Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 273545, 9 pages
doi:10.1155/2009/273545

Research Article

On the Identities of Symmetry for the ζ-Euler Polynomials of Higher Order

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Received 19 February 2009; Revised 31 May 2009; Accepted 18 June 2009

Recommended by Agacik Zafer

The main purpose of this paper is to investigate several further interesting properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$. From these symmetries, we can derive some recurrence identities for the ζ-Euler polynomials of higher order, which are closely related to the Frobenius-Euler polynomials of higher order. By using our identities of symmetry for the ζ-Euler polynomials of higher order, we can obtain many identities related to the Frobenius-Euler polynomials of higher order.

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1. Introduction/Definition

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integer, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1+q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.1}$$

(see [1]). Let us define the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) \tag{1.2}$$

(see [1–8]). From (1.2), we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \tag{1.3}$$

(see [9, 10]), where $f_1(x) = f(x+1)$. For $\zeta \in \mathbb{C}_p$ with $|1-\zeta|_p < 1$, let $f(x) = e^{xt}\zeta^x$. Then, we define the ζ-Euler numbers as follows:

$$\int_{\mathbb{Z}_p} \zeta^x e^{xt} d\mu_{-1}(x) = \frac{2}{\zeta e^t + 1} = \sum_{n=0}^{\infty} E_{n,\zeta} \frac{t^n}{n!}, \tag{1.4}$$

where $E_{n,\zeta}$ are called the ζ-Euler numbers. We can show that

$$\frac{2}{\zeta e^t + 1} = \frac{1+\zeta^{-1}}{e^t + \zeta^{-1}} \cdot \frac{2}{1+\zeta} = \frac{2}{1+\zeta} \sum_{n=0}^{\infty} H_n\left(-\zeta^{-1}\right) \frac{t^n}{n!}, \tag{1.5}$$

where $H_n(-\zeta^{-1})$ are the Frobenius-Euler numbers. By comparing the coefficients on both sides of (1.4) and (1.5), we see that

$$E_{n,\zeta} = \frac{2}{1+\zeta} H_n\left(-\zeta^{-1}\right). \tag{1.6}$$

Now, we also define the ζ-Euler polynomials as follows:

$$\frac{2}{\zeta e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\zeta}(x) \frac{t^n}{n!}. \tag{1.7}$$

In the viewpoint of (1.5), we can show that

$$\frac{2}{\zeta e^t + 1} e^{xt} = e^{xt} \frac{1+\zeta^{-1}}{e^t + \zeta^{-1}} \cdot \frac{2}{1+\zeta} = \frac{2}{1+\zeta} \sum_{n=0}^{\infty} H_n\left(-\zeta^{-1}, x\right) \frac{t^n}{n!}, \tag{1.8}$$

where $H_n(-\zeta^{-1}, x)$ are the nth Frobenius-Euler polynomials. From (1.7) and (1.8), we note that

$$E_{n,\zeta}(x) = \frac{2}{1+\zeta} H_n\left(-\zeta^{-1}, x\right) \tag{1.9}$$

(cf. [1–8, 11–18]). For each positive integer k, let $T_{k,\zeta}(n) = \sum_{\ell=0}^{n} (-1)^\ell \zeta^\ell \ell^k$. Then we have

$$\sum_{k=0}^{\infty} T_{k,\zeta}(n) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{n} (-1)^\ell \ell^k \zeta^\ell \right) \frac{t^k}{k!} = \sum_{\ell=0}^{n} (-1)^\ell \zeta^\ell e^{\ell t} = \frac{1 + (-1)^{n+1} e^{(n+1)t}}{\zeta e^t + 1}. \tag{1.10}$$

The ζ-Euler polynomials of order k, denoted $E_{n,\zeta}^{(k)}(x)$, are defined as

$$e^{xt}\left(\frac{2}{\zeta e^t+1}\right)^k=\left(\frac{2}{\zeta e^t+1}\right)\times\cdots\times\left(\frac{2}{\zeta e^t+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_{n,\zeta}^{(k)}(x)\frac{t^n}{n!}. \tag{1.11}$$

