Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 838347, 16 pages
doi:10.1155/2009/838347

Research Article

Stability of Quartic Functional Equations in the Spaces of Generalized Functions

Young-Su Lee[1] and Soon-Yeong Chung[2]

[1] *Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, Daejeon 305-701, South Korea*
[2] *Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 121-741, South Korea*

Correspondence should be addressed to Young-Su Lee, masuri@kaist.ac.kr

Received 6 August 2008; Revised 15 December 2008; Accepted 14 January 2009

Recommended by Rigoberto Medina

We consider the general solution of quartic functional equations and prove the Hyers-Ulam-Rassias stability. Moreover, using the pullbacks and the heat kernels we reformulate and prove the stability results of quartic functional equations in the spaces of tempered distributions and Fourier hyperfunctions.

1. Introduction

One of the interesting questions concerning the stability problems of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation? Such an idea was suggested in 1940 by Ulam [1]. The case of approximately additive mappings was solved by Hyers [2]. In 1978, Rassias [3] generalized Hyers' result to the unbounded Cauchy difference. During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [4–9]). The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds and this terminology is also applied to the cases of other functional equations. For instance, Rassias [10] investigated stability properties of the following functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y). \tag{1.1}$$

It is easy to see that $f(x)=x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4+(x-2y)^4+6x^4=4(x+y)^4+4(x-y)^4+24y^4. \tag{1.2}$$

For this reason, (1.1) is called a quartic functional equation. Also Chung and Sahoo [11] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1.1) if and only if $f(x) = A(x, x, x, x)$, where the function $A : \mathbb{R}^4 \to \mathbb{R}$ is symmetric and additive in each variable. Since the solution of (1.1) is even, we can rewrite (1.1) as

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \tag{1.3}$$

Lee et al. [12] obtained the general solution of (1.3) and proved the Hyers-Ulam-Rassias stability of this equation. Also Park [13] investigated the stability problem of (1.3) in the orthogonality normed space.

In this paper we consider the following quartic functional equation, which is a generalization of (1.3),

$$f(ax + y) + f(ax - y) = a^2 f(x + y) + a^2 f(x - y) + 2a^2(a^2 - 1)f(x) - 2(a^2 - 1)f(y), \tag{1.4}$$

for fixed integer a with $a \neq 0, \pm 1$. In the cases of $a = 0, \pm 1$ in (1.4), homogeneity property of quartic functional equations does not hold. We dispense with this cases henceforth, and assume that $a \neq 0, \pm 1$. In Section 2, we show that for each fixed integer a with $a \neq 0, \pm 1$, (1.4) is equivalent to (1.3). Moreover, using the idea of Găvruţa [14], we prove the Hyers-Ulam-Rassias stability of (1.4) in Section 3. Finally, making use of the pullbacks and the heat kernels, we reformulate and prove the Hyers-Ulam-Rassias stability of (1.4) in the spaces of some generalized functions such as $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions and $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions in Section 4.

2. General Solution of (1.4)

Throughout this section, we denote E_1 and E_2 by real vector spaces. It is well known [15] that a function $f : E_1 \to E_2$ satisfies the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.1}$$

if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all $x \in E_1$. The biadditive function B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \tag{2.2}$$

Stability problems of quadratic functional equations can be found in [16–19]. Similarly, a function $f : E_1 \to E_2$ satisfies the quartic functional equation (1.3) if and only if there exists a symmetric biquadratic function $F : E_1 \times E_1 \to E_2$ such that $f(x) = F(x, x)$ for all $x \in E_1$ (see [12]). We now present the general solution of (1.4) in the class of functions between real vector spaces.

Theorem 2.1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation* (1.3) *if and only if for each fixed integer a with $a \neq 0, \pm 1$, a mapping $f : E_1 \to E_2$ satisfies the functional equation* (1.4).

Proof. Suppose that f satisfies (1.3). Putting $x = y = 0$ in (1.3) we have $f(0) = 0$. Also letting $x = 0$ in (1.3) we get $f(-y) = f(y)$. Using an induction argument we may assume that (1.4) is true for all a with $1 < a \leq k$. Replacing y by $x + y$ and a by k in (1.4) we have

$$\begin{aligned} &f\big((k+1)x+y\big)+f\big((k-1)x-y\big) \\ &\quad = k^2 f(2x+y)+k^2 f(y)+2k^2\big(k^2-1\big)f(x)-2\big(k^2-1\big)f(x+y). \end{aligned} \tag{2.3}$$

Substituting y by $-y$ in (2.3) and using the evenness of f we get

$$\begin{aligned} &f\big((k+1)x-y\big)+f\big((k-1)x+y\big) \\ &\quad = k^2 f(2x-y)+k^2 f(y)+2k^2\big(k^2-1\big)f(x)-2\big(k^2-1\big)f(x-y). \end{aligned} \tag{2.4}$$

Adding (2.3) to (2.4) yields

$$\begin{aligned} &f\big((k+1)x+y\big)+f\big((k+1)x-y\big) \\ &\quad = -\big[f\big((k-1)x+y\big)+f\big((k-1)x-y\big)\big]+k^2\big[f(2x+y)+f(2x-y)\big] \\ &\qquad +4k^2\big(k^2-1\big)f(x)+2k^2 f(y)-2\big(k^2-1\big)\big[f(x+y)+f(x-y)\big]. \end{aligned} \tag{2.5}$$

