper we will use the following control law:

$$u(k) = L(k)x(k), \tag{2.1}$$

where $L(k)$ is a variable $m \times n$-matrix.

To formulate our results, let us introduce the following notation. Let A be a constant $n \times n$ matrix and let $\lambda_j(A)$, $j = 1, 2, \dots, n$, denote the eigenvalues of A, including their multiplicities. Put

$$g(A) = \left[N^2(A) - \sum_{j=1}^{n} \left|\lambda_j(A)\right|^2 \right]^{1/2}, \tag{2.2}$$

where $N(A)$ is the Hilbert-Schmidt (Frobenius) norm of A; that is, $N^2(A) = \operatorname{Trace}(AA^*)$.

The following relation

$$g(A) \leq \sqrt{\frac{1}{2}} N(A - A^*) \tag{2.3}$$

is true, and will be useful to obtain some estimates in this work.

Theorem A ([17, Theorem 3.7]). *For any $n \times n$-matrix A, the inequality*

$$\|A^m\| \leq \sum_{j=0}^{m_1} \frac{m! (\rho(A))^{m-j} (g(A))^j}{(m-j)! (j!)^{3/2}} \tag{2.4}$$

holds for every nonnegative integer m, where $\rho(A)$ is the spectral radius of A, and $m_1 = \min\{m, n-1\}$.

Remark 2.1. In general, the problem of obtaining a precise estimate for the norm of matrix-valued and operator-valued functions has been regularly discussed in the literature, for example, see Gel'fond and Shilov [24] and Daleckii and Krein [25].

The following concepts of stability will be used in formulating the main results of the paper (see, e.g., [26]).

Definition 2.2. The zero solution of system (1.4)–(1.2) is stable if for every $\varepsilon > 0$ and every $k_0 \in Z^+$, there is a number $\delta > 0$ (depending on ε and k_0) such that every solution $x(k)$ of the system with $\|\varphi(k)\| < \delta$ for all $r - k_0, r - k_0 + 1, \ldots, k_0$, satisfies the condition

$$\|x(k)\| < \varepsilon, \quad \forall k \in Z^+. \tag{2.5}$$

Definition 2.3. The zero solution of (1.4) is globally exponentially stable if there are constants $M > 0$ and $c_0 \in (0, 1)$ such that

$$\|x(k)\| \leq M c_0^k \max_{-r \leq s \leq 0} \|\varphi(s)\|, \quad (k \in Z^+) \tag{2.6}$$

for any solution $x(k)$ of (1.4) with the initial conditions (1.2).

Definition 2.4. The pair $(A(k), B(k))$ is said to be stabilizable for each $k \in Z^+$ if there is a matrix $L(k)$ such that all the eigenvalues of the matrix $C_L(k) = A(k) + B(k)L(k)$ are located inside the unit disk for every fixed $k \in Z^+$. Namely,

$$\rho_L = \sup_{k \in Z^+} \rho(C_L(k)) < 1. \tag{2.7}$$

Remark 2.5. The control $u(k) = L(k)x(k)$ is a feedback control of the system.

Definition 2.6. System (1.1) is said to be globally exponentially stabilizable (at $x = 0$) by means of the feedback law (2.1) if there is a variable matrix $L(k)$ such that the zero solution of (1.4) is globally exponentially stable.

3. Main Results

Now, we are ready to establish the main results of the paper, which will be valid for the system (1.1)-(1.2) with slowly varying coefficients.

Consider in C^n the equation

$$x(k + 1) = T(k)x(k) + A_1(k)x(k - r), \tag{3.1}$$

subject to the initial conditions (1.2), where $r \geq 1$ is a given integer and $T(k)$ is a variable $n \times n$-matrix.

