Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 956910, 8 pages
doi:10.1155/2009/956910

Research Article

A Note on the q-Euler Measures

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Received 6 March 2009; Accepted 20 May 2009

Recommended by Patricia J. Y. Wong

Properties of q-extensions of Euler numbers and polynomials which generalize those satisfied by E_k and $E_k(x)$ are used to construct q-extensions of p-adic Euler measures and define p-adic q-ℓ-series which interpolate q-Euler numbers at negative integers. Finally, we give Kummer Congruence for the q-extension of ordinary Euler numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic numbers $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$. In this paper, we use the notations of q-number as follows (see [1–37]):

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

The ordinary Euler numbers are defined as (see [1–37])

$$\sum_{k=0}^{\infty} E_k \frac{t^k}{k!} = \frac{2}{e^t+1}, \quad |t| < \pi, \tag{1.2}$$

where $2/(e^t + 1)$ is written as e^{Et} when E^k is replaced by E_k. From the definition of Euler number, we can derive

$$E_0 = 1, \qquad (E+1)^n + E_n = 0, \quad \text{if} \quad n > 0, \tag{1.3}$$

with the usual convention of replacing E^i by E_i.

Remark 1.1. The second kind Euler numbers are also defined as follows (see [25]):

$$\operatorname{sech} t = \frac{2}{e^t + e^{-t}} = \frac{2e^t}{e^{2t}+1} = \sum_{k=0}^{\infty} E_k^* \frac{t^k}{k!} \quad \left(|t| < \frac{\pi}{2}\right). \tag{1.4}$$

The Euler polynomials are also defined by

$$\frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \tag{1.5}$$

Thus, we have

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}. \tag{1.6}$$

In [7], q-Euler numbers, $E_{k,q}$, can be determined inductively by

$$E_{0,q} = 1, \qquad q\left(qE_q + 1\right)^k + E_{k,q} = 0 \quad \text{if} \quad k > 0, \tag{1.7}$$

where E_q^k must be replaced by $E_{k,q}$, symbolically. The q-Euler polynomials $E_{k,q}(x)$ are given by $(q^x E_q + [x]_q)^k$, that is,

$$E_{k,q}(x) = \left(q^x E_q + [x]_q\right)^k = \sum_{i=0}^{k} \binom{k}{i} E_{i,q} q^{ix} [x]_q^{k-i}. \tag{1.8}$$

Let d be a fixedodd positive integer. Then we have (see [7])

$$\frac{[2]_q}{[2]_{q^d}} [d]_q^n \sum_{a=0}^{d-1} q^a (-1)^a E_{n,q}\left(\frac{x+a}{d}\right) = E_{n,q}(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{1.9}$$

We use (1.9) to get bounded p-adic q-Euler measures and finally take the Mellin transform to define p-adic q-ℓ-series which interpolate q-Euler numbers at negative integers.

2. p-adic q-Euler Measures

Let d be a fixed odd positive integer, and let p be a fixed odd prime number. Define

$$X = X_d = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p), \tag{2.1}$$
$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod\ dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [1–37]).

Theorem 2.1. *Let $\mu_{k,q}^{(E)}$ be given by*

$$\mu_{k,q}^{(E)} \left(a + dp^N \mathbb{Z}_p \right) = \frac{\left[dp^N\right]_q^k}{\left[dp^N\right]_{-q}} q^a (-1)^a E_{k,q^{dp^N}} \left(\frac{a}{dp^N} \right), \quad \textit{for } k \in \mathbb{Z}_+,\ N \in \mathbb{N}. \tag{2.2}$$

Then $\mu_{k,q}^{(E)}$ extends to a $Q(q)$-valued measure on the compact open sets $U \subset X$. Note that $\mu_{0,q}^{(E)} = \mu_{-q}$, where $\mu_{-q}(a + dp^N \mathbb{Z}_p) = (-q)^a / [dp^N]_{-q}$ is fermionic measure on X (see [7]).