According to the inductive assumption for $a = k - 1$, (2.5) can be rewritten as

$$\begin{aligned} &f\big((k+1)x+y\big)+f\big((k+1)x-y\big) \\ &\quad = -\big[(k-1)^2 f(x+y)+(k-1)^2 f(x-y) \\ &\qquad\qquad +2(k-1)^2\big((k-1)^2-1\big)f(x)-2\big((k-1)^2-1\big)f(y)\big] \\ &\qquad +k^2\big[4f(x+y)+4f(x-y)+24f(x)-6f(y)\big] \\ &\qquad +4k^2(k^2-1)f(x)+2k^2 f(y)-2(k^2-1)\big[f(x+y)+f(x-y)\big] \\ &\quad = (k+1)^2 f(x+y)+(k+1)^2 f(x-y) \\ &\qquad +2(k+1)^2\big((k+1)^2-1\big)f(x)-2\big((k+1)^2-1\big)f(y) \end{aligned} \tag{2.6}$$

which proves the validity of (1.4) for $a = k + 1$. For a negative integer $a < -1$, replacing a by $-a > 1$ one can easily prove the validity of (1.4). Therefore (1.3) implies (1.4) for any fixed integer a with $a \neq 0, \pm 1$.

We now prove the converse. For each fixed integer a with $a \neq 0, \pm 1$, we assume that $f : E_1 \to E_2$ satisfies (1.4). Putting $x = y = 0$ in (1.4) we have $f(0) = 0$. Also letting $x = 0$ in (1.4) we get $f(-y) = f(y)$ for all $y \in X$. Setting $y = 0$ in (1.4) we obtain the homogeneity property $f(ax) = a^4 f(x)$ for all $x \in X$. Replacing y by $x + ay$ in (1.4) we have

$$\begin{aligned} &f\big(a(x+y)+x\big)+f\big(a(x-y)-x\big) \\ &\quad = a^2 f(2x+ay)+a^6 f(y)+2a^2(a^2-1)f(x)-2(a^2-1)f(x+ay). \end{aligned} \tag{2.7}$$

Interchanging y into $-y$ in (2.7) yields

$$\begin{aligned} &f\left(a(x-y)+x\right)+f\left(a(x+y)-x\right) \\ &\quad = a^2 f(2x-ay)+a^6 f(y)+2a^2\left(a^2-1\right)f(x)-2\left(a^2-1\right)f(x-ay). \end{aligned} \tag{2.8}$$

Replacing x and y by $x+y$ and x in (1.4) we get

$$\begin{aligned} &f\left(a(x+y)+x\right)+f\left(a(x+y)-x\right) \\ &\quad = a^2 f(2x+y)+a^2 f(y)+2a^2\left(a^2-1\right)f(x+y)-2\left(a^2-1\right)f(x). \end{aligned} \tag{2.9}$$

Substituting y by $-y$ in (2.9) gives

$$\begin{aligned} &f\left(a(x-y)+x\right)+f\left(a(x-y)-x\right) \\ &\quad = a^2 f(2x-y)+a^2 f(y)+2a^2\left(a^2-1\right)f(x-y)-2\left(a^2-1\right)f(x). \end{aligned} \tag{2.10}$$

Plugging (2.7) into (2.8), and using (2.9) and (2.10) we have

$$\begin{aligned} &a^2\left[f(2x+ay)+f(2x-ay)\right]+2a^6 f(y)+4a^2\left(a^2-1\right)f(x) \\ &\quad -2\left(a^2-1\right)\left[f(x+ay)+f(x-ay)\right] \\ &\quad = a^2\left[f(2x+y)+f(2x-y)\right]+2a^2 f(y)-4\left(a^2-1\right)f(x) \\ &\quad +2a^2\left(a^2-1\right)\left[f(x+y)+f(x-y)\right]. \end{aligned} \tag{2.11}$$

Replacing x and y by $2x$ and ay in (1.4), respectively, we get

$$\begin{aligned} &a^4 f(2x+y)+a^4 f(2x-y) \\ &\quad = a^2 f(2x+ay)+a^2 f(2x-ay)+2a^2\left(a^2-1\right)f(2x)-2a^4\left(a^2-1\right)f(y). \end{aligned} \tag{2.12}$$

Setting y by ay in (1.4) and dividing by a^2 we obtain

$$\begin{aligned} &a^2 f(x+y)+a^2 f(x-y) \\ &\quad = f(x+ay)+f(x-ay)+2\left(a^2-1\right)f(x)-2a^2\left(a^2-1\right)f(y). \end{aligned} \tag{2.13}$$

It follows from (2.12) and (2.13) that (2.11) can be rewritten in the form

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+2f(2x)-8f(x)-6f(y). \tag{2.14}$$

Using an induction argument in (2.14), it is easy to see that f satisfies the following functional equation

$$f(bx+y)+f(bx-y)=b^2f(x+y)+b^2f(x-y)+\frac{1}{6}b^2(b^2-1)f(2x) \\ -\frac{2}{3}b^2(b^2-1)f(x)-2(b^2-1)f(y) \tag{2.15}$$

for each fixed integer $b\neq 0,\pm 1$. Replacing b by a in (2.15), and comparing (1.4) with (2.15) we have $f(2x)=16f(x)$. Thus (2.14) implies (1.3). This completes the proof. □

3. Stability of (1.4)

Now we are going to prove the Hyers-Ulam-Rassias stability for quartic functional equations. Let X be a real vector space and let Y be a Banach space.

