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Research Article

Nonlocal Controllability for the Semilinear Fuzzy Integrodifferential Equations in n-Dimensional Fuzzy Vector Space

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We study the existence and uniqueness of solutions and controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space $(E_N)^n$ by using Banach fixed point theorem, that is, an extension of the result of J. H. Park et al. to n-dimensional fuzzy vector space.

1. Introduction

Many authors have studied several concepts of fuzzy systems. Diamond and Kloeden [1] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \qquad x(0) = x_0, \tag{1.1}$$

where $x(\cdot)$ and $u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [2] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 by using Kuhn-Tucker theorems. Fuzzy integrodifferential equations are a field of interest, due to their applicability to the analysis of phenomena with memory where imprecision is inherent. Balasubramaniam and Muralisankar [3] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. They considered the semilinear one-dimensional heat equation on a connected domain $(0,1)$ for material with memory. In one-dimensional fuzzy vector space E_N^1, Park et al. [4] proved the existence and uniqueness of fuzzy solutions and presented the sufficient condition of nonlocal controllability for the following semilinear fuzzy integrodifferential equation with nonlocal initial condition:

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] + f(t,x) + u(t), \quad t \in J = [0,T],$$
$$x(0) + g\big(t_1, t_2, \ldots, t_p, x(t_m)\big) = x_0 \in E_N, \quad m = 1,2,\ldots,p, \tag{1.2}$$

where $T > 0$, $A : J \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : J \times E_N \to E_N$ is a nonlinear continuous function, $g : J^p \times E_N \to E_N$ is a nonlinear continuous function, $G(t)$ is an $n \times n$ continuous matrix such that $dG(t)x/dt$ is continuous for $x \in E_N$ and $t \in J$ with $\|G(t)\| \le K$, $K > 0$, with all nonnegative elements, $u : J \to E_N$ is control function.

In [5], Kwun et al. proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equations by using successive iteration. In [6], Kwun et al. investigated the continuously initial observability for the semilinear fuzzy integrodifferential equations. Bede and Gal [7] studied almost periodic fuzzy-number-valued functions. Gal and N'Guérékata [8] studied almost automorphic fuzzy-number-valued functions.

In this paper, we study the the existence and uniqueness of solutions and controllability for the following semilinear fuzzy integrodifferential equations:

$$\frac{dx_i(t)}{dt} = A_i\left[x_i(t) + \int_0^t G(t-s)x_i(s)ds\right] + f_i(t, x_i(t)) + u_i(t) \text{ on } E_N^i,$$
$$x_i(0) + g_i(x_i) = x_{0_i} \in E_N^i \quad (i = 1,2,\ldots,n), \tag{1.3}$$

where $A_i : [0,T] \to E_N^i$ is fuzzy coefficient, E_N^i is the set of all upper semicontinuously convex fuzzy numbers on R with $E_N^i \neq E_N^j$ $(i \neq j)$, $f_i : [0,T] \times E_N^i \to E_N^i$ is a nonlinear regular fuzzy function, $g_i : E_N^i \to E_N^i$ is a nonlinear continuous function, $G(t)$ is $n \times n$ continuous matrix such that $dG(t)x_i/dt$ is continuous for $x_i \in E_N^i$ and $t \in [0,T]$ with $\|G(t)\| \le k$, $k > 0$, $u_i : [0,T] \to E_N^i$ is control function and $x_{0_i} \in E_N^i$ is initial value.

2. Preliminaries

A fuzzy set of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \ge \alpha\}$ for any $\alpha \in [0,1]$, its α-level set.

Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$.

Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_o) = 1$;

(2) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \le \lambda \le 1$;

(3) $u(x)$ is upper semicontinuous, that is, $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \ldots)$, $x_k \to x_0$;

(4) $[u]^0$ is compact.

We call $u \in E^n$ an n-dimension fuzzy number.

Wang et al. [9] defined n-dimensional fuzzy vector space and investigated its properties.

For any $u_i \in E$, $i = 1, 2, \ldots, n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, \ldots, u_n$ (i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, \ldots, u_n$) an n-dimension fuzzy vector, denote it as $(u_1, u_2, \ldots, u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}$) n-dimensional fuzzy vector space, and denote it as $(E)^n$.

Definition 2.1 (see [9]). If $u \in E^n$, and $[u]^\alpha$ is a hyperrectangle, that is, $[u]^\alpha$ can be represented by $\prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$, that is, $[u_{1l}^\alpha, u_{1r}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha]$ for every $\alpha \in [0,1]$, where $u_{il}^\alpha, u_{ir}^\alpha \in R$ with $u_{il}^\alpha \leq u_{ir}^\alpha$ when $\alpha \in (0,1]$, $i = 1, 2, \ldots, n$, then we call u a fuzzy n-cell number. We denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

Theorem 2.2 (see [9]). *For any $u \in L(E^n)$ with $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$, there exists a unique $(u_1, u_2, \ldots, u_n) \in (E)^n$ such that $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \ldots, n$ and $\alpha \in [0,1])$.*

Conversely, for any $(u_1, u_2, \ldots, u_n) \in (E)^n$ with $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha] (i = 1, 2, \ldots, n$ and $\alpha \in [0,1])$, there exists a unique $u \in L(E^n)$ such that $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$.

Note 1 (see [9]). Theorem 2.2 indicates that fuzzy n-cell numbers and n-dimension fuzzy vectors can represent each other, so $L(E^n)$ and $(E)^n$ may be regarded as identity. If $(u_1, u_2, \ldots, u_n) \in (E)^n$ is the unique n-dimension fuzzy vector determined by $u \in L(E^n)$, then we denote $u = (u_1, u_2, \ldots, u_n)$.

Let $(E_N^i)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n$, E_N^i $(i = 1, 2, \times, n)$ be fuzzy subset of R. Then $(E_N^i)^n \subseteq (E)^n$.