Proposition 3.1. *Suppose that*

(a) $p = \sup_{k \geq 0} \|A_1(k)\| < \infty$,

(b) *there is a constant* $q > 0$ *such that*

$$\left\|T(k) - T(j)\right\| \leq q\left|k - j\right|; \quad k, j \in Z^+, \tag{3.2}$$

(c) $S_0 =: S_0(T(\cdot), A_1(\cdot)) = \sum_{k=0}^{\infty}(qk + p)\sup_{l=0,1,\ldots}\|T^k(l)\| < 1.$

Then the zero solution of system (3.1)–(1.2) *is globally exponentially stable. Moreover, any solution of* (3.1) *satisfies the inequality*

$$\|x(k)\| \leq \frac{\beta_0 \|\varphi(0)\| + \gamma}{1 - S_0}, \quad k = 1, 2, \ldots, \tag{3.3}$$

where

$$\beta_0 = \sup_{l,k=0,1,\ldots} \left\| T^k(l) \right\| < \infty, \qquad \gamma = p \max_{-r \leq k \leq 0} \|\varphi(k)\| \sum_{k=0}^{\infty} \sup_{l=0,1,\ldots} \left\| T^k(l) \right\|. \tag{3.4}$$

Proof. Rewrite (3.1) in the form

$$x(k+1) - T(s)x(k) = (T(k) - T(s))x(k) + A_1(k)x(k-r), \tag{3.5}$$

with a fixed nonnegative integer s. The variation of constants formula yields

$$x(m+1) = T^{m+1}(s)\varphi(0) + \sum_{j=0}^{m} T^{m-j}(s)\left[(T(j) - T(s))x(j) + A_1(j)x(j-r)\right]. \tag{3.6}$$

Taking $s = m$, we have

$$x(m+1) = T^{m+1}(m)\varphi(0) + \sum_{j=0}^{m} T^{m-j}(m)\left[(T(j) - T(m))x(j) + A_1(j)x(j-r)\right]. \tag{3.7}$$

Hence,

$$\begin{aligned} \|x(m+1)\| &\leq \beta_0 \|\varphi(0)\| + \sum_{j=0}^{m} \left\| T^{m-j}(m) \right\| \left[q|m-j| \|x(j)\| + \|A_1(j)\| \|x(j-r)\| \right] \\ &\leq \beta_0 \|\varphi(0)\| + q \max_{0 \leq k \leq m} \|x(k)\| \sum_{k=0}^{m} \left\| T^k(m) \right\| k + \sum_{k=0}^{m} \left\| T^k(m) \right\| \max_{-r \leq j \leq k} \|x(j)\| \|A_1(k)\| \\ &\leq \beta_0 \|\varphi(0)\| + \max_{0 \leq k \leq m} \|x(k)\| \sum_{k=0}^{\infty} (qk + p) \sup_{l=0,1,\ldots} \left\| T^k(l) \right\| + \gamma. \end{aligned} \tag{3.8}$$

Thus,

$$\|x(m+1)\| \leq \beta_0 \|\varphi(0)\| + \gamma + \max_{0 \leq j \leq m} \|x(j)\| \sum_{k=0}^{\infty} (qk + p) \sup_{l=0,1,\ldots} \left\| T^k(l) \right\|. \tag{3.9}$$

Hence,

$$\max_{0\le k\le m+1}\|x(k)\| \le \beta_0\|\varphi(0)\| + \gamma + S_0 \max_{0\le k\le m+1}\|x(k)\|. \tag{3.10}$$

From this inequality we obtain

$$\max_{0\le k\le m+1}\|x(k)\| \le \frac{\beta_0\|\varphi(0)\| + \gamma}{1 - S_0}. \tag{3.11}$$

But, the right-hand side of this inequality does not depend on m. Thus, it follows that

$$\|x(k)\| \le \frac{\beta_0\|\varphi(0)\| + \gamma}{1 - S_0}, \quad \forall k = 1, 2, \dots. \tag{3.12}$$

This proves the global stability of the zero solution of (3.1)–(1.2).

To establish the global exponential stability of (3.1)–(1.2), we take the function

$$x_\alpha(k) = x(k)e^{\alpha k}, \tag{3.13}$$

with $\alpha > 0$ small enough, where $x(k)$ is a solution of (3.1).