Theorem 3.1. *Let $\phi : X^2 \to \mathbb{R}^+ := [0,\infty)$ be a mapping such that*

$$\sum_{j=0}^{\infty}\frac{\phi(a^jx,0)}{a^{4j}}, \qquad \left(\sum_{j=1}^{\infty}a^{4j}\phi\left(\frac{x}{a^j},0\right)\right) \tag{3.1}$$

converges and

$$\lim_{k\to\infty}\frac{\phi(a^kx,a^ky)}{a^{4k}}=0, \qquad \left(\lim_{k\to\infty}a^{4k}\phi\left(\frac{x}{a^k},\frac{y}{a^k}\right)=0\right) \tag{3.2}$$

for all $x,y\in X$. Suppose that a mapping $f : X\to Y$ satisfies the inequality

$$\|f(ax+y)+f(ax-y)-a^2f(x+y)-a^2f(x-y) \\ -2a^2(a^2-1)f(x)+2(a^2-1)f(y)\|\leq\phi(x,y) \tag{3.3}$$

for all $x,y\in X$. Then there exists a unique quartic mapping $T : X\to Y$ which satisfies quartic functional equation (1.4) *and the inequality*

$$\left\|f(x)-\frac{f(0)}{a^2+1}-T(x)\right\|\leq\frac{1}{2a^4}\sum_{j=0}^{\infty}\frac{\phi(a^jx,0)}{a^{4j}}, \\ \left(\left\|f(x)-\frac{f(0)}{a^2+1}-T(x)\right\|\leq\frac{1}{2a^4}\sum_{j=1}^{\infty}a^{4j}\phi\left(\frac{x}{a^j},0\right)\right) \tag{3.4}$$

for all $x\in X$. The mapping T is given by

$$T(x)=\lim_{k\to\infty}\frac{f(a^kx)}{a^{4k}}, \qquad \left(T(x)=\lim_{k\to\infty}a^{4k}f\left(\frac{x}{a^k}\right)\right) \tag{3.5}$$

for all $x \in X$. Also, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^4 T(x)$ for all $r \in \mathbb{R}$.

Proof. Putting $y = 0$ in (3.3) and then dividing the result by $2a^4$ we have

$$\left\| \frac{f(ax)}{a^4} - f(x) + \frac{(a^2-1)}{a^4} f(0) \right\| \le \frac{1}{2a^4} \phi(x,0) \tag{3.6}$$

which is rewritten as

$$\left\| g(x) - \frac{g(ax)}{a^4} \right\| \le \frac{1}{2a^4} \phi(x,0) \tag{3.7}$$

for all $x \in X$, where $g(x) := f(x) - (1/(a^2+1)) f(0)$. Making use of induction arguments and triangle inequalities we have

$$\left\| g(x) - \frac{g(a^k x)}{a^{4k}} \right\| \le \frac{1}{2a^4} \sum_{j=0}^{k-1} \frac{\phi(a^j x, 0)}{a^{4j}} \tag{3.8}$$

for all $k \in \mathbb{N}$, $x \in X$. Now we prove the sequence $\{g(a^k x)/a^{4k}\}$ is a Cauchy sequence. Replacing x by $a^l x$ in (3.8) and then dividing by a^{4l} we see that for $k \ge l > 0$,

$$\left\| \frac{g(a^l x)}{a^{4l}} - \frac{g(a^{k+l} x)}{a^{4(k+l)}} \right\| \le \frac{1}{2a^4} \sum_{j=0}^{k-1} \frac{\phi(a^{j+l} x, 0)}{a^{4(j+l)}}. \tag{3.9}$$

Since the right-hand side of (3.9) tends to 0 as $l \to \infty$, the sequence $\{g(a^k x)/a^{4k}\}$ is a Cauchy sequence. Therefore we may define

$$T(x) := \lim_{k \to \infty} \frac{g(a^k x)}{a^{4k}} \tag{3.10}$$

for all $x \in X$. Replacing x, y by $a^k x, a^k y$, respectively, in (3.3) and then dividing by a^{4k} we have

$$\begin{aligned} a^{-4k} \| & f(a^k(ax+y)) + f(a^k(ax-y)) - a^2 f(a^k(x+y)) - a^2 f(a^k(x-y)) \\ & - 2a^2(a^2-1) f(a^k x) + 2(a^2-1) f(a^k y) \| \le a^{-4k} \phi(a^k x, a^k y). \end{aligned} \tag{3.11}$$

