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Research Article

Almost Periodic Solutions of Prey-Predator Discrete Models with Delay

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The purpose of this article is to investigate the existence of almost periodic solutions of a system of almost periodic Lotka-Volterra difference equations which are a prey-predator system $x_1(n+1) = x_1(n)\exp\{b_1(n)-a_1(n)x_1(n)-c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\}$, $x_2(n+1) = x_2(n)\exp\{-b_2(n)-a_2(n)x_2(n) + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\}$ and a competitive system $x_i(n+1) = x_i(n)\exp\{b_i(n) - a_{ii}x_i(n) - \sum_{j=1,j\neq i}^{l}\sum_{s=-\infty}^{n} K_{ij}(n-s)x_j(s)\}$, by using certain stability properties, which are referred to as (K,ρ)-weakly uniformly asymptotic stable in hull and (K,ρ)-totally stable.

1. Introduction and Preliminary

For ordinary differential equations and functional differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular methods is to assume the certain stability properties [1–4]. Recently, Song and Tian [5] have shown the existence of periodic and almost periodic solutions for nonlinear Volterra difference equations by means of (K,ρ)-stability conditions. Their results are to extend results in Hamaya [2] to discrete Volterra equations. To the best of our knowledge, there are no relevant results on almost periodic solutions for discrete Lotka-Volterra models by means of our approach, except for Xia and Cheng's special paper [6]. However, they treated only nondelay case in [6]. We emphasize that our results extend [3, 6, 7] as a discret delay case.

In this paper, we will discuss the existence of almost periodic solutions for discrete Lotka-Volterra difference equations with time delay.

In what follows, we denote by R^m real Euclidean m-space, Z is the set of integers, Z^+ is the set of nonnegative integers, and $|\cdot|$ will denote the Euclidean norm in R^m. For any interval

$I \subset Z := (-\infty, \infty)$, we denote by $\mathrm{BS}(I)$ the set of all bounded functions mapping I into R^m, and set $|\phi|_I = \sup\{|\phi(s)| : s \in I\}$.

Now, for any function $x : (-\infty, a) \to R^m$ and $n < a$, define a function $x_n : Z^- = (-\infty, 0] \to R^m$ by $x_n(s) = x(n+s)$ for $s \in Z^-$. Let BS be a real linear space of functions mapping Z^- into R^m with sup-norm:

$$\mathrm{BS} = \left\{ \phi \mid \phi : Z^- \longrightarrow R^m \text{ with } |\phi| = \sup_{s \in Z^-} |\phi(s)| < \infty \right\}. \tag{1.1}$$

We introduce an almost periodic function $f(n,x) : Z \times D \to R^m$, where D is an open set in R^m.

Definition 1.1. It holds that $f(n,x)$ is said to be almost periodic in n uniformly for $x \in D$, if for any $\epsilon > 0$ and any compact set K in D, there exists a positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$|f(n+\tau, x) - f(n,x)| \le \epsilon \tag{1.2}$$

for all $n \in Z$ and all $x \in K$. Such a number τ in the above inquality is called an ϵ-translation number of $f(n,x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above difinition, we discuss the concept of the normality of almost periodic functions. Namely, let $f(n,x)$ be almost periodic in n uniformly for $x \in D$. Then, for any sequence $\{h'_k\} \subset Z$, there exist a subsequence $\{h_k\}$ of $\{h'_k\}$ and function $g(n,x)$ such that

$$f(n+h_k, x) \longrightarrow g(n,x) \tag{1.3}$$

uniformly on $Z \times K$ as $k \to \infty$, where K is a compact set in D. There are many properties of the discrete almost periodic functions [5, 8], which are corresponding properties of the continuous almost periodic functions $f(t,x) \in C(R \times D, R^m)$ [4]. We will denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n,x) = f(n+\tau, x), \quad \tau \in Z. \tag{1.4}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (1.4). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n+n_k, x) \to g(n,x)$ uniformly on $Z \times S$ for any compact subset S in R^m. By (1.3), if $f : Z \times D \to R^m$ is almost periodic in n uniformly for $x \in D$, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf. [4]).

Definition 1.2. It holds that $u(n)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a, \infty) \subset Z^+ = [0, \infty)$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n). \tag{1.5}$$

However, $u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{n_k\}$ for which $u(n+n_k)$ converges uniformly on $n; a \le n < \infty$.