Proof. It is sufficient to show that

$$\sum_{i=0}^{p-1} \mu_{k,q}^{(E)} \left(a + idp^N + dp^{N+1} \mathbb{Z}_p \right) = \mu_{k,q}^{(E)} \left(a + dp^N \mathbb{Z}_p \right). \tag{2.3}$$

By (1.9) and (2.2), we see that

$$\begin{aligned}
&\sum_{i=0}^{p-1} \mu_{k,q}^{(E)} \left(a + idp^N + dp^{N+1} \mathbb{Z}_p \right) \\
&\quad = \frac{\left[dp^{N+1}\right]_q^k}{\left[dp^{N+1}\right]_{-q}} \sum_{i=0}^{p-1} q^{a+idp^N} (-1)^{a+idp^N} E_{k,q^{dp^{N+1}}} \left(\frac{a + idp^N}{dp^{N+1}} \right) \\
&\quad = \frac{\left[dp^{N+1}\right]_q^k}{\left[dp^N\right]_{-q}} q^a (-1)^a \sum_{i=0}^{p-1} \left(q^{dp^N} \right)^i (-1)^i E_{k,(q^{dp^N})^p} \left(\frac{a/dp^N + i}{p} \right) \\
&\quad = \frac{\left[dp^N\right]_q^k}{\left[dp^N\right]_{-q}} q^a (-1)^a \frac{[2]_{q^{dp^N}}}{[2]_{q^{dp^{N+1}}}} [p]_{q^{dp^N}}^k \sum_{i=0}^{p-1} \left(q^{dp^N} \right)^i (-1)^i E_{k,(q^{dp^N})^p} \left(\frac{a/dp^N + i}{p} \right)
\end{aligned}$$

$$= \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a \frac{[2]_{q^{dp^N}}}{[2]_{(q^{dp^N})^p}} [p]_{q^{dp^N}}^k \sum_{i=0}^{p-1} \left(q^{dp^N}\right)^i (-1)^i E_{k,(q^{dp^N})^p}\left(\frac{a/dp^N + i}{p}\right)$$

$$= \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a E_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right) = \mu_{k,q}^{(E)}\left(a + dp^N \mathbb{Z}_p\right), \tag{2.4}$$

and we easily see that $|\mu_{k,q}^{(E)}|_p \leq M$ for some constant M. □

Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then we define the generalized q-Euler numbers attached to χ as follows:

$$E_{k,\chi,q} = \frac{[2]_q}{[2]_{q^d}} [d]_q^k = \sum_{x=0}^{d-1} q^x (-1)^x \chi(x) E_{k,q^d}\left(\frac{x}{d}\right). \tag{2.5}$$

The locally constant function χ on X can be integrated by the p-adic bounded q-Euler measure $\mu_{k,q}^{(E)}$ as follows:

$$\begin{aligned}
\int_X \chi(x) d\mu_{k,q}^{(E)}(x) &= \lim_{N\to\infty} \sum_{0\leq x<dp^N} \chi(x) \mu_{k,q}^{(E)}\left(x + dp^N \mathbb{Z}_p\right) \\
&= \lim_{N\to\infty} \frac{[dp^N]_q^k}{[dp^N]_{-q}} \sum_{0\leq a<d} \sum_{0\leq x<p^N} \chi(a+dx) q^{a+dx} (-1)^{a+dx} E_{k,q^{dp^N}}\left(\frac{a+xd}{dp^N}\right) \\
&= \frac{[2]_q}{[2]_{q^d}} [d]_q^k \sum_{0\leq a<d} \chi(a)(-1)^a q^a \lim_{N\to\infty} \frac{[p^N]_{q^d}^k}{[p^N]_{-q^d}} \\
&\quad \times \sum_{0\leq x<p^N} \left(q^d\right)^x (-1)^x E_{k,(q^d)^{p^N}}\left(\frac{a/d + x}{p^N}\right) \\
&= \frac{[2]_q}{[2]_{q^d}} [d]_q^k \sum_{0\leq a<d} \chi(a)(-1)^a q^a E_{k,q^d}\left(\frac{a}{d}\right) = E_{k,\chi,q}, \\
\int_{pX} \chi(x) d\mu_{k,q}^{(E)}(x) &= [p]_q^n \frac{[2]_q}{[2]_{q^p}} \frac{[2]_{q^p}}{[2]_{q^{pd}}} [d]_{q^p}^n \sum_{0\leq a<d} \chi(pa) q^{pa} (-1)^a E_{n,q^{dp}}\left(\frac{a}{d}\right) \\
&= \chi(p) [p]_q^n \frac{[2]_q}{[2]_{q^p}} \left\{ \frac{[2]_{q^p}}{[2]_{q^{pd}}} [d]_{q^p}^n \sum_{0\leq a<d} \chi(a) q^{pa} (-1)^a E_{n,q^{dp}}\left(\frac{a}{d}\right) \right\} \\
&= \chi(p) [p]_q^n \frac{[2]_q}{[2]_{q^p}} E_{n,\chi,q^p}.
\end{aligned} \tag{2.6}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\bmod\ 2)$. Then one has*

$$\int_X \chi(x) d\mu_{k,q}^{(E)}(x) = E_{k,\chi,q}, \qquad \int_{pX} \chi(x) d\mu_{k,q}^{(E)}(x) = \chi(p) [p]_q^k \frac{[2]_q}{[2]_{q^p}} E_{k,\chi,q^p},$$
$$\int_{X^*} \chi(x) d\mu_{k,q}^{(E)}(x) = E_{k,\chi,q} - \chi(p) [p]_q^k \frac{[2]_q}{[2]_{q^p}} E_{k,\chi,q^p}. \tag{2.7}$$