Taking the limit as $k \to \infty$, we verify that T satisfies (1.4) for all $x, y \in X$. Now letting $k \to \infty$ in (3.8) we have

$$\| g(x) - T(x) \| \le \frac{1}{2a^4} \sum_{j=0}^{\infty} \frac{\phi(a^j x, 0)}{a^{4j}} \tag{3.12}$$

for all $x \in X$. To prove the uniqueness, let us assume that there exists another quartic mapping $S : X \to Y$ which satisfies (1.4) and the inequality (3.12). Obviously, we have $T(a^k x) = a^{4k}T(x)$ and $S(a^k x) = a^{4k}S(x)$ for all $k \in \mathbb{N}, x \in X$. Thus, we have

$$\begin{aligned} \|T(x) - S(x)\| &= a^{-4k}\|T(a^k x) - S(a^k x)\| \\ &\le a^{-4k}(\|T(a^k x) - g(a^k x)\| + \|g(a^k x) - S(a^k x)\|) \\ &\le a^{-4}\sum_{j=k}^{\infty}\frac{\phi(a^j x, 0)}{a^{4j}} \end{aligned} \tag{3.13}$$

for all $x \in X$. Letting $k \to \infty$, we must have $S(x) = T(x)$ for all x. This completes the proof. □

Corollary 3.2. *Let a be fixed integer with $a \neq 0, \pm 1$ and let ϵ, p, q be real numbers such that $\epsilon \ge 0$ and either $0 \le p, q < 4$ or $p, q > 4$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\begin{aligned} &\|f(ax + y) + f(ax - y) - a^2 f(x + y) - a^2 f(x - y) \\ &\quad - 2a^2(a^2 - 1)f(x) + 2(a^2 - 1)f(y)\| \le \epsilon(\|x\|^p + \|y\|^q) \end{aligned} \tag{3.14}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $T : X \to Y$ which satisfies (1.4) and the inequality

$$\left\| f(x) - \frac{f(0)}{a^2 + 1} - T(x) \right\| \le \frac{\epsilon}{2|a^4 - |a|^p|}\|x\|^p \tag{3.15}$$

for all $x \in X$ and for all $x \in X \setminus \{0\}$ if $p < 0$. The mapping T is given by

$$\begin{aligned} &T(x) = \lim_{k\to\infty}\frac{f(a^k x)}{a^{4k}} \quad \text{if } p, q < 4, \\ &\left(T(x) = \lim_{k\to\infty} a^{4k} f\left(\frac{x}{a^k}\right) \text{ if } p, q > 4\right) \end{aligned} \tag{3.16}$$

for all $x \in X$.

Corollary 3.3. *Let a be fixed integer with $a \neq 0, \pm 1$ and $\epsilon \ge 0$ be a real number. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\begin{aligned} &\|f(ax + y) + f(ax - y) - a^2 f(x + y) - a^2 f(x - y) \\ &\quad - 2a^2(a^2 - 1)f(x) + 2(a^2 - 1)f(y)\| \le \epsilon \end{aligned} \tag{3.17}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $T : X \to Y$ defined by

$$T(x) = \lim_{k\to\infty}\frac{f(a^k x)}{a^{4k}} \tag{3.18}$$

which satisfies (1.4) *and the inequality*

$$\left\| f(x) - \frac{f(0)}{a^2+1} - T(x) \right\| \le \frac{\epsilon}{2(a^4-1)} \tag{3.19}$$

for all $x \in X$.

4. Stability of (1.4) in Generalized Functions

In this section, we reformulate and prove the stability theorem of the quartic functional equation (1.4) in the spaces of some generalized functions such as $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions and $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions. We first introduce briefly spaces of some generalized functions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \partial/\partial x_j$.

Definition 4.1 (see [20, 21]). We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \tag{4.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. A linear form u on $\mathcal{S}(\mathbb{R}^n)$ is said to be tempered distribution if there is a constant $C \ge 0$ and a nonnegative integer N such that

$$\left| \langle u, \varphi \rangle \right| \le C \sum_{|\alpha|, |\beta| \le N} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi \right| \tag{4.2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (4.1) a new space of test functions has emerged as follows.

Definition 4.2 (see [22]). We denote by $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{\left| x^\alpha \partial^\beta \varphi(x) \right|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{4.3}$$

for some positive constants A, B depending only on φ. We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some $A, B > 0$, and denote by $\mathcal{F}'(\mathbb{R}^n)$ the strong dual of $\mathcal{F}(\mathbb{R}^n)$ and call its elements Fourier hyperfunctions.