Substituting (3.13) in (3.1), we obtain

$$x_\alpha(k+1) = \widetilde{T}(k)x_\alpha(k) + \widetilde{A}_1(k)x_\alpha(k-r), \tag{3.14}$$

where

$$\widetilde{T}(k) = e^{\alpha}T(k), \qquad \widetilde{A}_1(k) = e^{(r+1)\alpha} A_1(k). \tag{3.15}$$

Applying the above reasoning to (3.14), according to inequality (3.3), it follows that $x_\alpha(k)$ is a bounded function. Consequently, relation (3.13) implies the global exponential stability of the zero solution of system (3.1)–(1.2). □

Computing the quantities β_0 and S_0, defined by

$$\beta_0 = \sup_{k,l=0,1,\dots} \left\|T^k(l)\right\|, \qquad S_0 = \sum_{k=0}^{\infty}(kq+p) \sup_{l=0,1,\dots}\left\|T^k(l)\right\| \tag{3.16}$$

is not an easy task. However, in this section we will improve the estimates to these formulae.

Proposition 3.2. *Assume that (a) and (b) hold, and in addition*

$$v_0 = \sup_{k\ge0} g(T(k)) < \infty, \qquad \rho_0 = \sup_{k\ge0} \rho(T(k)) < 1, \tag{3.17}$$

where $\rho(T(k))$ is the spectral radius of $T(k)$ for each $k \in Z^+$. If

$$\widetilde{S}_0 = \sum_{k=0}^{n-1} \frac{v_0^k}{\sqrt{k!}} \left[\frac{(k+1)q}{(1-\rho_0)^{k+2}} + \frac{p}{(1-\rho_0)^{k+1}} \right] < 1, \tag{3.18}$$

then the zero solution of system (3.1)–(1.2) *is globally exponentially stable.*

Proof. Let us turn now to inequality (3.3). Firstly we will prove the inequality

$$\sum_{k=0}^{\infty} (kq+p) \sup_{l=0,1,\ldots} \left\| T^k(l) \right\| \leq q\lambda_0 + p\lambda_1, \tag{3.19}$$

where

$$\lambda_0 = \sum_{k=0}^{n-1} \frac{(k+1)v_0^k}{\sqrt{k!}(1-\rho_0)^{k+2}}, \qquad \lambda_1 = \sum_{k=0}^{n-1} \frac{v_0^k}{\sqrt{k!}(1-\rho_0)^{k+1}}. \tag{3.20}$$

Consider

$$\theta_0 = \sum_{k=1}^{\infty} k \sup_{l=0,1,\ldots} \left\| T^k(l) \right\|. \tag{3.21}$$

By Theorem A, we have

$$\theta_0 \leq \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} \frac{kk!\rho_0^{k-j} v_0^j}{(k-j)!(j!)^{3/2}}. \tag{3.22}$$

But

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{kk! z^{k-j}}{(k-j)!} &\leq \sum_{k=0}^{\infty} \frac{(k+1) z^{k-j}}{(k-j)!} = \frac{d^{j+1}}{dz^{j+1}} \left(\sum_{k=0}^{\infty} z^{k+1} \right) \\ &= \frac{d^{j+1}}{dz^{j+1}} z(1-z)^{-1} = (j+1)!(1-z)^{-j-2}, \quad 0 < z < 1. \end{aligned} \tag{3.23}$$

Hence,

$$\theta_0 \leq \sum_{j=0}^{n-1} \frac{v_0^j}{(j!)^{3/2}} \sum_{k=0}^{\infty} \frac{kk!\rho_0^{k-j}}{(k-j)!} = \sum_{j=0}^{n-1} \frac{(j+1)v_0^j}{\sqrt{j!}(1-\rho_0)^{j+2}} = \lambda_0. \tag{3.24}$$