Then the values of $E_{n,\zeta}^{(k)}(x)$ at $x=0$ are called the ζ-Euler numbers of order k. When $k=1$, the polynomials or numbers are called the ζ-Euler polynomials or numbers. The purpose of this paper is to investigate some properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$. From the properties of symmetry for the multivariate p-adic fermionic integral on $\mathbb{Z}_p$, we derive some identities of symmetry for the ζ-Euler polynomials of higher order. By using our identities of symmetry for the ζ-Euler polynomials of higher order, we can obtain many identities related to the Frobenius-Euler polynomials of higher order.

2. On the Symmetry for the ζ-Euler Polynomials of Higher Order

Let $w_1, w_2\in\mathbb{N}$ with $w_1\equiv 1(\operatorname{mod} 2)$ and $w_2\equiv 1(\operatorname{mod}\ 2)$. Then we set

$$\begin{aligned}R^{(m)}(\mathrm{w}_1,w_2)={}&\frac{\int_{\mathbb{Z}_p^m}e^{w_1(x_1+x_2+\cdots+x_m+w_2x)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p}\zeta^{w_1w_2x}e^{w_1w_2xt}d\mu_{-1}(x)}\\&\times\int_{\mathbb{Z}_p^m}e^{w_2(x_1+x_2+\cdots+x_m+w_1y)t}\zeta^{w_2x_1+\cdots+w_2x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m),\end{aligned}\tag{2.1}$$

where

$$\int_{\mathbb{Z}_p^m}f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}f(x_1,\ldots,x_m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m). \tag{2.2}$$

Thus, we note that this expression for $R^{(m)}(w_1,w_2)$ is symmetry in w_1 and w_2. From (2.1), we have

$$\begin{aligned}R^{(m)}(w_1,w_2)={}&\left(\int_{\mathbb{Z}_p^m}e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt}\\&\times\left(\frac{\int_{\mathbb{Z}_p}e^{w_2x_mt}\zeta^{w_2x_m}d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p}e^{w_1w_2xt}\zeta^{w_1w_2x}d\mu_{-1}(x)}\right)\\&\times\left(\int_{\mathbb{Z}_p^{m-1}}e^{w_2(x_1+\cdots+x_{m-1})t}\zeta^{w_2x_1+\cdots+w_2x_{m-1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1})\right)e^{w_1w_2yt}.\end{aligned}\tag{2.3}$$

We can show that

$$\frac{\int_{\mathbb{Z}_p} e^{xt}\zeta^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{w_1xt}\zeta^{w_1x} d\mu_{-1}(x)} = \sum_{\ell=0}^{w_1-1}(-1)^{\ell}\zeta^{\ell}e^{\ell t} = \sum_{k=0}^{\infty}\left(T_{k,\zeta}(w_1-1)\right)\frac{t^k}{k!}. \tag{2.4}$$

By (1.4) and (1.11), we see that

$$\begin{aligned}
&\left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1x_1+\cdots+w_1x_m} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt} \\
&\quad= \left(\frac{2}{\zeta^{w_1}e^{w_1t}+1}\right)^m e^{w_1w_2xt} \\
&\quad= \sum_{n=0}^{\infty}E^{(m)}_{n,\zeta^{w_1}}(w_2x)w_1^n\frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

Thus, we have

$$E^{(m)}_{n,\zeta^{w_1}}(w_2x) = \sum_{\ell=0}^{n}\binom{n}{\ell}E^{(m)}_{\ell,\zeta^{w_1}}w_2^{n-\ell}x^{n-\ell}. \tag{2.6}$$