It can be verified that the seminorms (4.3) are equivalent to

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\ \alpha\in\mathbb{N}_0^n} \frac{\left|\partial^\alpha \varphi(x)\right| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \tag{4.4}$$

for some constants $h, k > 0$. It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{F}'(\mathbb{R}^n). \tag{4.5}$$

From the above inclusions it suffices to say that we consider (1.4) in the space $\mathcal{F}'(\mathbb{R}^n)$. Note that (3.14) itself makes no sense in the spaces of generalized functions. Following the notions as in [23–25], we reformulate the inequality (3.14) as

$$\begin{aligned}&\big\| u \circ A_1 + u \circ A_2 - a^2 u \circ B_1 - a^2 u \circ B_2 \\ &\quad - 2a^2\left(a^2-1\right) u \circ P_1 + 2\left(a^2-1\right) u \circ P_2 \big\| \le \epsilon\left(|x|^p + |y|^q\right),\end{aligned} \tag{4.6}$$

where $A_1(x,y) = ax+y,\ A_2(x,y) = ax-y,\ B_1(x,y) = x+y,\ B_2(x,y) = x-y,\ P_1(x,y) = x,\ P_2(x,y) = y$. Here $\circ$ denotes the pullbacks of generalized functions. Also $|\cdot|$ denotes the Euclidean norm and the inequality $\|v\| \le \psi(x,y)$ in (4.6) means that $|\langle v, \varphi\rangle| \le \|\psi\varphi\|_{L^1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}^{2n}$. We refer to (see [20, Chapter VI]) for pullbacks and to [21, 23–26] for more details of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$.

If $p < 0,\ q < 0$, the right side of (4.6) does not define a distribution. Thus, the inequality (4.6) makes no sense in this case. Also, if $p = 4,\ q = 4$, it is not known whether Hyers-Ulam-Rassias stability of (1.4) holds even in the classical case. Thus, we consider only the case $0 \le p, q < 4$ or $p, q > 4$.

In order to prove the stability problems of quartic functional equations in the space of $\mathcal{F}'(\mathbb{R}^n)$ we employ the n-dimensional heat kernel, that is, the fundamental solution $E_t(x)$ of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}^n_x \times \mathbb{R}^+_t$ given by

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right), & t > 0, \\ 0, & t \le 0. \end{cases} \tag{4.7}$$

Since for each $t > 0$, $E_t(\cdot)$ belongs to $\mathcal{F}(\mathbb{R}^n)$, the convolution

$$\tilde{u}(x,t) = \left(u * E_t\right)(x) = \left\langle u_y, E_t(x-y)\right\rangle, \quad x \in \mathbb{R}^n,\ t > 0 \tag{4.8}$$

is well defined for each $u \in \mathcal{F}'(\mathbb{R}^n)$, which is called the Gauss transform of u. In connection with the Gauss transform it is well known that the semigroup property of the heat kernel

$$\left(E_t * E_s\right)(x) = E_{t+s}(x) \tag{4.9}$$

holds for convolution. Semigroup property will be useful to convert inequality (3.3) into the classical functional inequality defined on upper-half plane. Moreover, the following result called heat kernel method holds [27].

Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $\widetilde{u}(x,t)$ is a C^∞-solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widetilde{u}(x,t) = 0 \tag{4.10}$$

satisfying

(i) There exist positive constants C, M and N such that

$$\left|\widetilde{u}(x,t)\right| \leq Ct^{-M}(1+|x|)^N \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{4.11}$$

(ii) $\widetilde{u} \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \widetilde{u}(x,t)\varphi(x)dx. \tag{4.12}$$

Conversely, every C^∞-solution $U(x,t)$ of the heat equation satisfying the growth condition (4.11) can be uniquely expressed as $U(x,t) = \widetilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$. Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results (see [28]). In this case, the estimate (4.11) is replaced by the following.

For every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\left|\widetilde{u}(x,t)\right| \leq C_\varepsilon \exp\left(\varepsilon\left(|x| + \frac{1}{t}\right)\right) \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{4.13}$$

We note that the Gauss transform

$$\psi_p(x,t) := \int |\xi|^p E_t(x-\xi)d\xi \tag{4.14}$$

is well defined and $\psi_p(x,t) \to |x|^p$ locally uniformly as $t \to 0^+$. Also $\psi_p(x,t)$ satisfies semi-homogeneity property

$$\psi_p(rx, r^2 t) = |r|^p \psi_p(x,t) \tag{4.15}$$

for all $r \in \mathbb{R}$.

We are now in a position to state and prove the main result of this paper.

Theorem 4.3. *Let a be fixed integer with $a \neq 0, \pm 1$ and let ϵ, p, q be real numbers such that $\epsilon \geq 0$ and either $0 \leq p, q < 4$ or $p, q > 4$. Suppose that u in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfies the inequality* (4.6). *Then there exists a unique quartic mapping $T(x)$ which satisfies* (1.4) *and the inequality*

$$\left\| u - \frac{c}{a^2+1} - T(x) \right\| \leq \frac{\epsilon}{2\left|a^4 - |a|^p\right|}|x|^p, \tag{4.16}$$

where $c := \limsup_{t \to 0^+} \widetilde{u}(0,t)$.