Definition 2.3 (see [9]). The complete metric D_L on $(E_N^i)^n$ is defined by

$$\begin{aligned} D_L(u, v) &= \sup_{0<\alpha\leq 1} d_L\left([u]^\alpha, [v]^\alpha\right) \\ &= \sup_{0<\alpha\leq 1} \max_{1\leq i\leq n}\left\{\left|u_{il}^\alpha - v_{il}^\alpha\right|, \left|u_{ir}^\alpha - v_{ir}^\alpha\right|\right\} \end{aligned} \tag{2.1}$$

for any $u, v \in (E_N^i)^n$, which satisfies $d_L(u + w, v + w) = d_L(u, v)$.

Definition 2.4. Let $u, v \in C([0,T] : (E_N^i)^n)$, then

$$H_1(u, v) = \sup_{0\leq t\leq T} D_L(u(t), v(t)). \tag{2.2}$$

Definition 2.5 (see [9]). The derivative $x'(t)$ of a fuzzy process $x \in (E_N^i)^n$ is defined by

$$[x'(t)]^{\alpha} = \prod_{i=1}^{n} \left[(x_{il}^{\alpha})'(t), (x_{ir}^{\alpha})'(t)\right] \tag{2.3}$$

provided that the equation defines a fuzzy $x'(t) \in (E_N^i)^n$.

Definition 2.6 (see [9]). The fuzzy integral $\int_b^a x(t)dt$, $a, b \in [0, \mathrm{T}]$ is defined by

$$\left[\int_b^a x(t)dt\right]^{\alpha} = \prod_{i=1}^{n} \left[\int_b^a x_{il}^{\alpha}(t)dt, \int_b^a x_{ir}^{\alpha}(t)dt\right] \tag{2.4}$$

provided that the Lebesgue integrals on the right-hand side exist.

3. Existence and Uniqueness

In this section we consider the existence and uniqueness of the fuzzy solution for (1.3) ($u \equiv 0$).

We define

$$\begin{aligned} A &= (A_1, A_2, \ldots, A_n), \\ x &= (x_1, x_2, \ldots, x_n), \\ f &= (f_1, f_2, \ldots, f_n), \\ u &= (u_1, u_2, \ldots, u_n), \\ g &= (g_1, g_2, \ldots, g_n), \\ x_0 &= (x_{0_1}, x_{0_2}, \ldots, x_{0_n}). \end{aligned} \tag{3.1}$$

Then

$$A, x, f, x_0, u, g \in \left(E_N^i\right)^n. \tag{3.2}$$

Instead of (1.3), we consider the following fuzzy integrodifferential equations in $(E_N^i)^n$:

$$\begin{gathered} \frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] + f(t, x(t)) + u(t) \text{ on } \left(E_N^i\right)^n \\ x(0) + g(x) = x_0 \in \left(E_N^i\right)^n \end{gathered} \tag{3.3}$$

with fuzzy coefficient $A : [0, T] \to (E_N^i)^n$, initial value $x_0 \in (E_N^i)^n$, and $u : [0, T] \to (E_N^i)^n$ is a control function. Given nonlinear regular fuzzy function $f : [0, T] \times (E_N^i)^n \to (E_N^i)^n$ satisfies a global Lipschitz condition, that is, there exists a finite $k > 0$ such that

$$d_L\left([f(s, x(s))]^{\alpha}, [f(s, y(s))]^{\alpha}\right) \le k d_L\left([x(s)]^{\alpha}, [y(s)]^{\alpha}\right) \tag{3.4}$$

for all $x(s), y(s) \in (E_N^i)^n$. The nonlinear function $g : (E_N^i)^n \to (E_N^i)^n$ is a continuous function and satisfies the Lipschitz condition

$$d_L\left([g(x(\cdot))]^\alpha, [g(y(\cdot))]^\alpha\right) \leq h d_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right) \tag{3.5}$$

for all $x(\cdot), y(\cdot) \in (E_N^i)^n$, h is a finite positive constant.

Definition 3.1. The fuzzy process $x : I = [0,T] \to (E_N^i)^n$ with α-level set $[x(t)]^\alpha = \Pi_{i=1}^n [x_i]^\alpha = \Pi_{i=1}^n [x_{il}^\alpha, x_{ir}^\alpha]$ is a fuzzy solution of (3.3) without nonhomogeneous term if and only if

$$\begin{gathered}
(x_{il}^\alpha)'(t) = \min\left\{ A_{ij}^\alpha(t)\left[x_{ik}^\alpha(t) + \int_0^t G(t-s)x_{ik}^\alpha(s)ds\right] : j,k = l,r\right\}, \\
(x_{ir}^\alpha)'(t) = \max\left\{ A_{ij}^\alpha(t)\left[x_{ik}^\alpha(t) + \int_0^t G(t-s)x_{ik}^\alpha(s)ds\right] : j,k = l,r\right\}, \\
x_{il}^\alpha(0) + g_{il}^\alpha(x_{il}^\alpha) = x_{0_{il}}^\alpha, \quad x_{ir}^\alpha(0) + g_{ir}^\alpha(x_{ir}^\alpha) = x_{0_{ir}}^\alpha, \quad i = 1,2,\ldots,n.
\end{gathered} \tag{3.6}$$

For the sequel, we need the following assumptions.

(H1) $S(t)$ is a fuzzy number satisfying, for $y \in (E_N^i)^n$, $(d/dt)\,S(t)y \in C^1(I : (E_N^i)^n) \bigcap C(I : (E_N^i)^n)$, the equation

$$\begin{aligned}
\frac{d}{dt}S(t)y &= A\left[S(t)y + \int_0^t G(t-s)S(s)y\,ds\right] \\
&= S(t)Ay + \int_0^t S(t-s)AG(s)y\,ds, \quad t \in I,
\end{aligned} \tag{3.7}$$

where

$$[S(t)]^\alpha = \prod_{i=1}^n [S_i(t)]^\alpha = \prod_{i=1}^n [S_{il}^\alpha(t), S_{ir}^\alpha(t)], \tag{3.8}$$

and $S_{ij}^\alpha(t)$ $(j = l,r)$ is continuous with $|S_{ij}^\alpha(t)| \leq c$, $c > 0$, for all $t \in I = [0,T]$.

(H2) $c\{h(1+T+cT) + kT(1+cT)\} < 1$.