From (2.3), (2.4), and (2.5), we can derive

$$\begin{aligned}
R^{(m)}(w_1,w_2) &= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_1}}(w_2x)w_1^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{k=0}^{\infty}T_{k,\zeta^{w_2}}(w_1-1)\frac{w_2^k}{k!}t^k\right)\left(\sum_{i=0}^{\infty}E^{(m-1)}_{i,\zeta^{w_2}}(w_1y)\frac{w_2^i}{i!}t^i\right) \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_1}}(w_2x)w_1^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)w_2^kw_2^{j-k}\frac{E^{(m-1)}_{j-k}(w_1y)}{k!(j-k)!}j!\right)\frac{t^j}{j!}\right) \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_1}}(w_2x)w_1^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2}}(w_1y)\right)w_2^j\frac{t^j}{j!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2}}(w_1y)\right)\frac{w_2^jw_1^{n-j}}{(n-j)!j!}E^{(m)}_{n-j,\zeta^{w_1}}(w_2x)n!\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j}E^{(m)}_{n-j,\zeta^{w_1}}(w_2x)\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2}}(w_1y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

By the same method, we also see that

$$
\begin{aligned}
R^{(m)}(w_1,w_2) &= \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t}\zeta^{w_2x_1+\cdots+w_2x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt} \\
&\times\left(\frac{\int_{\mathbb{Z}_p} e^{w_1x_mt}\zeta^{w_1x_m}d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt}\zeta^{w_1w_2x}d\mu_{-1}(x)}\right) \\
&\times\left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1})t}\zeta^{w_1x_1+\cdots+w_1x_{m-1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1})\right)e^{w_1w_2yt} \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_2}}(w_1x)w_2^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{k=0}^{\infty}T_{k,\zeta^{w_1}}(w_2-1)w_1^k\frac{t^k}{k!}\right)\left(\sum_{i=0}^{\infty}E^{(m-1)}_{i,\zeta^{w_1}}(w_2y)w_1^i\frac{t^i}{i!}\right) \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_2}}(w_1x)w_2^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{T_{k,\zeta^{w_1}}(w_2-1)}{k!}\frac{E^{(m-1)}_{j-k}(w_2y)}{(j-k)!}\right)w_1^jt^j\right) \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_2}}(w_1x)w_2^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{T_{k,\zeta^{w_1}}(w_2-1)E^{(m-1)}_{j-k}(w_2y)}{k!(j-k)!}j!\right)\frac{w_1^jt^j}{j!}\right) \\
&= \left(\sum_{\ell=0}^{\infty}E^{(m)}_{\ell,\zeta^{w_2}}(w_1x)w_2^{\ell}\frac{t^{\ell}}{\ell!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}T_{k,\zeta^{w_1}}(w_2-1)E^{(m-1)}_{j-k,\zeta^{w_1}}(w_2y)\right)w_1^j\frac{t^j}{j!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{j}\binom{j}{k}T_{k,\zeta^{w_1}}(w_2-1)E^{(m-1)}_{j-k,\zeta^{w_1}}(w_2y)\right)\frac{w_1^jw_2^{n-j}}{j!(n-j)!}E^{(m)}_{n-j,\zeta^{w_2}}(w_1x)n!\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j}E^{(m)}_{n-j,\zeta^{w_2}}(w_1x)\sum_{k=0}^{j}\binom{j}{k}T_{k,\zeta^{w_1}}(w_2-1)E^{(m-1)}_{j-k,\zeta^{w_1}}(w_2y)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.8}
$$

By comparing the coefficients on both sides of (2.7) and (2.8), we obtain the following.

Theorem 2.1. *For $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1(\mathrm{mod}\ 2)$, $w_2\equiv 1(\mathrm{mod}\ 2)$, and $n\geq 0, m\geq 1$, one has*

$$
\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}w_2^jw_1^{n-j}E^{(m)}_{n-j,\zeta^{w_1}}(w_2x)\sum_{k=0}^{j}T_{k,\zeta^{w_2}}(w_1-1)\binom{j}{k}E^{(m-1)}_{j-k,\zeta^{w_2}}(w_1y) \\
&\quad=\sum_{j=0}^{n}\binom{n}{j}w_1^jw_2^{n-j}E^{(m)}_{n-j,\zeta^{w_2}}(w_1x)\sum_{k=0}^{j}\binom{j}{k}T_{k,\zeta^{w_1}}(w_2-1)E^{(m-1)}_{j-k,\zeta^{w_1}}(w_2y).
\end{aligned}
\tag{2.9}
$$