Proceeding in a similar way, we obtain

$$\sum_{k=0}^{\infty} \sup_{l=0,1,\ldots} \left\| T^k(l) \right\| \leq \lambda_1. \tag{3.25}$$

These relations yield inequality (3.19). Consequently,

$$\|x(k)\| \leq (1 - q\lambda_0 - p\lambda_1)^{-1}(M_0\|\varphi(0)\| + \gamma), \tag{3.26}$$

where

$$M_0 = \sup_{k\geq 1}\left(\sum_{j=0}^{n-1}\frac{k!\rho_0^{k-j}v_0^j}{(k-j)!(j!)^{3/2}}\right) < \infty, \quad \text{by condition (3.17)}. \tag{3.27}$$

Relation (3.26) proves the global stability of the zero solution of system (3.1)–(1.2). Establishing the exponential stability of this equation is enough to apply the same arguments of the Proposition 3.1. □

Theorem 3.3. *Under the assumption (a), let* $(A(k), B(k))$ *be stabilizable for each fixed* $k \in Z^+$ *with respect to a matrix function* $L(k)$*, satisfying the following conditions:*

(i) $\rho_L = \sup_{k\geq 0}\rho(C_L(k)) < 1$,

(ii) $q_L = \sup_{k\geq 0}\|C_L(k+1) - C_L(k)\| < \infty$, *and*

(iii) $v_L = \sup_{k\geq 0}g(C_L(k)) < \infty$.

If,

$$S(C_L, A_1) = \sum_{k=0}^{n-1}\frac{v_L^k}{\sqrt{k!}}\left[\frac{(k+1)q_L}{(1-\rho_L)^{k+2}} + \frac{p}{(1-\rho_L)^{k+1}}\right] < 1, \tag{3.28}$$

then system (1.1)-(1.2) *is globally exponentially stabilizable by means of the feedback law* (2.1).

Proof. Rewrite (1.4) in the form

$$x(k+1) = T(k)x(k) + A_1(k)x(k-r), \tag{3.29}$$

where $T(k) = A(k) + B(k)L(k)$.

According to (i), (ii), and (iii), the conditions (b) and (3.17) hold. Furthermore, condition (3.28) assures the existence of a matrix function $L(k)$ such that condition (3.18) is fulfilled. Thus, from Proposition 3.2, the result follows. □

Put

$$\sigma(A(\cdot), B(\cdot); A_1(\cdot)) \equiv \min_L S(C_L, A_1), \tag{3.30}$$

where the minimum is taken over all $m \times n$ matrices $L(k)$ satisfying (i), (ii), and (iii).

Corollary 3.4. *Suppose that (a) holds, and the pair* $(A(k), B(k))$ *is stabilizable for each fixed* $k \in Z^+$. *If*

$$\sigma(A(\cdot), B(\cdot); A_1(\cdot)) < 1, \tag{3.31}$$

then the system (1.1)-(1.2) *is globally exponentially stabilizable by means of the feedback law* (2.1).

Now, consider in C^n *the discrete-time control system*

$$x(k+1) = Ax(k) + A_1(k)x(k-r) + Bu(k), \tag{3.32}$$

subject to the same initial conditions (1.2)*, where* A *and* B *are constant matrices. In addition, one assumes that the pair* (A, B) *is stabilizable, that is, there is a constant matrix* L *such that all the eigenvalues of* $C_L = A + BL$ *are located inside the unit disk. Hence,* $\rho(C_L) < 1$. *In this case,* $q_L = 0$ *and* $v_L = g(C_L)$. *Thus,*

$$S(C_L, A_1) = p\sum_{k=0}^{n-1} \frac{(g(A+BL))^k}{\sqrt{k!}(1-\rho(A+BL))^{k+1}} < 1. \tag{3.33}$$

Hence, Theorem 3.3 implies the following corollary.