In view of Definition 3.1 and (H1), (3.3) can be expressed as

$$\begin{gathered}
x(t) = S(t)\left(x_0 - g(x)\right) + \int_0^t S(t-s)\left(f(s,x(s)) + u(s)\right)ds, \\
x(0) + g(x) = x_0.
\end{gathered} \tag{3.9}$$

Theorem 3.2. *Let $T > 0$. If hypotheses (H1)-(H2) are hold, then for every $x_0 \in (E_N^i)^n$, (3.9) $(u \equiv 0)$ have a unique fuzzy solution $x \in C([0,T] : (E_N^i)^n)$.*

Proof. For each $x(t) \in (E_N^i)^n$ and $t \in [0,T]$, define $(G_0x)(t) \in (E_N^i)^n$ by

$$(G_0x)(t) = S(t)\big(x_0 - g(x)\big) + \int_0^t S(t-s)f(s,x(s))ds. \tag{3.10}$$

Thus, $G_0x : [0,T] \to (E_N^i)^n$is continuous, so G_0 is a mapping from $C([0,T] : (E_N^i)^n)$ into itself. By Definitions 2.3 and 2.4, some properties of d_L, and inequalities (3.4) and (3.5), we have following inequalities. For $x, y \in C([0,T] : (E_N^i)^n)$,

$$\begin{aligned}
&d_L\big([(G_0x)(t)]^\alpha, [(G_0y)(t)]^\alpha\big)\\
&\quad= d_L\left(\left[S(t)\big(x_0 - g(x)\big) + \int_0^t S(t-s)f(s,x(s))ds\right]^\alpha,\right.\\
&\qquad\qquad\left.\left[S(t)\big(x_0 - g(y)\big) + \int_0^t S(t-s)f(s,y(s))ds\right]^\alpha\right)\\
&\quad= d_L\left(\left[-S(t)g(x) + \int_0^t S(t-s)f(s,x(s))ds\right]^\alpha,\right.\\
&\qquad\qquad\left.\left[-S(t)g(y) + \int_0^t S(t-s)f(s,y(s))ds\right]^\alpha\right)\\
&\quad\le d_L\big([S(t)g(x)]^\alpha, [S(t)g(y)]^\alpha\big) + \int_0^t d_L\big([S(t-s)f(s,x(s))]^\alpha, [S(t-s)f(s,y(s))]^\alpha\big)ds\\
&\quad= \max_{1\le i\le n}\big\{\big|S_{il}^\alpha(t)\big(g_{il}^\alpha(x) - g_{il}^\alpha(y)\big)\big|, \big|S_{ir}^\alpha(t)\big(g_{ir}^\alpha(x) - g_{ir}^\alpha(y)\big)\big|\big\}\\
&\qquad+ \int_0^t \max_{1\le i\le n}\big\{\big|S_{il}^\alpha(t-s)\big(f_{il}^\alpha(s,x(s)) - f_{il}^\alpha(s,y(s))\big)\big|, \big|S_{ir}^\alpha(t-s)\big(f_{ir}^\alpha(s,x(s)) - f_{ir}^\alpha(s,y(s))\big)\big|\big\}ds\\
&\quad\le c\max_{1\le i\le n}\big\{\big|\big(g_{il}^\alpha(x) - g_{il}^\alpha(y)\big)\big|, \big|\big(g_{ir}^\alpha(x) - g_{ir}^\alpha(y)\big)\big|\big\}\\
&\qquad+ c\int_0^t \max_{1\le i\le n}\big\{\big|f_{il}^\alpha(s,x(s)) - f_{il}^\alpha(s,y(s))\big|, \big|f_{ir}^\alpha(s,x(s)) - f_{ir}^\alpha(s,y(s))\big|\big\}ds\\
&\quad= cd_L\big([g(x)]^\alpha, [g(y)]^\alpha\big) + c\int_0^t d_L\big([f(s,x(s))]^\alpha, [f(s,y(s))]^\alpha\big)ds\\
&\quad\le chd_L\big([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\big) + ck\int_0^t d_L\big([x(s)]^\alpha, [y(s)]^\alpha\big)ds.
\end{aligned} \tag{3.11}$$

Therefore

$$
\begin{aligned}
D_L&\big((G_0x)(t),(G_0y)(t)\big)\\
&=\sup_{0<\alpha\le 1} d_L\big([(G_0x)(t)]^\alpha,[(G_0y)(t)]^\alpha\big)\\
&\le ch\sup_{0<\alpha\le 1} d_L\big([x(\cdot)]^\alpha,[y(\cdot)]^\alpha\big)+ck\sup_{0<\alpha\le 1}\int_0^t d_L\big([x(s)]^\alpha,[y(s)]^\alpha\big)ds\\
&\le chD_L\big(x(\cdot),y(\cdot)\big)+ck\int_0^t D_L\big(x(s),y(s)\big)ds.
\end{aligned}
\tag{3.12}
$$

Hence

$$
\begin{aligned}
H_1(G_0x,G_0y)&=\sup_{0\le t\le T} D_L\big((G_0x)(t),(G_0y)(t)\big)\\
&\le ch\sup_{0\le t\le T} D_L\big(x(\cdot),y(\cdot)\big)+ck\sup_{0\le t\le T}\int_0^t D_L\big(x(s),y(s)\big)ds\\
&\le ch\,H_1(x,y)+ckT\,H_1(x,y)\\
&=c(h+kT)\,H_1(x,y).
\end{aligned}
\tag{3.13}
$$

By hypothesis (H2), G_0 is a contraction mapping.

Using the Banach fixed point theorem, (3.9) have a unique fixed point $x\in C([0,T]:(E_N^i)^n)$. □

4. Controllability

In this section, we show the nonlocal controllability for the control system (1.3).

Definition 4.1. Equation (1.3) is nonlocal controllable. Then there exists $u(t)$ such that the fuzzy solution $x(t)$ for (3.9) as $x(T)=x^1-g(x)$ (i.e., $[x(T)]^\alpha=[x^1-g(x)]^\alpha$) where $x^1\in(E_N^i)^n$ is target set.