Let $k \in \mathbb{Z}_+$. From (2.2), we note that

$$\begin{aligned}
\mu_{k,q}^{(E)}\left(a + dp^N \mathbb{Z}_p\right) &= \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a E_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right) \\
&= \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a \sum_{i=0}^{k} \binom{k}{i} E_{i,q^{dp^N}} q^{ai} \left[\frac{a}{dp^N}\right]_{q^{dp^N}}^{k-i} \\
&= \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a \sum_{i=0}^{k} \binom{k}{i} E_{i,q^{dp^N}} q^{ai} \frac{[a]_q^{k-i}}{[dp^N]_q^{k-i}} \\
&= \frac{(-q)^a}{[dp^N]_{-q}} [a]_q^k + \frac{[dp^N]_q^k}{[dp^N]_{-q}} q^a (-1)^a \sum_{i=1}^{k} \binom{k}{i} E_{i,q^{dp^N}} q^{ai} \frac{[a]_q^{k-i}}{[dp^N]_q^{k-i}}.
\end{aligned} \tag{2.8}$$

Thus, we have

$$d\mu_{k,q}^{(E)}(x) = [x]_q^k d\mu_{-q}(x). \tag{2.9}$$

Therefore, we obtain the following theorem and corollary.

Theorem 2.3. *For $k \geq 0$, one has*

$$d\mu_{k,q}^{(E)}(x) = [x]_q^k d\mu_{-q}(x). \tag{2.10}$$

Corollary 2.4. *For $k \geq 0$, one has*

$$\int_X d\mu_{k,q}^{(E)}(x) = \int_X [x]_q^k d\mu_{-q}(x) = E_{k,q}. \tag{2.11}$$

3. p-adic q-ℓ-Series

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Let ω denote the Teichmüller character mod p. For $x \in X^*$, we set $\langle x \rangle_q = [x]_q/\omega(x)$. Note that $|\langle x \rangle_q - 1|_p < p^{-1/(p-1)}$, and $\langle x \rangle_q^s$ is defined by $\exp(s \log_p \langle x \rangle_q)$, for $|s|_p \leq 1$. For $s \in \mathbb{Z}_p$,we define

$$\ell_{p,q}(s,\chi) = \int_{X^*} \langle x \rangle_q^{-s} \chi(x) d\mu_{-q}(x). \tag{3.1}$$

Thus, we have

$$\begin{aligned} \ell_{p,q}\left(-k, \chi\omega^k\right) &= \int_{X^*} [x]_q^k \chi(x) d\mu_{-q}(x) = \int_{X^*} \chi(x) d\mu_{k,q}^{(E)}(x) \\ &= E_{k,\chi,q} - \chi(p) [p]_q^k \frac{[2]_q}{[2]_{q^p}} E_{k,\chi,q^p}, \quad \text{for} \quad k \in \mathbb{Z}_+ . \end{aligned} \tag{3.2}$$

Since $|\langle x \rangle_q - 1|_p < p^{-1/(p-1)}$ for $x \in X^*$, we have $\langle x \rangle^{p^n} \equiv 1 (\text{mod } p^n)$. Let $k \equiv k^{'}$ $(\text{mod } p^n(p-1))$. Then we have

$$\ell_{p,q}\left(-k, \chi\omega^k\right) \equiv \ell_{p,q}\left(-k^{'}, \chi\omega^{k^{'}}\right) (\text{mod } p^n). \tag{3.3}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *Let $k \equiv k' (\text{mod } (p-1)p^n)$. Then one has*

$$E_{k,\chi,q} - \frac{[2]_q}{[2]_{q^p}} \chi(p) [p]_q^k E_{k,\chi,q^p} \equiv E_{k',\chi,q} - \frac{[2]_q}{[2]_{q^p}} \chi(p) [p]_q^{k'} E_{k',\chi,q^p} \; (\text{mod } p^n). \tag{3.4}$$

Acknowledgments

This paper was supported by Jangjeon Mathematical Society.

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