Corollary 3.5. *Let* (A, B) *be a stabilizable pair of constant matrices, with respect to a constant matrix* L *satisfying the condition*

$$p\sum_{k=0}^{n-1} \frac{(g(A+BL))^k}{\sqrt{k!}(1-\rho(A+BL))^{k+1}} < 1. \tag{3.34}$$

Then system (3.32)-(1.2)*, under condition (a), is globally exponentially stabilizable by means of the feedback law* (2.1).

Example 3.6. Consider the control system in R^2:

$$x(k+1) = A(k)x(k) + A_1(k)x(k-2) + B(k)u(k), \tag{3.35}$$

where $A(k) = \begin{bmatrix} a_1(k) & a_2(k) \\ 1 & 0 \end{bmatrix}$, $A_1(k) = \begin{bmatrix} d_1(k) & d_2(k) \\ d_3(k) & d_4(k) \end{bmatrix}$, and $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$, subject to the initial conditions

$$x(k) = \varphi(k), \quad k = -2, -1, 0, \tag{3.36}$$

where $\varphi(k)$ is a given function with values in R^2, $a_1(k), a_2(k)$ are positive scalar-valued bounded sequences with the property

$$\tilde{q} \equiv \sup_{k\geq 0}\{|a_1(k+1) - a_1(k)| + |a_2(k+1) - a_2(k)|\} < \infty, \tag{3.37}$$

and $d_i(k)$, $i = 1, \ldots, 4$, are positive scalar-valued sequences with

$$p = \sup_{k \geq 0} \left(\sum_{i=1}^{4} |d_i(k)| \right) < \infty. \tag{3.38}$$

In the present case, the pair $(A(k), B)$ is controllable. Take

$$L(k) = L = \begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}. \tag{3.39}$$

Then

$$C_L(k) = \begin{bmatrix} \beta(k) & \omega(k) \\ 1 & 0 \end{bmatrix}, \tag{3.40}$$

where $\beta(k) = bl_1 + a_1(k)$ and $\omega(k) = bl_2 + a_2(k) > 0$.

By inequality

$$g(A) \leq \frac{1}{\sqrt{2}} N(A^* - A), \tag{3.41}$$

it follows that

$$v_L \leq v(l_1, l_2) := 1 + \sup_{k \geq 0} \omega(k). \tag{3.42}$$

Assume that

$$\rho_L = \rho(l_1 l_2) = \sup_{k \geq 0} \left\{ \left| \frac{\beta(k)}{2} + \left(\frac{\beta^2(k)}{4} - \omega(k) \right)^{1/2} \right| \right\} < 1. \tag{3.43}$$

Since $B(k)$ and $L(k)$ are constants, by (3.37) we have $q_L = \tilde{q}$. Hence, according to (3.28),

$$\begin{aligned} S(C_L, A_1) \leq S(l_1, l_2) = {} & \frac{\tilde{q}}{(1 - \rho(l_1, l_2))^2} + \frac{p}{(1 - \rho(l_1, l_2))} \\ & + v(l_1, l_2) \left\{ \frac{2\tilde{q}}{(1 - \rho(l_1, l_2))^3} + \frac{p}{(1 - \rho(l_1, l_2))^2} \right\}. \end{aligned} \tag{3.44}$$

If $\tilde{q}$ and p are small enough such that for some l_1 and l_2 we have $S(l_1, l_2) < 1$, then by Theorem 3.3, system (3.35)-(3.36), under conditions (3.37) and (3.38), is globally exponentially stabilizable.

In the same way, Theorem 3.3 can be extended to the discrete-time control system with multiple delays

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{N} A_i(k)x(k-r_i) + B(k)u(k), \tag{3.45}$$

$$x(k) = \varphi(k), \quad k \in \{-r_N, -r_N+1, \ldots, 0\}, \tag{3.46}$$

where A_i $(i = 1, \ldots, N)$ are variable $n \times n$ matrices, $1 \le r_1 \le r_2 \le \cdots \le r_N; N \ge 1$.