Define the fuzzy mapping $\widetilde{\beta}:\widetilde{P}(R^n)\to(E_N^i)^n$ by

$$
\widetilde{\beta}^\alpha(v)=\begin{cases}\displaystyle\int_0^T S^\alpha(T-s)v(s)ds, & v\subset\overline{\Gamma}_u,\\[2ex] 0, & \text{otherwise,}\end{cases}
\tag{4.1}
$$

where $\overline{\Gamma}_u$ is closed support of u. Then there exists

$$\widetilde{\beta}_i : \widetilde{P}(R) \longrightarrow E_N^i \quad (i = 1, 2, \ldots, n) \tag{4.2}$$

such that

$$\widetilde{\beta}_i^{\alpha}(v_i) = \begin{cases} \int_0^T S_i^{\alpha}(T-s) v_i(s) ds, & v_i(s) \subset \overline{\Gamma}_{u_i}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.3}$$

Then $\widetilde{\beta}_{ij}^{\alpha}$ $(j = l, r)$ exists such that

$$\begin{aligned} \widetilde{\beta}_{il}^{\alpha}(v_{il}) &= \int_0^T S_{il}^{\alpha}(T-s) v_{il}(s) ds, \quad v_{il}(s) \in \left[u_{il}^{\alpha}(s), u_i^1\right], \\ \widetilde{\beta}_{ir}^{\alpha}(v_{ir}) &= \int_0^T S_{ir}^{\alpha}(T-s) v_{ir}(s) ds, \quad v_{ir}(s) \in \left[u_i^1, u_{ir}^{\alpha}(s)\right]. \end{aligned} \tag{4.4}$$

We assume that $\widetilde{\beta}_{il}^{\alpha}, \widetilde{\beta}_{ir}^{\alpha}$ are bijective mappings.

We can introduce α-level set of $u(s)$ of (3.4)-(3.5)

$$\begin{aligned} [u(s)]^{\alpha} &= \prod_{i=1}^{n} \; [u_i(s)]^{\alpha} \\ &= \prod_{i=1}^{n} \left[u_{il}^{\alpha}(s), u_{ir}^{\alpha}(s)\right] \\ &= \prod_{i=1}^{n} \Bigg[\left(\widetilde{\beta}_{il}^{\alpha}\right)^{-1} \Bigg(\left(\left(x^1\right)_{il}^{\alpha} - g_{il}^{\alpha}\left(x_{il}^{\alpha}\right)\right) - S_{il}^{\alpha}(T)\left(x_{0_{il}}^{\alpha} - g_{il}^{\alpha}\left(x_{il}^{\alpha}\right)\right) \\ &\qquad - \int_0^T S_{il}^{\alpha}(T-s) f_{il}^{\alpha}\left(s, x_{il}^{\alpha}(s)\right) ds \Bigg), \\ &\qquad \left(\widetilde{\beta}_{ir}^{\alpha}\right)^{-1} \Bigg(\left(\left(x^1\right)_{ir}^{\alpha} - g_{ir}^{\alpha}\left(x_{ir}^{\alpha}\right)\right) - S_{ir}^{\alpha}(T)\left(x_{0_{ir}}^{\alpha} - g_{ir}^{\alpha}\left(x_{ir}^{\alpha}\right)\right) \\ &\qquad - \int_0^T S_{ir}^{\alpha}(T-s) f_{ir}^{\alpha}\left(s, x_{ir}^{\alpha}(s)\right) ds \Bigg) \Bigg]. \end{aligned} \tag{4.5}$$

Then substituting this expression into (3.9) yields α-level of $x(T)$.

For each $i = 1, 2, \ldots, n$,

$$[x_i(T)]^\alpha = \Bigg[S_{il}^\alpha(T)\left(x_{0_{il}}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) + \int_0^T S_{il}^\alpha(T-s) f_{il}^\alpha(s, x_{il}^\alpha(s))ds$$
$$+ \int_0^T S_{il}^\alpha(T-s)\left(\tilde{\beta}_{il}^\alpha\right)^{-1}\Bigg(\left(\left(x^1\right)_{il}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right) - S_{il}^\alpha(T)\left(x_{0_{il}}^\alpha - g_{il}^\alpha(x_{il}^\alpha)\right)$$
$$- \int_0^T S_{il}^\alpha(T-s) f_{il}^\alpha(s, x_{il}^\alpha(s))ds\Bigg)ds,$$
$$S_{ir}^\alpha(T)\left(x_{0_{ir}}^\alpha - g_{ir}^\alpha(x_{ir}^\alpha)\right) + \int_0^T S_{ir}^\alpha(T-s) f_{ir}^\alpha(s, x_{ir}^\alpha(s))ds \tag{4.6}$$
$$+ \int_0^T S_{ir}^\alpha(T-s)\left(\tilde{\beta}_{ir}^\alpha\right)^{-1}\Bigg(\left(\left(x^1\right)_{ir}^\alpha - g_{ir}^\alpha(x_{ir}^\alpha)\right) - S_{ir}^\alpha(T)\left(x_{0_{ir}}^\alpha - g_{ir}^\alpha(x_{ir}^\alpha)\right)$$
$$- \int_0^T S_{ir}^\alpha(T-s) f_{ir}^\alpha(s, x_{ir}^\alpha(s))ds\Bigg)ds\Bigg]$$
$$= \left[\left(x^1 - g(x)\right)_{il}^\alpha, \left(x^1 - g(x)\right)_{ir}^\alpha\right] = \left[\left(x^1 - g(x)\right)_i\right]^\alpha.$$

Therefore

$$[x(T)]^\alpha = \prod_{i=1}^n \; [x_i(T)]^\alpha = \prod_{i=1}^n \left[\left(x^1 - g(x)\right)_i\right]^\alpha = \left[x^1 - g(x)\right]^\alpha. \tag{4.7}$$

We now set

$$\Phi x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s) f(s, x(s))ds$$
$$+ \int_0^t S(t-s)\tilde{\beta}^{-1}\left(x^1 - g(x) - S(T)(x_0 - g(x)) - \int_0^T S(T-s) f(s, x(s))ds\right)ds, \tag{4.8}$$

where the fuzzy mapping $\tilde{\beta}^{-1}$ satisfies above statements.