Let $y = 0$ and $m = 1$ in (2.9). Then we have

$$\sum_{j=0}^{n}\binom{n}{j}w_1^{n-j}w_2^{j}E_{n-j,\zeta^{w_1}}(w_2x)T_{k,\zeta^{w_2}}(w_1-1) = \sum_{j=0}^{n}\binom{n}{j}w_1^{j}w_2^{n-j}E_{n-j,\zeta^{w_2}}(w_1x)T_{k,\zeta^{w_1}}(w_2-1). \tag{2.10}$$

From (2.10), we note that

$$\sum_{i=0}^{n}\binom{n}{i}w_1^{i}w_2^{n-i}E_{i,\zeta^{w_1}}(w_2x)T_{n-i,\zeta^{w_2}}(w_1-1) = \sum_{i=0}^{n}\binom{n}{i}w_1^{n-i}w_2^{i}E_{i,\zeta^{w_2}}(w_1x)T_{n-i,\zeta^{w_1}}(w_2-1). \tag{2.11}$$

If we take $w_2 = 1$ in (2.11), then we have

$$E_{n,\zeta}(w_1x) = \sum_{i=0}^{n}\binom{n}{i}w_1^{i}E_{i,\zeta^{w_1}}(x)T_{n-i,\zeta}(w_1-1). \tag{2.12}$$

From (2.3), we note that

$$\begin{aligned}
R^{(m)}(w_1,w_2) &= \left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt} \\
&\quad\times\left(\frac{\int_{\mathbb{Z}_p} e^{w_2x_mt}\zeta^{w_2x_m}d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt}\zeta^{w_1w_2x}d\mu_{-1}(x)}\right) \\
&\quad\times\left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1})t}\zeta^{w_2x_1+\cdots+w_2x_{m-1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1})\right)e^{w_1w_2yt} \\
&= \left(\int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m)t}\zeta^{w_1x_1+\cdots+w_1x_m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m)\right)e^{w_1w_2xt} \\
&\quad\times\left(\sum_{i=0}^{w_1-1}(-1)^{i}e^{w_2it}\zeta^{w_2i}\right) \\
&\quad\times\left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1})t}\zeta^{w_2x_1+\cdots+w_2x_{m-1}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1})\right)e^{w_1w_2yt}
\end{aligned}$$

$$= \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} \int_{\mathbb{Z}_p^m} e^{w_1(x_1+\cdots+x_m+(w_2/w_1)i+w_2 x)t} \zeta^{w_1 x_1+\cdots+w_1 x_m} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) \right)$$
$$\times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_2(x_1+\cdots+x_{m-1}+w_1 y)t} \zeta^{w_2 x_1+\cdots+w_2 x_{m-1}} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1}) \right)$$
$$= \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} \sum_{k=0}^{\infty} E_{k,\zeta^{w_1}}^{(m)} \left(\frac{w_2}{w_1} i + w_2 x \right) w_1^k \frac{t^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_2}}^{(m-1)} (w_1 y) w_2^{\ell} \frac{t^{\ell}}{\ell!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{k,\zeta^{w_1}}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \right) \frac{w_1^k}{k!} E_{n-k,\zeta^{w_2}}^{(m-1)} (w_1 y) \frac{w_2^{n-k}}{(n-k)!} n! \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k,\zeta^{w_2}}^{(m-1)} (w_1 y) \sum_{i=0}^{w_1-1} (-1)^i \zeta^{w_2 i} E_{k,\zeta^{w_1}}^{(m)} \left(w_2 x + \frac{w_2}{w_1} i \right) \right) \frac{t^n}{n!}. \tag{2.13}$$

