Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 463169, 11 pages
doi:10.1155/2009/463169

Research Article

On Boundedness of Solutions of the Difference Equation $x_{n+1} = (px_n + qx_{n-1})/(1 + x_n)$ for $q > 1 + p > 1$

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Received 4 February 2009; Revised 19 April 2009; Accepted 2 June 2009

Recommended by Agacik Zafer

We study the boundedness of the difference equation $x_{n+1} = (px_n + qx_{n-1})/(1 + x_n)$, $n = 0, 1, \ldots,$ where $q > 1 + p > 1$ and the initial values $x_{-1}, x_0 \in (0, +\infty)$. We show that the solution $\{x_n\}_{n=-1}^{\infty}$ of this equation converges to $\overline{x} = q + p - 1$ if $x_n \geq \overline{x}$ or $x_n \leq \overline{x}$ for all $n \geq -1$; otherwise $\{x_n\}_{n=-1}^{\infty}$ is unbounded. Besides, we obtain the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions $\{x_n\}_{n=-1}^{\infty}$ of this equation are bounded, which answers the open problem 6.10.12 proposed by Kulenović and Ladas (2002) .

1. Introduction

In this paper, we study the following difference equation:

$$x_{n+1} = \frac{px_n + qx_{n-1}}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $p, q \in (0, +\infty)$ with $q > 1 + p$ and the initial values $x_{-1}, x_0 \in (0, +\infty)$.

The global behavior of (1.1) for the case $p + q < 1$ is certainly folklore. It can be found, for example, in [1] (see also a precise result in [2]).

The global stability of (1.1) for the case $p+q = 1$ follows from the main result in [3] (see also Lemma 1 in Stević's paper [4]). Some generalizations of Copson's result can be found, for example, in papers [5–8]. Some more sophisticated results, such as finding the asymptotic behavior of solutions of (1.1) for the case $p + q = 1$ (even when $p = 0$) can be found, for example, in papers [4](see also [8–11]). Some other properties of (1.1) have been also treated in [4].

The case $q = 1+p$ was treated for the first time by Stević's in paper [12]. The main trick from [12] has been later used with a success for many times; see, for example, [13–15].

Some existing results for (1.1) are summarized as follows[16].

Theorem A. (1) *If $p + q \le 1$, then the zero equilibrium of* (1.1) *is globally asymptotically stable.*

(2) *If $q = 1$, then the equilibrium $\overline{x} = p$ of* (1.1) *is globally asymptotically stable.*

(3) *If $1 < q < 1 + p$, then every positive solution of* (1.1) *converges to the positive equilibrium $\overline{x} = p + q - 1$.*

(4) *If $q = 1 + p$, then every positive solution of* (1.1) *converges to a period-two solution.*

(5) *If $q > 1 + p$, then* (1.1) *has unbounded solutions.*

In [16], Kulenović and Ladas proposed the following open problem.

Open problem B (see Open problem 6.10.12of [16])

Assume that $q \in (1, +\infty)$.

(a) Find the set B of all initial conditions $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the solutions $\{x_n\}_{n=-1}^{\infty}$ of (1.1) are bounded.

(b) Let $(x_{-1}, x_0) \in B$. Investigate the asymptotic behavior of $\{x_n\}_{n=-1}^{\infty}$.

In this paper, we will obtain the following results: let $p, q \in (0, +\infty)$ with $q > 1 + p$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1.1) with the initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$. If $x_n \ge \overline{x}$ for all $n \ge -1$ (or $x_n \le \overline{x}$ for all $n \ge -1$), then $\{x_n\}_{n=-1}^{\infty}$ converges to $\overline{x} = q + p - 1$. Otherwise $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

For closely related results see [17–34].