Notice that $\Phi x(T) = x^1 - g(x)$, which means that the control $u(t)$ steers (3.9) from the origin to $x^1 - g(x)$ in time T provided that we can obtain a fixed point of the operator Φ.

(H3) Assume that the linear system of (3.9) ($f \equiv 0$) is controllable.

Theorem 4.2. *Suppose that hypotheses (H1)–(H3) are satisfied. Then* (3.9) *are nonlocal controllable.*

Proof. We can easily check that Φ is continuous function from $C([0,T]:(E_N^i)^n)$ to itself. By Definitions 2.3 and 2.4, some properties of d_L, and inequalities (3.4) and (3.5), we have the following inequalities. For any $x,y\in C([0,T]:(E_N^i)^n)$,

$$
\begin{aligned}
&d_L([\Phi x(t)]^\alpha,[\Phi y(t)]^\alpha)\\
&= d_L\Bigg(\Bigg[S(t)(x_0-g(x))+\int_o^t S(t-s)f(s,x(s))ds+\int_0^t S(t-s)\widetilde{\beta}^{-1}\\
&\qquad\times\left(x^1-g(x)-S(T)(x_0-g(x))-\int_0^T S(T-s)f(s,x(s))ds\right)ds\Bigg]^\alpha,\\
&\qquad\Bigg[S(t)(x_0-g(y))+\int_0^t S(t-s)f(s,y(s))ds+\int_0^t S(t-s)\widetilde{\beta}^{-1}\\
&\qquad\times\left(x^1-g(y)-S(T)(x_0-g(y))-\int_0^T S(T-s)f(s,y(s))ds\right)ds\Bigg]^\alpha\Bigg)\\
&\le d_L([S(t)g(x)]^\alpha,[S(t)g(y)]^\alpha)+\int_0^t d_L([S(t-s)f(s,x(s))]^\alpha,[S(t-s)f(s,y(s))]^\alpha)ds\\
&\quad+\int_0^t d_L\left(\left[S(t-s)\widetilde{\beta}^{-1}g(x)\right]^\alpha,\left[S(t-s)\widetilde{\beta}^{-1}g(y)\right]^\alpha\right)ds\\
&\quad+\int_0^t d_L\left(\left[S(t-s)\widetilde{\beta}^{-1}S(T)g(x)\right]^\alpha,\left[S(t-s)\widetilde{\beta}^{-1}S(T)g(y)\right]^\alpha\right)ds\\
&\quad+\int_0^t d_L\left(\left[S(t-s)\widetilde{\beta}^{-1}\int_0^T S(T-s)f(s,x(s))ds\right]^\alpha,\left[S(t-s)\widetilde{\beta}^{-1}\int_0^T S(T-s)f(s,y(s))ds\right]^\alpha\right)ds\\
&=\max_{1\le i\le n}\{|S_{il}^\alpha(t)(g_{il}^\alpha(x)-g_{il}^\alpha(y))|,|S_{ir}^\alpha(t)(g_{ir}^\alpha(x)-g_{ir}^\alpha(y))|\}\\
&\quad+\int_0^t\max_{1\le i\le n}\{|S_{il}^\alpha(t-s)(f_{il}^\alpha(s,x(s))-f_{il}^\alpha(s,y(s)))|,|S_{ir}^\alpha(t-s)(f_{ir}^\alpha(s,x(s))-f_{ir}^\alpha(s,y(s)))|\}ds\\
&\quad+\int_0^t\max_{1\le i\le n}\left\{\left|S_{il}^\alpha(t-s)\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}(g_{il}^\alpha(x)-g_{il}^\alpha(y))\right|,\left|S_{ir}^\alpha(t-s)\left(\widetilde{\beta}_{ir}^\alpha\right)^{-1}(g_{ir}^\alpha(x)-g_{ir}^\alpha(y))\right|\right\}ds\\
&\quad+\int_0^t\max_{1\le i\le n}\Bigg\{\left|S_{il}^\alpha(t-s)\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}S_{il}^\alpha(T)(g_{il}^\alpha(x)-g_{il}^\alpha(y))\right|,\\
&\qquad\qquad\left|S_{ir}^\alpha(t-s)\left(\widetilde{\beta}_{ir}^\alpha\right)^{-1}S_{ir}^\alpha(T)(g_{ir}^\alpha(x)-g_{ir}^\alpha(y))\right|\Bigg\}ds\\
&\quad+\int_0^t\max_{1\le i\le n}\Bigg\{\left|S_{il}^\alpha(t-s)\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}\left(\int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s,x(s))ds-\int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s,y(s))ds\right)\right|,\\
&\qquad\qquad\left|S_{ir}^\alpha(t-s)\left(\widetilde{\beta}_{il}^\alpha\right)^{-1}\left(\int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s,x(s))ds-\int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s,y(s))ds\right)\right|\Bigg\}ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \max_{1\leq i\leq n}\left\{\left|g_{il}^{\alpha}(x)-g_{il}^{\alpha}(y)\right|,\left|g_{ir}^{\alpha}(x)-g_{ir}^{\alpha}(y)\right|\right\}\\
&\quad + c\int_{0}^{t}\max_{1\leq i\leq n}\left\{\left|f_{il}^{\alpha}(s,x(s))-f_{il}^{\alpha}(s,y(s))\right|,\left|f_{ir}^{\alpha}(s,x(s))-f_{ir}^{\alpha}(s,y(s))\right|\right\}ds\\
&\quad + c\int_{0}^{t}\max_{1\leq i\leq n}\left\{\left|g_{il}^{\alpha}(x)-g_{il}^{\alpha}(y)\right|,\left|g_{ir}^{\alpha}(x)-g_{ir}^{\alpha}(y)\right|\right\}ds\\
&\quad + c^{2}\int_{0}^{t}\max_{1\leq i\leq n}\left\{\left|g_{il}^{\alpha}(x)-g_{il}^{\alpha}(y)\right|,\left|g_{ir}^{\alpha}(x)-g_{ir}^{\alpha}(y)\right|\right\}ds\\
&\quad + c^{2}\int_{0}^{t}\int_{0}^{T}\max_{1\leq i\leq n}\left\{\left|f_{il}^{\alpha}(s,x(s))-f_{il}^{\alpha}(s,y(s))\right|,\left|f_{ir}^{\alpha}(s,x(s))-f_{ir}^{\alpha}(s,y(s))\right|\right\}ds\,ds\\
&= c d_{L}\left([g(x)]^{\alpha},[g(y)]^{\alpha}\right)+c\int_{0}^{t}d_{L}\left([f(s,x(s))]^{\alpha},[f(s,y(s))]^{\alpha}\right)ds\\
&\quad + c\int_{0}^{t}d_{L}\left([g(x)]^{\alpha},[g(y)]^{\alpha}\right)ds+c^{2}\int_{0}^{t}d_{L}\left([g(x)]^{\alpha},[g(y)]^{\alpha}\right)ds\\
&\quad + c^{2}\int_{0}^{t}\int_{0}^{T}d_{L}\left([f(s,x(s))]^{\alpha},[f(s,y(s))]^{\alpha}\right)ds\,ds\\
&\leq ch\left\{d_{L}\left([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}\right)+(1+c)\int_{0}^{t}d_{L}\left([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}\right)ds\right\}\\
&\quad + ck\left\{\int_{0}^{t}d_{L}\left([x(s)]^{\alpha},[y(s)]^{\alpha}\right)ds+c\int_{0}^{t}\int_{0}^{T}d_{L}\left([x(s)]^{\alpha},[y(s)]^{\alpha}\right)ds\,ds\right\}.
\end{aligned}
\tag{4.9}
$$