Proof. Define $v := u \circ A_1 + u \circ A_2 - a^2 u \circ B_1 - a^2 u \circ B_2 - 2a^2(a^2-1)u \circ P_1 + 2(a^2-1)u \circ P_2$. Convolving the tensor product $E_t(\xi)E_s(\eta)$ of n-dimensional heat kernels in v we have

$$\begin{aligned}\left|\left[v * \left(E_t(\xi)E_s(\eta)\right)\right](x,y)\right| &= \left|\langle v, E_t(x-\xi)E_s(y-\eta)\rangle\right| \\ &\le \epsilon\left\|\left(|\xi|^p + |\eta|^q\right)E_t(x-\xi)E_s(y-\eta)\right\|_{L^1} \\ &= \epsilon\left(\psi_p(x,t) + \psi_q(y,s)\right).\end{aligned} \tag{4.17}$$

On the other hand, we figure out

$$\begin{aligned}\left[(u \circ A_1) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \langle u \circ A_1, E_t(x-\xi)E_s(y-\eta)\rangle \\ &= \left\langle u_\xi, |a|^{-n}\int E_t\left(x - \frac{\xi-\eta}{a}\right)E_s(y-\eta)d\eta\right\rangle \\ &= \left\langle u_\xi, |a|^{-n}\int E_t\left(\frac{ax+y-\xi-\eta}{a}\right)E_s(\eta)d\eta\right\rangle \\ &= \left\langle u_\xi, \int E_{a^2t}(ax+y-\xi-\eta)E_s(\eta)d\eta\right\rangle \\ &= \langle u_\xi, (E_{a^2t} * E_s)(ax+y-\xi)\rangle \\ &= \langle u_\xi, E_{a^2t+s}(ax+y-\xi)\rangle \\ &= \tilde{u}(ax+y, a^2t+s)\end{aligned} \tag{4.18}$$

and similarly we get

$$\begin{aligned}\left[(u \circ A_2) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \tilde{u}(ax-y, a^2t+s), \\ \left[(u \circ B_1) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \tilde{u}(x+y, t+s), \\ \left[(u \circ B_2) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \tilde{u}(x-y, t+s), \\ \left[(u \circ P_1) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \tilde{u}(x, t), \\ \left[(u \circ P_2) * \left(E_t(\xi)E_s(\eta)\right)\right](x,y) &= \tilde{u}(y, s),\end{aligned} \tag{4.19}$$

where $\tilde{u}$ is the Gauss transform of u. Thus, inequality (4.6) is converted into the classical functional inequality

$$\begin{aligned}&\left|\tilde{u}(ax+y, a^2t+s) + \tilde{u}(ax-y, a^2t+s) - a^2\tilde{u}(x+y, t+s)\right. \\ &\qquad \left.- a^2\tilde{u}(x-y, t+s) - 2a^2(a^2-1)\tilde{u}(x,t) + 2(a^2-1)\tilde{u}(y,s)\right| \\ &\quad \le \epsilon\left(\psi_p(x,t) + \psi_q(y,s)\right)\end{aligned} \tag{4.20}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$. In view of (4.20), it can be verified that

$$c := \limsup_{s \to 0^+} \tilde{u}(0, s) \tag{4.21}$$

exists.

We first prove the case $0 \le p, q < 4$. Choose a sequence $\{s_j\}$ of positive numbers which tends to 0 as $j \to \infty$ such that $\tilde{u}(0, s_j) \to c$ as $j \to \infty$. Letting $y = 0$, $s = s_j \to 0^+$ in (4.20) and dividing the result by $2a^4$ we get

$$\left| \frac{\tilde{u}(ax, a^2t)}{a^4} - \tilde{u}(x,t) + \frac{(a^2-1)}{a^4} c \right| \le \frac{\epsilon}{2a^4} \psi_p(x,t) \tag{4.22}$$

which is written in the form

$$\left| \tilde{v}(x,t) - \frac{\tilde{v}(ax, a^2t)}{a^4} \right| \le \frac{\epsilon}{2a^4} \psi_p(x,t) \tag{4.23}$$

for all $x \in \mathbb{R}^n,\ t > 0$, where $\tilde{v}(x,t) := \tilde{u}(x,t) - (c/(a^2+1))$. By virtue of the semi-homogeneous property of ψ_p, substituting x, t by ax, a^2t, respectively, in (4.23) and dividing the result by a^4 we obtain

$$\left| \frac{\tilde{v}(ax, a^2t)}{a^4} - \frac{\tilde{v}(a^2x, a^4t)}{a^8} \right| \le \frac{\epsilon}{2a^4} |a|^{p-4} \psi_p(x,t). \tag{4.24}$$

Using induction arguments and triangle inequalities we have

$$\left| \tilde{v}(x,t) - \frac{\tilde{v}(a^k x, a^{2k}t)}{a^{4k}} \right| \le \frac{\epsilon}{2a^4} \psi_p(x,t) \sum_{j=0}^{k-1} |a|^{(p-4)j} \tag{4.25}$$

for all $k \in \mathbb{N},\ x \in \mathbb{R}^n,\ t > 0$. Let us prove the sequence $\{a^{-4k}\tilde{v}(a^k x, a^{2k}t)\}$ is convergent for all $x \in \mathbb{R}^n,\ t > 0$. Replacing x, t by $a^l x, a^{2l}t$, respectively, in (4.25) and dividing the result by a^{4l} we see that

$$\left| \frac{\tilde{v}(a^l x, a^{2l}t)}{a^{4l}} - \frac{\tilde{v}(a^{k+l} x, a^{2(k+l)}t)}{a^{4(k+l)}} \right| \le \frac{\epsilon}{2a^4} \psi_p(x,t) \sum_{j=0}^{k-1} |a|^{(p-4)(j+l)}. \tag{4.26}$$