2. Some Definitions and Lemmas

In this section, let $q > 1 + p > 1$ and $\overline{x} = q + p - 1$ be the positive equilibrium of (1.1). Write $D = (0, +\infty) \times (0, +\infty)$ and define $f : D \to D$ by, for all $(x, y) \in D$,

$$f(x, y) = \left(y, \frac{py + qx}{1 + y} \right). \tag{2.1}$$

It is easy to see that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of (1.1), then $f^n(x_{-1}, x_0) = (x_{n-1}, x_n)$ for any $n \ge 0$. Let

$$\begin{aligned} A_1 &= (0, \overline{x}) \times (0, \overline{x}), \qquad A_2 = (\overline{x}, +\infty) \times (\overline{x}, +\infty), \\ A_3 &= (0, \overline{x}) \times (\overline{x}, +\infty), \qquad A_4 = (\overline{x}, +\infty) \times (0, \overline{x}), \\ R_0 &= \{\overline{x}\} \times (0, \overline{x}), \qquad L_0 = \{\overline{x}\} \times (\overline{x}, +\infty), \\ R_1 &= (0, \overline{x}) \times \{\overline{x}\}, \qquad L_1 = (\overline{x}, +\infty) \times \{\overline{x}\}. \end{aligned} \tag{2.2}$$

Then $D = (\cup_{i=1}^{4} A_i) \cup L_0 \cup L_1 \cup R_0 \cup R_1 \cup \{(\overline{x}, \overline{x})\}$. The proof of Lemma 2.1 is quite similar to that of Lemma 1 in [35] and hence is omitted.

Lemma 2.1. *The following statements are true.*

(1) $f : D \to f(D)$ *is a homeomorphism.*

(2) $f(L_1) = L_0$ *and* $f(L_0) \subset A_4$.

(3) $f(R_1) = \{\overline{x}\} \times (p, \overline{x})$ *and* $f(R_0) \subset A_3$.

(4) $f(A_3) \subset A_4$ *and* $f(A_4) \subset A_3$.

(5) $A_2 \cup L_1 \subset f(A_2) \subset A_2 \cup L_1 \cup A_4$ *and* $A_1 \cup R_1 \subset f(A_1) \subset A_1 \cup R_1 \cup A_3$.

Lemma 2.2. *Let* $q > 1 + p > 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of* (1.1).

(1) *If* $\lim_{n\to+\infty} x_{2n} = a \in (0, +\infty)$ *and* $a \neq p$, *then* $\lim_{n\to+\infty} x_{2n+1} = a = \overline{x}$.

(2) *If* $\lim_{n\to+\infty} x_{2n-1} = b \in (0, +\infty)$ *and* $b \neq p$, *then* $\lim_{n\to+\infty} x_{2n} = b = \overline{x}$.

Proof. We show only (1) because the proof of (2) follows from (1) by using the change $y_n = x_{n-1}$ and the fact that (1) is autonomous. Since $\lim_{n\to+\infty} x_{2n} = a \in (0, +\infty)$ and $a \neq p$, by (1.1) we have

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} \frac{q x_{2n} - x_{2n+2}}{x_{2n+2} - p} = \frac{(q-1)a}{a-p}. \tag{2.3}$$

Also it follows from (1.1) that

$$a = \lim_{n\to+\infty} x_{2n} = \lim_{n\to+\infty} \frac{q x_{2n-1} - x_{2n+1}}{x_{2n+1} - p} = \frac{(q-1)^2 a}{(q-1)a - p(a-p)}, \tag{2.4}$$

from which we have $a = \overline{x}$ and $\lim_{n\to+\infty} x_{2n+1} = a = \overline{x}$. This completes the proof. □

Lemma 2.3. *Let* $q > 1 + p > 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of* (1.1) *with the initial values* $(x_{-1}, x_0) \in A_4$. *If there exists some* $n \geq 0$ *such that* $x_{2n-1} \geq x_{2n+1}$, *then* $x_{2n} \geq x_{2n+2}$.