Therefore

$$
\begin{aligned}
&D_{L}(\Phi x(t),\Phi y(t))\\
&\quad = \sup_{0<\alpha\leq 1}d_{L}\left([\Phi x(t)]^{\alpha},[\Phi y(t)]^{\alpha}\right)\\
&\quad \leq ch\left\{\sup_{0<\alpha\leq 1}d_{L}\left([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}\right)+(1+c)\int_{0}^{t}\sup_{0<\alpha\leq 1}d_{L}\left([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}\right)ds\right\}\\
&\qquad + ck\left\{\int_{0}^{t}\sup_{0<\alpha\leq 1}d_{L}\left([x(s)]^{\alpha},[y(s)]^{\alpha}\right)ds+c\int_{0}^{t}\int_{0}^{T}\sup_{0<\alpha\leq 1}d_{L}\left([x(s)]^{\alpha},[y(s)]^{\alpha}\right)ds\,ds\right\}\\
&\quad = ch\left\{D_{L}(x(\cdot),y(\cdot))+(1+c)\int_{0}^{t}D_{L}(x(\cdot),y(\cdot))ds\right\}\\
&\qquad + ck\left\{\int_{0}^{t}D_{L}(x(s),y(s))ds+c\int_{0}^{t}\int_{0}^{T}D_{L}(x(s),y(s))ds\,ds\right\}.
\end{aligned}
\tag{4.10}
$$

Hence

$$
\begin{aligned}
H_1(\Phi x, \Phi y) &= \sup_{0\le t\le T} D_L(\Phi x(t), \Phi y(t)) \\
&\le ch\left\{ \sup_{0\le t\le T} D_L(x(\cdot), y(\cdot)) + (1+c)\sup_{0\le t\le T}\int_0^t D_L(x(\cdot), y(\cdot))ds \right\} \\
&\quad + ck\left\{ \sup_{0\le t\le T}\int_0^t D_L(x(s), y(s))ds + c\sup_{0\le t\le T}\int_0^t\int_0^T D_L(x(s), y(s))ds\, ds \right\} \\
&\le ch\{H_1(x,y) + (1+c)T H_1(x,y)\} + ck\left\{T H_1(x,y) + cT^2 H_1(x,y)\right\} \\
&= c\{h(1+T+cT) + kT(1+cT)\}H_1(x,y).
\end{aligned}
\tag{4.11}
$$

By hypothesis (H2), Φ is a contraction mapping. Using the Banach fixed point theorem, (4.8) has a unique fixed point $x \in C([0,T] : (E_N^i)^n)$. □

5. Example

Consider the two semilinear one-dimensional heat equations on a connected domain $(0,1)$ for material with memory on E_N^i, $i = 1,2$, boundary condition $x_i(t,0) = x_i(t,1) = 0$, $i = 1,2$ and with initial conditions $x_i(0,z_i) + \sum_{k=1}^{p}(c_k)_i x_i(t_k, z_i) = x_{0_i}(z_i)$, where $x_{0_i}(z_i) \in E_N^i$, $\sum_{k=1}^{p}(c_k)_i x_i(t_k, z_i) = g_i(x_i)$, $i = 1,2$. Let $x_i(t,z_i)$, $i = 1,2$, be the internal energy and let $f_i(t, x_i(t,z_i)) = \tilde{2}t x_i(t,z_i)^2$, $i = 1,2$, be the external heat.

Let

$$
\begin{gathered}
A = (A_1, A_2) = \left(\tilde{2}\frac{\partial^2}{\partial z_1^2}, \tilde{2}\frac{\partial^2}{\partial z_2^2}\right), \\
f(t, x(t)) = (f_1(t, x_1(t)), f_2(t, x_2(t))) = \left(\tilde{2}t x_1(t,z_1)^2, \tilde{2}t x_2(t,z_2)^2\right), \\
g(x) = (g_1(x_1), g_2(x_2)) = \left(\sum_{k=1}^{p}(c_k)_1 x_1(t_k, z_1), \sum_{k=1}^{p}(c_k)_2 x_2(t_k, z_2)\right), \\
x(0) + g(x) = (x_1(0) + g_1(x), x_2(0) + g_2(x)), \qquad x_0 = (x_{0_1}, x_{0_2}) = \left(\tilde{0}, \tilde{0}\right), \\
G(t-s) = \left(e^{-(t-s)}, e^{-(t-s)}\right),
\end{gathered}
\tag{5.1}
$$

then the balance equations become

$$
\begin{gathered}
\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] + f(t, x(t)) \text{ on } \left(E_N^i\right)^2, \\
x(0) + g(x) = x_0 \in \left(E_N^i\right)^2.
\end{gathered}
\tag{5.2}
$$

