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Research Article

Estimation on Certain Nonlinear Discrete Inequality and Applications to Boundary Value Problem

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We investigate certain sum-difference inequalities in two variables which provide explicit bounds on unknown functions. Our result enables us to solve those discrete inequalities considered by Sheng and Li (2008). Furthermore, we apply our result to a boundary value problem of a partial difference equation for estimation.

1. Introduction

Various generalizations of the Gronwall inequality [1, 2] are fundamental tools in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of differential equations and integral equation. There are a lot of papers investigating them (such as [3–8]). Along with the development of the theory of integral inequalities and the theory of difference equations, more attentions are paid to some discrete versions of Gronwall-Bellman-type inequalities (such as [9–11]). Some recent works can be found, for example, in [12–17] and some references therein.

We first introduce two lemmas which are useful in our main result.

Lemma 1.1 (the Bernoulli inequality [18]). *Let* $0 \le \alpha \le 1$ *and* $z \ge -1$, *then* $(1+z)^{\alpha} \le 1 + \alpha z$.

Lemma 1.2 (see [19]). *Assume that* $u(n)$, $a(n)$, $b(n)$ *are nonnegative functions and* $a(n)$ *is nonincreasing for all natural numbers, if for all natural numbers,*

$$u(n) \le a(n) + \sum_{s=n+1}^{\infty} b(s)u(s), \tag{1.1}$$

then for all natural numbers,

$$u(n) \le a(n) \prod_{s=n+1}^{\infty} (1 + b(s)). \tag{1.2}$$

Sheng and Li [16] considered the inequalities

$$\begin{aligned} u^p(n) &\le a(n) + b(n) \sum_{s=n+1}^{\infty} \left[f(s)u^p(s) + g(s)u^q(s) \right], \\ u^p(n) &\le a(n) + b(n) \sum_{s=n+1}^{\infty} \left[f(s)u^q(s) + L(s, u(s)) \right], \\ u^p(n) &\le a(n) + b(n) \sum_{s=n+1}^{\infty} \left[f(s)u^p(s) + L\left(s, u^q(s)\right) \right], \end{aligned} \tag{1.3}$$

where $0 \le L(n, x) - L(n, y) \le K(n, y)(x - y)$ for $x \ge y \ge 0$.

In this paper, we investigate certain new nonlinear discrete inequalities in two variables:

$$u^p(m, n) \le a(m, n) + b(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s, t)u^p(s, t) + g(s, t)u^q(s, t) \right], \tag{1.4}$$

$$u^p(m, n) \le a(m, n) + b(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s, t)u^q(s, t) + L(s, t, u(s, t)) \right], \tag{1.5}$$

$$u^p(m, n) \le a(m, n) + b(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s, t)u^p(s, t) + L\left(s, t, u^q(s, t)\right) \right], \tag{1.6}$$

where $0 \le L(m, n, x) - L(m, n, y) \le K(m, n, y)(x - y)$ for $x \ge y \ge 0$.

Furthermore, we apply our result to a boundary value problem of a partial difference equation for estimation. Our paper gives, in some sense, an extension of a result of [16].

2. Main Result

Throughout this paper, let $\mathbf{R}$ denote the set of all real numbers, let $\mathbf{R}_+ = [0, \infty)$ be the given subset of $\mathbf{R}$, and $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers. For functions $w(m), z(m, n),\ m, n \in \mathbf{N}_0$, their first-order differences are defined by $\Delta w(m) = w(m + 1) - w(m)$, $\Delta_1 w(m, n) = w(m + 1, n) - w(m, n)$, and $\Delta_2 z(m, n) = z(m, n + 1) - z(m, n)$. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. In what follows, we assume all functions which appear in the inequalities to be real-value, p and q are constants, and $p \ge 1,\ 0 \le q \le p$.

Lemma 2.1. *Assume that* $v(m,n)$, $h(m,n)$, *and* $F(m,n)$ *are nonnegative functions defined for* $m,n \in \mathbf{N}_0$, *and* $h(m,n)$ *is nonincreasing in each variable, if*

$$v(m,n) \le h(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F(s,t)v(s,t), \quad m,n \in \mathbf{N}_0, \tag{2.1}$$

then

$$v(m,n) \le h(m,n) \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} F(s,t)\right), \quad m,n \in \mathbf{N}_0. \tag{2.2}$$

Proof. Define a function $\theta(m,n)$ by

$$\theta(m,n) = h(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F(s,t)v(s,t), \quad m,n \in \mathbf{N}_0. \tag{2.3}$$

The function $h(m,n)$ is nonincreasing in each variable, so is $\theta(m,n)$, we have

$$\theta(m,n) \le h(m,n) + \sum_{s=m+1}^{\infty} \left(\sum_{t=n+1}^{\infty} F(s,t)\right)\theta(s,n), \quad m,n \in \mathbf{N}_0. \tag{2.4}$$