Proof. Since $(x_{-1}, x_0) \in A_4$, it follows from Lemma 2.1 that $(x_{2n-1}, x_{2n}) \in A_4$ for any $n \geq 0$. Without loss of generality we may assume that $n = 0$, that is, $x_{-1} \geq x_1$. Now we show $x_0 \geq x_2$. Suppose for the sake of contradiction that $x_0 < x_2$, then

$$x_{-1} \geq x_1 = \frac{p x_0 + q x_{-1}}{1 + x_0}, \tag{2.5}$$

$$x_0 < x_2 = \frac{p x_1 + q x_0}{1 + x_1}. \tag{2.6}$$

By (2.5) we have

$$x_0 \geq \frac{x_{-1}(q-1)}{x_{-1} - p}, \tag{2.7}$$

and by (2.6) we get

$$(q-1-p)x_0^2+\left(p^2+q-1-qx_{-1}\right)x_0+pqx_{-1}>0. \tag{2.8}$$

Claim 1. If $x_{-1}\geq\overline{x}$, then

$$\left(p^2+q-1-qx_{-1}\right)^2-4(q-1-p)pqx_{-1}\geq 0. \tag{2.9}$$

Proof of Claim 1

Let $g(x)=(p^2+q-1-qx)^2-4(q-1-p)pqx\ (x\geq\overline{x})$, then we have

$$\begin{aligned} g'(x)&=2q\left(1+qx-p^2-q\right)-4pq(q-1-p)\\ &\geq 2q\left[(q-1)^2+p^2+p(1-q)+p\right]\\ &=2q\left[(q-1)(q-p-1)+p^2+p\right]\\ &>0. \end{aligned} \tag{2.10}$$

Since $x_{-1}\geq\overline{x}$, it follows

$$\begin{aligned} &\left(p^2+q-1-qx_{-1}\right)^2-4(q-1-p)pqx_{-1}\\ &\quad\geq\left(q^2+qp-2q+1-p^2\right)^2-4(q-1-p)qp(q+p-1)\\ &\quad=\left(q^2-2q+1-p^2\right)^2+2qp\left(q^2-2q+1-p^2\right)\\ &\qquad+(qp)^2-4\left(q^2-2q+1-p^2\right)pq\\ &\quad=\left(q^2-2q+1-p^2-pq\right)^2\\ &\quad\geq 0. \end{aligned} \tag{2.11}$$

This completes the proof of Claim 1.

By (2.8), we have

$$x_0>\lambda_1=\frac{(1+qx_{-1}-p^2-q)+\sqrt{(1+qx_{-1}-p^2-q)^2-4pq(q-1-p)x_{-1}}}{2(q-1-p)} \tag{2.12}$$

or

$$x_0 < \lambda_2 = \frac{(1 + qx_{-1} - p^2 - q) - \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)}. \tag{2.13}$$

Claim 2. We have

$$\lambda_1 \geq \overline{x}, \tag{2.14}$$

$$\lambda_2 \leq \frac{x_{-1}(q-1)}{x_{-1} - p}. \tag{2.15}$$

Proof of Claim 2

Since

$$\begin{aligned} &\sqrt{[1 + q(q + p - 1) - p^2 - q]^2 - 4pq(q - 1 - p)(p + q - 1)} \\ &\quad = q^2 - p^2 - 2q + 1 - qp \\ &\quad = 2(q + p - 1)(q - 1 - p) - \left[1 + q(q + p - 1) - p^2 - q\right], \end{aligned} \tag{2.16}$$

we have

$$\begin{aligned} \lambda_1 &= \frac{(1 + qx_{-1} - p^2 - q) + \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)} \\ &\geq \frac{(1 + q\overline{x} - p^2 - q) + \sqrt{(1 + q\overline{x} - p^2 - q)^2 - 4pq(q - 1 - p)\overline{x}}}{2(q - 1 - p)} \\ &= \frac{[1 + q(q + p - 1) - p^2 - q] + \sqrt{[1 + q(q + p - 1) - p^2 - q]^2 - 4pq(q - 1 - p)(p + q - 1)}}{2(q - 1 - p)} \\ &\geq (q + p - 1) = \overline{x}. \end{aligned} \tag{2.17}$$

The proof of (2.14) is completed.