The α-level sets of fuzzy numbers are the following: $[\tilde{0}]^\alpha = [\alpha - 1, 1 - \alpha]$, $[\tilde{2}]^\alpha = [\alpha + 1, 3 - \alpha]$ for all $\alpha \in [0,1]$. Then α-level set of $f(t, x(\mathrm{t}))$ is

$$
\begin{aligned}
&[f(t,x(t))]^\alpha \\
&\quad= \left[\tilde{2}tx_1(t)^2\right]^\alpha \times \left[\tilde{2}tx_2(t)^2\right]^\alpha \\
&\quad= \left[\tilde{2}\right]^\alpha \cdot t\left[x_1(t)^2\right]^\alpha \times \left[\tilde{2}\right]^\alpha \cdot t\left[x_2(t)^2\right]^\alpha \\
&\quad= [\alpha+1, 3-\alpha]\cdot t\left[\left(x_{1l}^\alpha(t)\right)^2, \left(x_{1r}^\alpha(t)\right)^2\right] \times [\alpha+1, 3-\alpha]\cdot t\left[\left(x_{2l}^\alpha(t)\right)^2, \left(x_{2r}^\alpha(t)\right)^2\right] \\
&\quad= \left[(\alpha+1)t\left(x_{1l}^\alpha(t)\right)^2, (3-\alpha)t\left(x_{1r}^\alpha(t)\right)^2\right] \times \left[(\alpha+1)t\left(x_{2l}^\alpha(t)\right)^2, (3-\alpha)t\left(x_{2r}^\alpha(t)\right)^2\right].
\end{aligned}
\tag{5.3}
$$

Further, we have

$$
\begin{aligned}
&d_L\left([f(t,x(t))]^\alpha, f(t,y(t))]^\alpha\right) \\
&\quad= d_L\left(\left[(\alpha+1)t\left(x_{il}^\alpha(t)\right)^2, (3-\alpha)t\left(x_{ir}^\alpha(t)\right)^2\right], \left[(\alpha+1)t\left(y_{il}^\alpha(t)\right)^2, (3-\alpha)t\left(y_{ir}^\alpha(t)\right)^2\right]\right) \\
&\quad= t\max_{1\le i\le 2}\left\{(\alpha+1)\left|\left(x_{il}^\alpha(t)\right)^2 - \left(y_{il}^\alpha(t)\right)^2\right|, (3-\alpha)\left|\left(x_{ir}^\alpha(t)\right)^2 - \left(y_{ir}^\alpha(t)\right)^2\right|\right\} \\
&\quad\le T(3-\alpha)\max_{1\le i\le 2}\left\{\left|x_{il}^\alpha(t) - y_{il}^\alpha(t)\right|\left|x_{il}^\alpha(t) + y_{il}^\alpha(t)\right|, \left|x_{ir}^\alpha(t) - y_{ir}^\alpha(t)\right|\left|x_{ir}^\alpha(t) + y_{ir}^\alpha(t)\right|\right\} \\
&\quad\le 3T\left|x_{ir}^\alpha(t) + y_{ir}^\alpha(t)\right| \times \max_{1\le i\le 2}\left\{\left|x_{il}^\alpha(t) - y_{il}^\alpha(t)\right|, \left|x_{ir}^\alpha(t) - y_{ir}^\alpha(t)\right|\right\} \\
&\quad= kd_L\left([x(t)]^\alpha, [y(t)]^\alpha\right), \\
&d_L\left([g(x(\cdot))]^\alpha, [g(y(\cdot))]^\alpha\right) \\
&\quad= d_L\left(\left[\sum_{k=1}^{p} c_k(x(t_k))\right]^\alpha, \left[\sum_{k=1}^{p} c_k(y(t_k))\right]^\alpha\right) \\
&\quad= \max_{1\le i\le 2}\left\{\left|\sum_{k=1}^{p}(c_k)_i\left(x_{il}^\alpha(t_k)\right) - \sum_{k=1}^{p}(c_k)_i\left(y_{il}^\alpha(t_k)\right)\right|, \left|\sum_{k=1}^{p}(c_k)_i\left(x_{ir}^\alpha(t_k)\right) - \sum_{k=1}^{p}(c_k)_i\left(y_{ir}^\alpha(t_k)\right)\right|\right\} \\
&\quad\le \left|\sum_{k=1}^{p} c_k\right| \max_{1\le i\le 2}\left\{\left|x_{il}^\alpha(t_k) - y_{il}^\alpha(t_k)\right|, \left|x_{ir}^\alpha(t_k) - y_{ir}^\alpha(t_k)\right|\right\} \\
&\quad= \left|\sum_{k=1}^{p} c_k\right| d_L\left([x(t_k)]^\alpha, [y(t_k)]^\alpha\right) \\
&\quad\le \left|\sum_{k=1}^{p} c_k\right| \max_k d_L\left([x(t_k)]^\alpha, [y(t_k)]^\alpha\right) \\
&\quad= hd_L\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right),
\end{aligned}
\tag{5.4}
$$

where k and h satisfy the inequality (3.4) and (3.5), respectively. Choose T such that $T < (1 - ch)/ck$. Then all conditions stated in Theorem 3.2 are satisfied, so problem (5.2) has a unique fuzzy solution.

Let target set be $x^1 = (x_1^1, x_2^1) = (\widetilde{2}, \widetilde{3})$. The α-level set of fuzzy numbers is $\widetilde{3}[\widetilde{3}]^\alpha = [\alpha + 2, 4 - \alpha]$.

From the definition of fuzzy solution,

$$
\begin{aligned}
x_{il}^\alpha(t) &= S_{il}^\alpha(t)\left((x_0)_{il}^\alpha - \sum_{k=1}^{p}(c_k)_i\left(x_{il}^\alpha(t_k)\right)\right) \\
&\quad + \int_0^t S_{il}^\alpha(t-s)(\alpha+1)s\left(x_{il}^\alpha(s)\right)^2 ds + \int_0^t S_{il}^\alpha(t-s)u_{il}^\alpha(s)ds, \\
x_{ir}^\alpha(t) &= S_{ir}^\alpha(t)\left((x_0)_{ir}^\alpha - \sum_{k=1}^{p}(c_k)_i\left(x_{ir}^\alpha(t_k)\right)\right) \\
&\quad + \int_0^t S_{ir}^\alpha(t-s)(3-\alpha)s\left(x_{ir}^\alpha(s)\right)^2 ds + \int_0^t S_{ir}^\alpha(t-s)u_{ir}^\alpha(s)ds,
\end{aligned}
\tag{5.5}
$$

where $i = 1, 2$.