Letting $l \to \infty$, we have $\{a^{-4k}\tilde{v}(a^k x, a^{2k}t)\}$ is a Cauchy sequence. Therefore we may define

$$g(x,t) = \lim_{k \to \infty} a^{-4k} \tilde{v}(a^k x, a^{2k}t) \tag{4.27}$$

for all $x \in \mathbb{R}^n,\ t > 0$. On the other hand, replacing x, y, t, s by $a^k x, a^k y, a^{2k} t, a^{2k} s$ in (4.20), respectively, and then dividing the result by a^{4k} we get

$$\begin{aligned}
&a^{-4k}\big|\widetilde{u}\big(a^k(ax+y), a^{2k}(a^2t+s)\big) + \widetilde{u}\big(a^k(ax-y), a^{2k}(a^2t+s)\big) \\
&\quad - a^2\widetilde{u}\big(a^k(x+y), a^{2k}(t+s)\big) - a^2\widetilde{u}\big(a^k(x+y), a^{2k}(t+s)\big) \\
&\quad - 2a^2(a^2-1)\widetilde{u}\big(a^k x, a^{2k} t\big) + 2(a^2-1)\widetilde{u}\big(a^k y, a^{2k} s\big)\big| \\
&\quad \le a^{-4k}\epsilon\big(\psi_p\big(a^k x, a^{2k}t\big) + \psi_q\big(a^k y, a^{2k}s\big)\big) \\
&\quad = \epsilon\big(|a|^{(p-4)k}\psi_p(x,t) + |a|^{(q-4)k}\psi_q(y,s)\big).
\end{aligned} \tag{4.28}$$

Now letting $k \to \infty$ we see by definition of g that g satisfies

$$\begin{aligned}
&g\big(ax+y, a^2t+s\big) + g\big(ax-y, a^2t+s\big) \\
&\quad = a^2 g(x+y, t+s) + a^2 g(x-y, t+s) \\
&\qquad + 2a^2\big(a^2-1\big)g(x,t) - 2\big(a^2-1\big)g(y,s)
\end{aligned} \tag{4.29}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$. Letting $k \to \infty$ in (4.25) yields

$$\big|\widetilde{v}(x,t) - g(x,t)\big| \le \frac{\epsilon}{2\big(a^4 - |a|^p\big)}\psi_p(x,t). \tag{4.30}$$

To prove the uniqueness of $g(x,t)$, we assume that $h(x,t)$ is another function satisfying (4.29) and (4.30). Setting $y = 0$ and $s \to 0^+$ in (4.29) we have

$$g\big(a^k x, a^{2k} t\big) = a^{4k} g(x,t), \qquad h\big(a^k x, a^{2k} t\big) = a^{4k} h(x,t) \tag{4.31}$$

for all $k \in \mathbb{R},\ x \in \mathbb{R}^n,\ t > 0$. Then it follows from (4.30) and (4.31) that

$$\begin{aligned}
\big|g(x,t) - h(x,t)\big| &= a^{-4k}\big|g\big(a^k x, a^{2k}t\big) - h\big(a^k x, a^{2k}t\big)\big| \\
&\le a^{-4k}\big|g\big(a^k x, a^{2k}t\big) - \widetilde{v}\big(a^k x, a^{2k}t\big)\big| \\
&\quad + a^{-4k}\big|\widetilde{v}\big(a^k x, a^{2k}t\big) - h\big(a^k x, a^{2k}t\big)\big| \\
&\le \frac{\epsilon}{|a|^{(4-p)k}\big(a^4 - |a|^p\big)}\psi_p(x,t)
\end{aligned} \tag{4.32}$$

for all $k \in \mathbb{N},\ x \in \mathbb{R}^n,\ t > 0$. Letting $k \to \infty$, we have $g(x,t) = h(x,t)$ for all $x \in \mathbb{R}^n,\ t > 0$. This proves the uniqueness.

It follows from the inequality (4.30) that we get

$$\big|\langle \widetilde{v}(x,t) - g(x,t), \varphi\rangle\big| \le \frac{\epsilon}{2\big(a^4 - |a|^p\big)}\langle \psi_p(x,t), \varphi\rangle \tag{4.33}$$

for all test functions φ. Since $g(x,t)$ is given by the uniform limit of the sequence $a^{-4k}\tilde{v}(a^k x, a^{2k}t)$, $g(x,t)$ is also continuous on $\mathbb{R}^n \times (0,\infty)$. In view of (4.29), it follows from the continuity of g that for each $x \in \mathbb{R}^n$

$$T(x) := \lim_{t\to 0^+} g(x,t) \tag{4.34}$$

exists. Letting $t = s \to 0^+$ in (4.29) we have $T(x)$ satisfies quartic functional equation (1.4). Letting $t \to 0^+$ we have the inequality

$$\left\| u - \frac{c}{a^2+1} - T(x) \right\| \le \frac{\epsilon}{2\left(a^4 - |a|^p\right)} |x|^p. \tag{4.35}$$