Using Lemma 1.2, the desired inequality (2.2) is obtained from (2.1), (2.3), and (2.4). This completes the proof of Lemma 2.1. □

Theorem 2.2. *Suppose that* $a(m,n) \ge 0$ *and* $b(m,n)$, $f(m,n)$, $g(m,n)$, $u(m,n)$ *are nonnegative functions defined for* $m,n \in \mathbf{N}_0$, $u(m,n)$ *satisfies the inequality* (1.4). *Then*

$$u(m,n) \le a^{1/p}(m,n) + \frac{1}{p}a^{1/p-1}(m,n)b(m,n)h(m,n) \prod_{s=m+1}^{\infty} \left(1 + \sum_{t=n+1}^{\infty} H(s,t)\right), \tag{2.5}$$

where

$$\begin{aligned} h(m,n) &= \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s,t)a(s,t) + g(s,t)a^{q/p}(s,t)\right], \\ H(m,n) &= b(m,n)\left[f(m,n) + \frac{q}{p}a^{q/p-1}(m,n)g(m,n)\right]. \end{aligned} \tag{2.6}$$

Proof. Define a function $v(m,n)$ by

$$v(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right], \quad m,n \in \mathbf{N}_0. \tag{2.7}$$

From (1.4), we have

$$u^p(m,n) \le a(m,n) + b(m,n)v(m,n)$$
$$= a(m,n)\left(1 + \frac{b(m,n)v(m,n)}{a(m,n)}\right). \tag{2.8}$$

By applying Lemma 1.1, from (2.8), we obtain

$$u(m,n) \le a^{1/p}(m,n) + \frac{1}{p}a^{1/p-1}(m,n)b(m,n)v(m,n), \tag{2.9}$$

$$u^q(m,n) \le a^{q/p}(m,n) + \frac{q}{p}a^{q/p-1}(m,n)b(m,n)v(m,n). \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned} v(m,n) \le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Big[& f(s,t)(a(s,t)+b(s,t)v(s,t)) \\ &+ g(s,t)\left(a^{q/p}(s,t) + \frac{q}{p}a^{q/p-1}(s,t)b(s,t)v(s,t)\right)\Big] \\ = h(m,n) + \sum_{s=m+1}^{\infty}&\sum_{t=n+1}^{\infty} H(s,t)v(s,t), \quad m,n \in \mathbf{N}_0, \end{aligned} \tag{2.11}$$

where we note the definitions of $h(m,n)$ and $H(m,n)$ in (2.6). From (2.6), we see $h(m,n)$ is nonnegative and nonincreasing in each variable. By applying Lemma 2.1, the desired inequality (3.3) is obtained from (2.9) and (2.11). This completes the proof of Theorem 2.2. □

Theorem 2.3. *Suppose that $a(m,n) \ge 0$ and $b(m,n)$, $f(m,n)$, $u(m,n)$ are nonnegative functions defined for $m,n \in \mathbf{N}_0$, $L : \mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies*

$$0 \le L(m,n,x) - L(m,n,y) \le K(m,n,y)(x-y), \quad x \ge y \ge 0, \tag{2.12}$$

where $K : \mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{R}_+ \to \mathbf{R}_+$, and $u(m,n)$ satisfies the inequality (1.5). *Then*

$$u(m,n) \le a^{1/p}(m,n) + \frac{1}{p}a^{1/p-1}(m,n)b(m,n)G(m,n)\prod_{s=m+1}^{\infty}\left(1 + \sum_{t=n+1}^{\infty}F(s,t)\right), \tag{2.13}$$

where

$$G(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)a^{q/p}(s,t) + L\left(s,t,a^{1/p}(s,t)\right)\right], \tag{2.14}$$

$$F(m,n) = b(m,n)\left[\frac{q}{p}a^{q/p-1}(m,n)f(m,n) + \frac{1}{p}K\left(m,n,a^{1/p}(m,n)\right)a^{1/p-1}(m,n)\right]. \tag{2.15}$$

Proof. Define a function $v(m,n)$ by

$$v(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)u^{q}(s,t) + L(s,t,u(s,t))\right], \quad m,n \in \mathbf{N}_0. \tag{2.16}$$

Then, as in the proof of Theorem 2.2, we have (2.8), (2.9), and (2.10). By (2.12),

$$\begin{aligned}
&\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L(s,t,u(s,t)) \\
&\quad \le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[L\left(s,t,a^{1/p}(s,t) + \frac{1}{p}a^{1/p-1}(s,t)b(s,t)v(s,t)\right) \\
&\qquad\qquad - L\left(s,t,a^{1/p}(s,t)\right) + L\left(s,t,a^{1/p}(s,t)\right)\Bigg] \\
&\quad \le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L\left(s,t,a^{1/p}(s,t)\right) \\
&\qquad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} K\left(s,t,a^{1/p}(s,t)\right)\frac{1}{p}a^{1/p-1}(s,t)b(s,t)v(s,t).
\end{aligned} \tag{2.17}$$