Denote

$$\widetilde{p} = \sum_{i=1}^{N} \sup_{k\ge 0} \|A_i(k)\|, \qquad \widetilde{\gamma} = \widetilde{p} \max_{-r_N \le k \le 0} \|\varphi(k)\| \sum_{k=0}^{\infty} \sup_{l=0,1,\ldots} \left\|T^k(l)\right\|. \tag{3.47}$$

Theorem 3.7. *Let* $(A(k), B(k))$ *be stabilizable for each* $k \in Z^+$ *with respect to a matrix function* $L(k)$ *satisfying the conditions (i), (ii), and (iii). In addition, assume that*

$$\widetilde{p} = \sum_{i=1}^{N} \sup_{k\ge 0} \|A_i(k)\| < \infty. \tag{3.48}$$

If

$$\widetilde{S}(C_L, \Sigma) = \sum_{k=0}^{n-1} \frac{v_L^k}{\sqrt{k!}} \left[\frac{(k+1)q_L}{(1-\rho_L)^{k+2}} + \frac{\widetilde{p}}{(1-\rho_L)^{k+1}} \right] < 1, \tag{3.49}$$

then system (3.45)-(3.46) *is globally exponentially stabilizable by means of the feedback law* (2.1). *Moreover, any solution of* (3.45)-(3.46) *satisfies the inequality*

$$\|x(k)\| \le \frac{M_0\|\varphi(0)\| + \widetilde{\gamma}}{1 - \widetilde{S}(C_L, \Sigma)}, \quad \text{for } k \ge 1. \tag{3.50}$$

As an application, one consider, the stabilization of the nonlinear discrete-time control system

$$x(k+1) = A(k)x(k) + A_1(k)x(k-r) + B(k)u(k) + f(k, x(k), x(k-r), u(k)), \tag{3.51}$$

$$x(k) = \phi(k), \quad k \in \{-r, -r+1, \ldots, 0\}, \tag{3.52}$$

where $f : Z^+ \times C^n \times C^n \times C^m \to C^n$ $(m \le n)$ *is a given nonlinear function satisfying*

$$\|f(k, x, y, u)\| \le a\|x\| + b\|y\| + c\|u\|, \tag{3.53}$$

for some positive numbers a, b, *and* c.

One recalls that nonlinear control system (3.51)-(3.52) *is stabilizable by a feedback control* $u(k) = L(k)x(k)$*, where* $L(k)$ *is a matrix, if the closed-loop system*

$$x(k+1) = [A(k) + L(k)B(k)]x(k) + A_1(k)x(k-r) + f(k, x(k), x(k-r), L(k)x(k)), \quad (3.54)$$

is asymptotically stable.

Theorem 3.8. *Under* (3.53)*, let* $(A(k), B(k))$ *be stabilizable for each* $k \in Z^+$*, with respect to a matrix function* $L(k)$ *satisfying conditions (i), (ii), and (iii). In addition, assume that*

$$p^* = \sup_{k\geq 0}(\|A_1(k)\| + b) < \infty. \quad (3.55)$$

If

$$S(C_L, f) = \sum_{k=0}^{n-1} \frac{v_L^k}{\sqrt{k!}} \left[\frac{(k+1)q_L}{(1-\rho_L)^{k+2}} + \frac{a+p^*}{(1-\rho_L)^{k+1}} \right] + c\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{k!\rho_L^{k-j} v_L^j}{(k-j)!(j!)^{3/2}} \|L(k)\| \right) < 1, \quad (3.56)$$

then system (3.51)-(3.52) *is globally exponentially stabilizable by means of the feedback law* (2.1)*.*

Proof. Rewrite (3.54) in the form

$$x(k+1) = T(k)x(k) + A_1(k)x(k-r) + f(k, x(k), x(k-r), L(k)x(k)), \quad (3.57)$$

where $T(k) = A(k) + B(k)L(k)$.

Thus, by reasoning as in Theorem 3.3, and using the estimates established in Proposition 3.2, the result follows. □

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