Thus the α-level of $u(s)$ is

$$
\begin{aligned}
u_{1l}^\alpha(s) &= \left(\widetilde{\beta}_{1l}^\alpha\right)^{-1}\Bigg((\alpha+1) - \sum_{k=1}^{p}(c_k)_1\left(x_{il}^\alpha(t_k)\right) \\
&\quad - \left[S_{1l}^\alpha(T)\left((x_0)_{1l}^\alpha - \sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right)\right) + \int_0^T (\alpha+1)S_{1l}^\alpha(T-s)s\left(x_{1l}^\alpha(s)\right)^2 ds\right]\Bigg), \\
u_{1r}^\alpha(s) &= \left(\widetilde{\beta}_{1r}^\alpha\right)^{-1}\Bigg((3-\alpha) - \sum_{k=1}^{p}(c_k)_1\left(x_{ir}^\alpha(t_k)\right) \\
&\quad - \left[S_{1r}^\alpha(T)\left((x_0)_{1r}^\alpha - \sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\right) + \int_0^T (3-\alpha)S_{1r}^\alpha(T-s)s\left(x_{1r}^\alpha(s)\right)^2 ds\right]\Bigg), \\
u_{2l}^\alpha(s) &= \left(\widetilde{\beta}_{2l}^\alpha\right)^{-1}\Bigg((\alpha+2) - \sum_{k=1}^{p}(c_k)_2\left(x_{il}^\alpha(t_k)\right) \\
&\quad - \left[S_{2l}^\alpha(T)\left((x_0)_{2l}^\alpha - \sum_{k=1}^{p}(c_k)_2\left(x_{2l}^\alpha(t_k)\right)\right) + \int_0^T (\alpha+1)S_{2l}^\alpha(T-s)s\left(x_{2l}^\alpha(s)\right)^2 ds\right]\Bigg), \\
u_{2r}^\alpha(s) &= \left(\widetilde{\beta}_{2r}^\alpha\right)^{-1}\Bigg((4-\alpha) - \sum_{k=1}^{p}(c_k)_2\left(x_{ir}^\alpha(t_k)\right) \\
&\quad - \left[S_{2r}^\alpha(T)\left((x_0)_{2r}^\alpha - \sum_{k=1}^{p}(c_k)\left(x_{2r}^\alpha(t_k)\right)\right) + \int_0^T (3-\alpha)S_{2r}^\alpha(T-s)s\left(x_{2r}^\alpha(s)\right)^2 ds\right]\Bigg).
\end{aligned}
\tag{5.6}
$$

Then α-level of $x(T)=(x_1(T),x_2(T))$ is

$$\begin{aligned}
&[x_1(T)]^\alpha\\
&\quad=\left[x_{1l}^\alpha(T),x_{1r}^\alpha(T)\right]\\
&\quad=\Bigg[S_{1l}^\alpha(T)\left((x_0)_{1l}^\alpha-\sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right)\right)+\int_0^T(\alpha+1)S_{1l}^\alpha(T-s)s\left(x_{1l}^\alpha(s)\right)^2ds\\
&\qquad+\widetilde{\beta}_{1l}^\alpha\left(\widetilde{\beta}_{1l}^\alpha\right)^{-1}\Bigg((\alpha+1)-\sum_{k=1}^{p}(c_k)_1\left(x_{il}^\alpha(t_k)\right)\\
&\qquad\qquad-\Bigg\{S_{1l}^\alpha(T)\left((x_0)_{1l}^\alpha-\sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right)\right)\\
&\qquad\qquad+\int_0^T(\alpha+1)S_{1l}^\alpha(T-s)s\left(x_{1l}^\alpha(s)\right)^2ds\Bigg\}\Bigg)ds,\\
&\qquad S_{1r}^\alpha(T)\left((x_0)_{1r}^\alpha-\sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\right)+\int_0^T(3-\alpha)S_{1r}^\alpha(T-s)s\left(x_{1r}^\alpha(s)\right)^2ds\\
&\qquad+\widetilde{\beta}_{1r}^\alpha\left(\widetilde{\beta}_{1r}^\alpha\right)^{-1}\Bigg((3-\alpha)-\sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\\
&\qquad\qquad-\Bigg\{S_{1r}^\alpha(T)\left((x_0)_{1r}^\alpha-\sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\right)\\
&\qquad\qquad+\int_0^T(3-\alpha)S_{1r}^\alpha(T-s)s\left(x_{1r}^\alpha(s)\right)^2ds\Bigg\}\Bigg)ds\Bigg]\\
&\quad=\left[(\alpha+1)-\sum_{k=1}^{p}(c_k)_1\left(x_{1l}^\alpha(t_k)\right),(3-\alpha)-\sum_{k=1}^{p}(c_k)_1\left(x_{1r}^\alpha(t_k)\right)\right]=\left[\widetilde{2}-\sum_{k=1}^{p}(c_k)_1(x_1(t_k))\right]^\alpha.
\end{aligned}\tag{5.7}$$

Similarly

$$[x_2(T)]^\alpha=\left[x_{2l}^\alpha(T),x_{2r}^\alpha(T)\right]=\left[\widetilde{3}-\sum_{k=1}^{p}(c_k)_2(x_2(t_k))\right]^\alpha.\tag{5.8}$$

Hence

$$\begin{aligned}
x(T)&=(x_1(T),x_2(T))\\
&=\left(\widetilde{2}-\sum_{k=1}^{p}(c_k)_1(x_1(t_k)),\widetilde{3}-\sum_{k=1}^{p}(c_k)_2(x_2(t_k))\right)=x^1-g(x).
\end{aligned}\tag{5.9}$$

Then all the conditions stated in Theorem 4.2 are satisfied, so system (5.2) is nonlocal controllable on $[0,T]$.

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