Now we consider the case $p,q > 4$. For this case, replacing x,t by x/a, t/a^2 in (4.23), respectively, and letting $s \to 0^+$ and then multiplying the result by a^4 we have

$$\left| \tilde{v}(x,t) - a^4 \tilde{v}\left(\frac{x}{a}, \frac{t}{a^2}\right) \right| \le \frac{\epsilon}{2a^4} |a|^{4-p} \psi_p(x,t). \tag{4.36}$$

Using induction argument and triangle inequality we obtain

$$\left| \tilde{v}(x,t) - a^{4k} \tilde{v}\left(\frac{x}{a^k}, \frac{t}{a^{2k}}\right) \right| \le \frac{\epsilon}{2a^4} \psi_p(x,t) \sum_{j=1}^{k} |a|^{(4-p)j} \tag{4.37}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. Following the similar method in case of $0 < p,q < 4$, we see that

$$g(x,t) := \lim_{k\to\infty} a^{4k} \tilde{v}\left(\frac{x}{a^k}, \frac{t}{a^{2k}}\right) \tag{4.38}$$

is the unique function satisfying (4.29) so that $T(x) := \lim_{t\to 0^+} g(x,t)$ exists. Letting $k \to \infty$ in (4.37) we get

$$\left| \tilde{v}(x,t) - g(x,t) \right| \le \frac{\epsilon}{2\left(|a|^p - a^4\right)} \psi_p(x,t). \tag{4.39}$$

Now letting $t \to 0^+$ in (4.39) we have the inequality

$$\left\| u - \frac{c}{a^2+1} - T(x) \right\| \le \frac{\epsilon}{2\left(|a|^p - a^4\right)} |x|^p. \tag{4.40}$$

This completes the proof. □

As an immediate consequence, we have the following corollary.

Corollary 4.4. *Let a be fixed integer with $a \neq 0, \pm 1$ and $\epsilon \geq 0$ be a real number. Suppose that u in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfies the inequality*

$$\left\| u \circ A_1 + u \circ A_2 - a^2 u \circ B_1 - a^2 u \circ B_2 - 2a^2 (a^2 - 1) u \circ P_1 + 2(a^2 - 1) u \circ P_2 \right\| \leq \epsilon. \tag{4.41}$$

Then there exists a unique quartic mapping $T(x)$ which satisfies (1.4) and the inequality

$$\left\| u - \frac{c}{a^2 + 1} - T(x) \right\| \leq \frac{\epsilon}{2(a^4 - 1)}, \tag{4.42}$$

where $c := \limsup_{t \to 0^+} \widetilde{u}(0, t)$.

Acknowledgments

The first author was supported by the second stage of the Brain Korea 21 Project, The Development Project of Human Resources in Mathematics, KAIST, in 2009. The second author was supported by the Special Grant of Sogang University in 2005.

References

[1] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[4] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[5] V. A. Faĭziev, Th. M. Rassias, and P. K. Sahoo, "The space of (ψ, γ)-additive mappings on semigroups," *Transactions of the American Mathematical Society*, vol. 354, no. 11, pp. 4455–4472, 2002.

[6] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[7] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[8] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[9] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[10] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *Glasnik Matematicki Series III*, vol. 34, no. 2, pp. 243–252, 1999.

[11] J. K. Chung and P. K. Sahoo, "On the general solution of a quartic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 565–576, 2003.

[12] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.

[13] C.-G. Park, "On the stability of the orthogonally quartic functional equation," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 63–70, 2005.

[14] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[15] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[16] C. Borelli and G. L. Forti, "On a general Hyers-Ulam stability result," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 2, pp. 229–236, 1995.
[17] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.
[18] J. R. Lee, J. S. An, and C. Park, "On the stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2008, Article ID 628178, 8 pages, 2008.
[19] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.
[20] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, vol. 256 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1983.
[21] L. Schwartz, *Théorie des Distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X, Hermann, Paris, France, 1966.
[22] J. Chung, S.-Y. Chung, and D. Kim, "A characterization for Fourier hyperfunctions," *Publications of the Research Institute for Mathematical Sciences*, vol. 30, no. 2, pp. 203–208, 1994.
[23] J. Chung and S. Lee, "Some functional equations in the spaces of generalized functions," *Aequationes Mathematicae*, vol. 65, no. 3, pp. 267–279, 2003.
[24] J. Chung, "Stability of functional equations in the spaces of distributions and hyperfunctions," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 177–186, 2003.
[25] J. Chung, S.-Y. Chung, and D. Kim, "The stability of Cauchy equations in the space of Schwartz distributions," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 107–114, 2004.
[26] Y.-S. Lee, "Stability of a quadratic functional equation in the spaces of generalized functions," *Journal of Inequalities and Applications*, vol. 2008, Article ID 210615, 12 pages, 2008.
[27] T. Matsuzawa, "A calculus approach to hyperfunctions. III," *Nagoya Mathematical Journal*, vol. 118, pp. 133–153, 1990.
[28] K. W. Kim, S.-Y. Chung, and D. Kim, "Fourier hyperfunctions as the boundary values of smooth solutions of heat equations," *Publications of the Research Institute for Mathematical Sciences*, vol. 29, no. 2, pp. 289–300, 1993.