Now we show (2.15). Let

$$h(x) = pq(x - p)^2 - (x - p)(q - 1)\left(1 + qx - p^2 - q\right) + (q - 1)^2(q - 1 - p)x. \tag{2.18}$$

Note that $2pq - 2q(q-1) < 0$; it follows that if $x \ge \overline{x}$, then

$$\begin{aligned} h'(x) &= 2pq(x-p) - \left[(q-1)\left(1 + qx - p^2 - q\right) + q(q-1)(x-p) - (q-1)^2(q-1-p)\right] \\ &\le 2pq(q-1) - \left[(q-1)\left(2pq - q - p^2 + q^2 - p\right)\right] \\ &= (q-1)(q+p)(p+1-q) < 0, \end{aligned} \tag{2.19}$$

which implies that $h(x)$ is decreasing for $x \ge \overline{x}$. Since $x_{-1} \ge \overline{x}$ and

$$\begin{aligned} h(\overline{x}) &= pq(q-1)^2 - (q-1)(q-1)\left[1 + q(q+p-1) - p^2 - q\right] \\ &\quad + (q-1)^2(q-1-p)(q+p-1) = 0, \end{aligned} \tag{2.20}$$

it follows that

$$\begin{aligned} h(x_{-1}) &= pq(x_{-1}-p)^2 - (x_{-1}-p)(q-1)\left(1 + qx_{-1} - p^2 - q\right) \\ &\quad + (q-1)^2(q-1-p)x_{-1} \le h(\overline{x}) = 0. \end{aligned} \tag{2.21}$$

Thus

$$\begin{aligned} &(q-1)^2\left[\left(1 + qx_{-1} - p^2 - q\right)^2 - 4pq(q-1-p)x_{-1}\right] \\ &\quad \ge 4p^2q^2(x_{-1}-p)^2 - 4pq(x_{-1}-p)(q-1)\left(1 + qx_{-1} - p^2 - q\right) \\ &\quad + (q-1)^2\left(1 + qx_{-1} - p^2 - q\right)^2. \end{aligned} \tag{2.22}$$

This implies that

$$\begin{aligned} &(q-1)\sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q-1-p)x_{-1}} \\ &\quad \ge 2pq(x_{-1}-p) - (q-1)\left(1 + qx_{-1} - p^2 - q\right). \end{aligned} \tag{2.23}$$

Finally we have

$$\begin{aligned} \frac{x_{-1}(q-1)}{x_{-1}-p} &\ge \frac{4(q-1-p)pqx_{-1}}{2(q-1-p)\left[(1 + qx_{-1} - p^2 - q) + \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q-1-p)x_{-1}}\right]} \\ &= \frac{(1 + qx_{-1} - p^2 - q) - \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q-1-p)x_{-1}}}{2(q-1-p)} = \lambda_2. \end{aligned} \tag{2.24}$$

The proof of (2.15) is completed.

Note that $x_0 < \overline{x}$ since $(x_{-1}, x_0) \in A_4$. By (2.12), (2.13), (2.14), and (2.15), we see $x_0 < x_{-1}(q-1)/(x_{-1}-p)$, which contradicts to (2.7). The proof of Lemma 2.3 is completed. □

3. Main Results

In this section, we investigate the boundedness of solutions of (1.1). Let $q > 1 + p > 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1.1) with the initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$, then we see that $(x_{n+1} - \overline{x})(x_n - \overline{x}) < 0$ for some $n \geq -1$ or $x_n \geq \overline{x}$ for all $n \geq -1$ or $x_n \leq \overline{x}$ for all $n \geq -1$.

Theorem 3.1. *Let $q > 1 + p > 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of* (1.1) *such that $x_n \geq \overline{x}$ for all $n \geq -1$ or $x_n \leq \overline{x}$ for all $n \geq -1$, then $\{x_n\}_{n=-1}^{\infty}$ converges to $\overline{x} = q + p - 1$.*

Proof.