It follows from (2.8), (2.9), (2.10), and (2.17) that

$$\begin{aligned}
v(m,n) &\le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)a^{q/p}(s,t) + L\left(s,t,a^{1/p}(s,t)\right)\right] \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\frac{q}{p}f(s,t)a^{q/p-1}(s,t) + \frac{1}{p}K\left(s,t,a^{1/p}(s,t)\right)a^{1/p-1}(s,t)\right]b(s,t)v(s,t) \\
&= G(m,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}F(s,t)v(s,t),
\end{aligned} \tag{2.18}$$

where we note the definitions of $G(m,n)$ and $F(m,n)$ in (2.14) and (2.15). From (2.14) we see $G(m,n)$ is nonnegative and nonincreasing in each variable. By applying Lemma 2.1, the desired inequality (2.19) is obtained from (2.9) and (2.18). This completes the proof of Theorem 2.3. □

Theorem 2.4. *Suppose that* $a(m,n)$, $b(m,n)$, $f(m,n)$, $u(m,n)$, $L(m,n,x)$, $K(m,n,x)$ *are the same as in Theorem 2.3,* $u(m,n)$ *satisfies the inequality* (1.6). *Then*

$$u(m,n) \le a^{1/p}(m,n) + \frac{1}{p}a^{1/p-1}(m,n)b(m,n)G(m,n)\prod_{s=m+1}^{\infty}\left(1 + \sum_{t=n+1}^{\infty}F(s,t)\right), \tag{2.19}$$

where

$$J(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)a(s,t) + L\left(s,t,a^{q/p}(s,t)\right)\right], \tag{2.20}$$

$$M(m,n) = b(m,n)\left[f(m,n) + \frac{q}{p}K\left(m,n,a^{q/p}(m,n)\right)a^{q/p-1}(m,n)\right]. \tag{2.21}$$

Proof. Define a function $v(m,n)$ by

$$v(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)u^{p}(s,t) + L\left(s,t,u^{q}(s,t)\right)\right], \quad m,n \in \mathbf{N}_0. \tag{2.22}$$

Then, as in the proof of Theorem 2.2, we have (2.8), (2.9), and (2.10). By (2.12),

$$\begin{aligned}
&\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L\left(s,t,u^{q}(s,t)\right) \\
&\quad\le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[L\left(s,t,a^{q/p}(s,t) + \frac{q}{p}a^{q/p-1}(s,t)b(s,t)v(s,t)\right)\right. \\
&\qquad\qquad \left. - L\left(s,t,a^{q/p}(s,t)\right) + L\left(s,t,a^{q/p}(s,t)\right)\right] \\
&\quad\le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L\left(s,t,a^{q/p}(s,t)\right) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} K\left(s,t,a^{q/p}(s,t)\right)\frac{q}{p}a^{q/p-1}(s,t)b(s,t)v(s,t).
\end{aligned} \tag{2.23}$$

It follows from (2.8), (2.9), (2.10), and (2.23) that

$$\begin{aligned}
v(m,n) &\le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)a(s,t) + L\left(s,t,a^{q/p}(s,t)\right)\right] \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t) + \frac{q}{p}K\left(s,t,a^{q/p}(s,t)\right)a^{q/p-1}(s,t)\right]b(s,t)v(s,t) \\
&= J(m,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} M(s,t)v(s,t),
\end{aligned} \tag{2.24}$$

where $J(m,n)$ and $M(m,n)$ are defined by (2.20) and (2.21), respectively. From (2.20), we see $J(m,n)$ is nonnegative and nonincreasing in each variable. By applying Lemma 2.1, the desired inequality (2.19) is obtained from (2.9) and (2.24). This completes the of Theorem 2.4. □

3. Applications to Boundary Value Problem

In this section, we apply our result to the following boundary value problem (simply called BVP) for the partial difference equation:

$$\begin{aligned} &\Delta_1\Delta_2 z^p(m,n) = F(m,n,z(m,n)), \quad m,n \in \mathbf{N}_0, \\ &z(m,\infty) = a_1(m), \quad z(\infty,n) = a_2(n), \quad m,n \in \mathbf{N}_0, \end{aligned} \tag{3.1}$$

$F : \Lambda \times \mathbf{R} \to \mathbf{R}$ satisfies

$$|F(m,n,u)| \le f(m,n)\left|u^p\right| + g(m,n)\left|u^q\right|, \tag{3.2}$$

where p and q are constants, $p \ge 1$, $0 \le q \le p$, functions $f, g : \mathbf{N}_0 \times \mathbf{N}_0 \to \mathbf{R}_+$ are given, and functions $a_1, a_2 : \mathbf{N}_0 \to \mathbf{R}_+$ are nonincreasing. In what follows, we apply our main result to give an estimation of solutions of (3.1).