By the symmetric property of $R^{(m)}(w_1, w_2)$ in w_1, w_2, we also see that

$$R^{(m)}(w_1, w_2) = \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t}$$
$$\times \left(\frac{\int_{\mathbb{Z}_p} e^{w_1 x_m t} \zeta^{w_1 x_m} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} \zeta^{w_1 w_2 x} d\mu_{-1}(x)} \right)$$
$$\times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1})t} \zeta^{w_1(x_1+\cdots+x_{m-1})} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1}) \right) e^{w_1 w_2 y t}$$
$$= \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) \right) e^{w_1 w_2 x t}$$
$$\times \left(\sum_{i=0}^{w_2-1} (-1)^i e^{w_1 i t} \zeta^{w_1 i} \right)$$
$$\times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1}+w_2 y)t} \zeta^{w_1(x_1+\cdots+x_{m-1})} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1}) \right)$$
$$= \sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} \left(\int_{\mathbb{Z}_p^m} e^{w_2(x_1+\cdots+x_m+(w_1/w_2)i+w_1 x)t} \zeta^{w_2(x_1+\cdots+x_m)} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_m) \right)$$
$$\times \left(\int_{\mathbb{Z}_p^{m-1}} e^{w_1(x_1+\cdots+x_{m-1}+w_2 y)t} \zeta^{w_1 x_1+\cdots+w_1 x_{m-1}} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_{m-1}) \right)$$
$$= \left(\sum_{i=0}^{w_2-1} (-1)^i \zeta^{w_1 i} \sum_{k=0}^{\infty} E_{k,\zeta^{w_2}}^{(m)} \left(\frac{w_1}{w_2} i + w_1 x \right) w_2^k \frac{t^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} E_{\ell,\zeta^{w_1}}^{(m-1)} (w_2 y) w_1^{\ell} \frac{t^{\ell}}{\ell!} \right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}E_{k,\zeta^{w_2}}^{(m)}\left(w_1x+\frac{w_1}{w_2}i\right)\right)\frac{w_2^k}{k!}E_{n-k}^{(m-1)}(w_2y)\frac{w_1^{n-k}}{(n-k)!}n!\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1}}^{(m-1)}(w_2y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}E_{k,\zeta^{w_2}}^{(m)}\left(w_1x+\frac{w_1}{w_2}i\right)\right)\frac{t^n}{n!}. \tag{2.14}$$

By comparing the coefficients on both sides of (2.13) and (2.14), we obtain the following theorem.

Theorem 2.2. *For* $w_1, w_2 \in \mathbb{N}$ *with* $w_1 \equiv 1 \pmod 2$ *and* $w_2 \equiv 1 \pmod 2$ *, one has*

$$\sum_{k=0}^{n}\binom{n}{k}w_1^k w_2^{n-k}E_{n-k,\zeta^{w_2}}^{(m-1)}(w_1y)\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}E_{k,\zeta^{w_1}}^{(m)}\left(w_2x+\frac{w_2}{w_1}i\right)$$
$$= \sum_{k=0}^{n}\binom{n}{k}w_2^k w_1^{n-k}E_{n-k,\zeta^{w_1}}^{(m-1)}(w_2y)\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}E_{k,\zeta^{w_2}}^{(m)}\left(w_1x+\frac{w_1}{w_2}i\right). \tag{2.15}$$

Let $y = 0$ and $m = 1$, we have

$$w_1^n\sum_{i=0}^{w_1-1}(-1)^i\zeta^{w_2 i}E_{n,\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}i\right) = w_2^n\sum_{i=0}^{w_2-1}(-1)^i\zeta^{w_1 i}E_{n,\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right). \tag{2.16}$$

From (2.16), we can derive

$$\sum_{i=0}^{w_1-1}(-1)^i\zeta^{i}E_{n,\zeta^{w_1}}\left(x+\frac{1}{w_1}i\right) = \frac{1}{w_1^n}E_{n,\zeta}(w_1x). \tag{2.17}$$

Acknowledgment

The present research has been conducted by the research grant of the Kwangwoon University in 2009.

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