Case 1. $0 < x_n \leq \overline{x}$ for any $n \geq -1$. If $0 < x_{2n} \leq q - 1$ for some n, then

$$x_{2n+1} - x_{2n-1} = \frac{px_{2n} + qx_{2n-1} - x_{2n-1} - x_{2n-1}x_{2n}}{1 + x_{2n}} > 0. \tag{3.1}$$

If $q - 1 < x_{2n} \leq \overline{x}$ for some n, then

$$\frac{px_{2n}}{x_{2n} - q + 1} \geq \frac{p\overline{x}}{\overline{x} - q + 1} = \overline{x} \geq x_{2n-1}, \tag{3.2}$$

which implies that $px_{2n} \geq x_{2n-1}(x_{2n} - q + 1)$ and

$$x_{2n+1} - x_{2n-1} = \frac{px_{2n} + qx_{2n-1} - x_{2n-1} - x_{2n-1}x_{2n}}{1 + x_{2n}} \geq 0. \tag{3.3}$$

Thus $\overline{x} \geq x_{2n+1} \geq x_{2n-1}$ for any $n \geq 0$. In similar fashion, we can show $\overline{x} \geq x_{2n+2} \geq x_{2n}$ for any $n \geq 0$. Let $\lim_{n\to+\infty} x_{2n+1} = a$ and $\lim_{n\to+\infty} x_{2n} = b$, then

$$a = \frac{pb + qa}{1 + b}, \qquad b = \frac{pa + qb}{1 + a}, \tag{3.4}$$

which implies $a = b = \overline{x}$.

Case 2. $x_n \geq \overline{x} = p + q - 1$ for any $n \geq -1$. Since $f(x, y) = (py + qx)/(1 + y)$ $(x > p/q)$ is decreasing in y, it follows that for any $n \geq -1$,

$$\begin{aligned} x_{n+2} &= \frac{px_{n+1} + qx_n}{1 + x_{n+1}} \\ &\leq \frac{p\overline{x} + qx_n}{1 + \overline{x}} \leq x_n. \end{aligned} \tag{3.5}$$

In similar fashion, we can show that $\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n} = \overline{x}$. This completes the proof. □

Lemma 3.2 (see [20, Theorem 5]). *Let I be a set, and let $F : I \times I \to I$ be a function $F(u, v)$ which decreases in u and increases in v, then for every positive solution $\{x_n\}_{n=-1}^{+\infty}$ of equation $x_{n+1} = F(x_n, x_{n-1})$, $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ do exactly one of the following.*

(1) *They are both monotonically increasing.*

(2) *They are both monotonically decreasing.*

(3) *Eventually, one of them is monotonically increasing, and the other is monotonically decreasing.*

Remark 3.3. Using arguments similar to ones in the proof of Lemma 3.2, Stević proved Theorem 2 in [25]. Beside this, this trick have been used by Stević in [18, 28, 29].

Theorem 3.4. *Let $q > 1 + p > 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of* (1.1) *such that $(x_{n+1} - \overline{x})(x_n - \overline{x}) < 0$ for some $n \geq -1$, then $\{x_n\}_{n=-1}^{\infty}$ is unbounded.*

Proof. We may assume without loss of generality that $(x_0 - \overline{x})(x_{-1} - \overline{x}) < 0$ and $(x_{-1}, x_0) \in A_4$ (the proof for $(x_{-1}, x_0) \in A_3$ is similar). From Lemma 2.1 we see $(x_{2n-1}, x_{2n}) \in A_4$ for all $n \geq 0$.If $\{x_{2n}\}_{n=0}^{\infty}$ is eventually increasing, then it follows from Lemma 2.3 that $\{x_{2n-1}\}_{n=0}^{\infty}$ is eventually increasing. Thus $\lim_{n\to+\infty} x_{2n-1} = b > \overline{x}$ and $\lim_{n\to+\infty} x_{2n} = a \leq \overline{x}$, it follows from Lemma 2.2 that $b = \infty$.

If $\{x_{2n}\}_{n=0}^{\infty}$ is not eventually increasing, then there exists some $N \geq 0$ such that

$$x_{2N} \geq x_{2N+2} = \frac{px_{2N+1} + qx_{2N}}{1 + x_{2N+1}}, \tag{3.6}$$

from which we obtain $x_{2N} \geq px_{2N+1}/(1 + x_{2N+1} - q) \geq p$, since $x_{2N+1} \geq \overline{x} = p + q - 1$ and $q > 1$.