Corollary 3.1. *All solutions $z(m,n)$ of BVP* (3.1) *have the estimate*

$$u(m,n) \le a^{1/p}(m,n) + \frac{1}{p}a^{1/p-1}(m,n)h(m,n)\prod_{s=m+1}^{\infty}\left(1 + \sum_{t=n+1}^{\infty} H(s,t)\right), \tag{3.3}$$

where

$$\begin{aligned} a(m,n) &= \left|a_1(m) + a_2(n)\right|, \\ h(m,n) &= \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)a(s,t) + g(s,t)a^{q/p}(s,t)\right], \\ H(m,n) &= f(m,n) + \frac{q}{p}a^{q/p-1}(m,n)g(m,n). \end{aligned} \tag{3.4}$$

Proof. Clearly, the difference equation of BVP (3.1) is equivalent to

$$z^p(m,n) = a_1(m) + a_2(n) + \sum_{s=m}^{\infty}\sum_{t=n}^{\infty} F(s,t,z(s,t)). \tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\left|z^p(m,n)\right| \le \left|a_1(m) + a_2(n)\right| + \sum_{s=m}^{\infty}\sum_{t=n}^{\infty}\left[f(s,t)\left|z^p(s,t)\right| + g(s,t)\left|z^q(s,t)\right|\right]. \tag{3.6}$$

Let $a(m,n) = |a_1(m) + a_2(n)|$. Equation (3.6) is of the form (1.4), here $b(m,n) = 1$. Applying our Theorem 2.2 to inequality (3.6), we obtain the estimate of $z(m,n)$ as given in Corollary 3.1. □

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References

[1] R. Bellman, "The stability of solutions of linear differential equations," *Duke Mathematical Journal*, vol. 10, no. 4, pp. 643–647, 1943.
[2] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Annals of Mathematics*, vol. 20, no. 4, pp. 292–296, 1919.
[3] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.
[4] O. Lipovan, "Integral inequalities for retarded Volterra equations," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 349–358, 2006.
[5] Q.-H. Ma and E.-H. Yang, "On some new nonlinear delay integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 864–878, 2000.
[6] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, vol. 197 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1998.
[7] W.-S. Wang, "A generalized sum-difference inequality and applications to partial difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 695495, 12 pages, 2008.
[8] W. Zhang and S. Deng, "Projected Gronwall-Bellman's inequality for integrable functions," *Mathematical and Computer Modelling*, vol. 34, no. 3-4, pp. 393–402, 2001.
[9] T. E. Hull and W. A. J. Luxemburg, "Numerical methods and existence theorems for ordinary differential equations," *Numerische Mathematik*, vol. 2, no. 1, pp. 30–41, 1960.
[10] B. G. Pachpatte and S. G. Deo, "Stability of discrete-time systems with retarded argument," *Utilitas Mathematica*, vol. 4, pp. 15–33, 1973.
[11] D. Willett and J. S. W. Wong, "On the discrete analogues of some generalizations of Gronwall's inequality," *Monatshefte für Mathematik*, vol. 69, pp. 362–367, 1965.
[12] W.-S. Cheung and J. Ren, "Discrete non-linear inequalities and applications to boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 708–724, 2006.
[13] W. N. Li and W. Sheng, "Some nonlinear integral inequalities on time scales," *Journal of Inequalities and Applications*, vol. 2007, Article ID 70465, 15 pages, 2007.
[14] B. G. Pachpatte, "On some new inequalities related to certain inequalities in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 128–144, 1995.
[15] P. Y. H. Pang and R. P. Agarwal, "On an integral inequality and its discrete analogue," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 2, pp. 569–577, 1995.
[16] W. Sheng and W. N. Li, "Bounds on certain nonlinear discrete inequalities," *Journal of Mathematical Inequalities*, vol. 2, no. 2, pp. 279–286, 2008.
[17] W.-S. Wang and C.-X. Shen, "On a generalized retarded integral inequality with two variables," *Journal of Inequalities and Applications*, vol. 2008, Article ID 518646, 9 pages, 2008.
[18] D. S. Mitrinović, *Analytic Inequalities*, vol. 16 of *Die Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1970.
[19] B. G. Pachpatte, "On some fundamental finite difference inequalities," *Tamkang Journal of Mathematics*, vol. 32, no. 3, pp. 217–223, 2001.