Since $f(y, x) = (py + qx)/(1 + y) = p + (qx - p)/(1 + y)$ $(x \geq p, y \geq p)$ is increasing in x and is decreasing in y, we have that $x_{2n} \geq p$ for any $n \geq N$. It follows from Lemma 3.2 that $\{x_{2n}\}_{n=0}^{\infty}$ is eventually decreasing. Thus $\lim_{n\to+\infty} x_{2n} = a < \overline{x}$ and $\lim_{n\to+\infty} x_{2n-1} = b \geq \overline{x}$. It follows from Lemma 2.2 that $b = \infty$. This completes the proof. □

By Theorems 3.1 and 3.4 we have the following.

Corollary 3.5. *Let $q > 1 + p > 1$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive bounded solution of* (1.1), *then $x_{n-1} \geq x_n \geq \overline{x}$ for all $n \geq 0$ or $\overline{x} \geq x_n \geq x_{n-1}$ for all $n \geq 0$.*

Now one can find out the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions of (1.1) are bounded. Let $P_0 = \overline{A_2}, Q_0 = \overline{A_1}$. For any $n \geq 1$, let

$$P_n = f^{-1}(P_{n-1}), \qquad Q_n = f^{-1}(Q_{n-1}). \tag{3.7}$$

It follows from Lemma 2.1 that $P_1 = f^{-1}(P_0) \subset P_0,\ Q_1 = f^{-1}(Q_0) \subset Q_0$, which implies

$$P_n \subset P_{n-1}, \qquad Q_n \subset Q_{n-1} \tag{3.8}$$

for any $n \geq 1$.

Let S be the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions $\{x_n\}_{n=-1}^{\infty}$ of (1.1) are bounded. Then we have the following theorem.

Theorem 3.6. $S = [\bigcap_{n=0}^{\infty} Q_n] \cup [\bigcap_{n=0}^{\infty} P_n]$ $(\subset A_1 \cup A_2 \cup \{(\overline{x}, \overline{x})\})$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1.1) with the initial values $(x_{-1}, x_0) \in S$.

If $(x_{-1}, x_0) \in \bigcap_{n=0}^{\infty} Q_n$, then $f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_1}$ for any $n \geq 0$, which implies $x_n \leq \overline{x}$ for any $n \geq -1$. It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \overline{x}$.

If $(x_{-1}, x_0) \in \bigcap_{n=0}^{\infty} P_n$, then $f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_2}$, which implies $x_n \geq \overline{x}$ for any $n \geq -1$. It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \overline{x}$.

Now assume that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (1.1) with the initial values $(x_{-1}, x_0) \in D - S$.

If $(x_{-1}, x_0) \in A_3 \bigcup A_4 \bigcup L_0 \bigcup L_1 \bigcup R_0 \bigcup R_1$, then it follows from Lemma 2.1 that $f^2\ (x_{-1}, x_0) = (x_1, x_2) \in \{(x, y) : (x - \overline{x})(y - \overline{x}) < 0\}$, which along with Theorem 3.4 implies that $\{x_n\}$ is unbounded.

If $(x_{-1}, x_0) \in \overline{A_2} - \bigcap_{n=0}^{\infty} P_n$, then there exists $n \geq 0$ such that $(x_{-1}, x_0) \in P_n - P_{n+1} = f^{-n}(\overline{A_2}) - f^{-n-1}(\overline{A_2})$. Thus $f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_2} - f^{-1}(\overline{A_2})$. By Lemma 2.1, we obtain $f^{n+1}(x_{-1}, x_0) \in L_1 \bigcup A_4$ and $f^{n+3}(x_{-1}, x_0) = (x_{n+2}, x_{n+3}) \in A_4$, which along with Theorem 3.4 implies that $\{x_n\}$ is unbounded.

If $(x_{-1}, x_0) \in \overline{A_1} - \bigcap_{n=1}^{\infty} Q_n$, then there exists $n \geq 0$ such that $(x_{-1}, x_0) \in Q_n - Q_{n+1} = Q_n - f^{-1}(Q_n)$ and $f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_1} - f^{-1}(\overline{A_1})$. Again by Lemma 2.1 and Theorem 3.4, we have that $\{x_n\}$ is unbounded. This completes the proof. □

Acknowledgment

Project Supported by NNSF of China (10861002) and NSF of Guangxi (0640205, 0728002).

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