2. Prey-Predator Model

We will consider the existence of a strictly positive component-wise almost periodic solution of a system of Volterra difference equations:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{b_1(n) - a_1(n)x_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right\},\\ x_2(n+1) &= x_2(n)\exp\left\{-b_2(n) - a_2(n)x_2(n) + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\right\}, \end{aligned} \tag{F}$$

which describes a model of the dynamics of a prey-predator discrete system in mathematical ecology. In (F), setting $a_i(n)$, $b_i(n)$, and $c_i(n)R$-valued bounded almost periodic in Z:

$$\begin{gathered} a_i = \inf_{n\in Z} a_i(n), \qquad A_i = \sup_{n\in Z} a_i(n), \qquad b_i = \inf_{n\in Z} b_i(n),\\ B_i = \sup_{n\in Z} b_i(n), \qquad c_i = \inf_{n\in Z} c_i(n) \qquad C_i = \sup_{n\in Z} c_i(n) \quad (i=1,2), \end{gathered} \tag{2.1}$$

and $K_i : Z^+ = [0,\infty) \to R^+$ $(i=1,2)$ denote delay kernels such that

$$K_i(s) \ge 0, \quad \sum_{s=0}^{\infty} K_i(s) = 1, \quad \sum_{s=0}^{\infty} sK_i(s) < \infty, \quad (i=1,2). \tag{2.2}$$

Under the above assumptions, it follows that for any $(n_0, \phi_i) \in Z^+ \times \mathrm{BS}, (i=1,2)$, there is a unique solution $u(n) = (u_1(n), u_2(n))$ of (F) through $(n_0, \phi_i), (i=1,2)$, if it remains bounded.

We set

$$\begin{gathered} \alpha_1 = \exp\frac{\{B_1 - 1\}}{a_1}, \qquad \alpha_2 = \exp\frac{\{-b_2 + C_1\alpha_1 - 1\}}{a_2},\\ \beta_1 = \min\left\{\exp\frac{\{b_1 - A_1\alpha_1 - C_2\alpha_2\}(b_1 - C_2\alpha_2)}{A_1}, \frac{\{b_1 - C_2\alpha_2\}}{A_1}\right\},\\ \beta_2 = \min\left\{\exp\frac{\{-B_2 - A_2\alpha_2 + c_1\beta_1\}(-B_2 + c_1\beta_1)}{A_2}, \frac{\{-B_2 + c_1\beta_1\}}{A_2}\right\} \end{gathered} \tag{2.3}$$

(cf. [6, Application 4.1]).

We now make the following assumptions:

(i) $a_i > 0,\ b_i > 0\ (i = 1, 2),\ c_1 > 0, c_2 \geq 0$,

(ii) $b_1 > C_2\alpha_2,\ B_2 < c_1\beta_1$,

(iii) there exists a positive constant m such that

$$a_i > C_i + m \quad (i = 1, 2). \tag{2.4}$$

Then, we have $0 < \beta_i < \alpha_i$ for each $i = 1, 2$.

We can show the following lemmas.

Lemma 2.1. *If $x(n) = (x_1(n), x_2(n))$ is a solution of (F) through $(n_0, \phi_i), (i = 1, 2)$ such that $\beta_i \leq \phi_i(s) \leq \alpha_i\ (i = 1, 2)$ for all $s \leq 0$, then one has $\beta_i \leq x_i(n) \leq \alpha_i\ (i = 1, 2)$ for all $n \geq n_0$.*

Proof. First, we claim that

$$\limsup_{n\to\infty}\ x_1(n) \leq \alpha_1. \tag{2.5}$$

To do this, we first assume that there exists an $l_0 \geq n_0$ such that $x_1(l_0 + 1) \geq x_1(l_0)$. Then, it follows from the first equation of (F) that

$$b_1(l_0) - a_1(l_0)x_1(l_0) - c_2(l_0)\sum_{s=-\infty}^{l_0} K_2(l_0 - s)x_2(s) \geq 0. \tag{2.6}$$

Hence,

$$x_1(l_0) \leq \frac{b_1(l_0) - c_2(l_0)\sum_{s=-\infty}^{l_0} K_2(l_0 - s)x_2(s)}{a_1(l_0)} \leq \frac{b_1(l_0)}{a_1(l_0)} \leq \frac{B_1}{a_1}. \tag{2.7}$$

It follows that

$$\begin{aligned} x_1(l_0 + 1) &= x_1(l_0)\exp\left\{b_1(l_0) - a_1(l_0)x_1(l_0) - c_2(l_0)\sum_{s=-\infty}^{l_0} K_2(l_0 - s)x_2(s)\right\} \\ &\leq x_1(l_0)\exp\{B_1 - a_1x_1(l_0)\} \leq \frac{\exp\{B_1 - 1\}}{a_1} := \alpha_1, \end{aligned} \tag{2.8}$$

where we use the fact that

$$\max_{x\in R}\{x\exp(p - qx)\} = \frac{\exp(p - 1)}{q}, \quad \text{for } p, q > 0. \tag{2.9}$$

Now we claim that

$$x_1(n) \leq \alpha_1, \quad \text{for } n \geq l_0. \tag{2.5*}$$

By way of contradiction, we assume that there exists a $p_0 > l_0$ such that $x_1(p_0) > \alpha_1$. Then, $p_0 \geq l_0 + 2$. Let $\widehat{p}_0 \geq l_0 + 2$ be the smallest integer such that $x_1(\widehat{p}_0) > \alpha_1$. Then, $x_1(\widehat{p}_0 - 1) \leq x_1(\widehat{p}_0)$. The above argument shows that $x_1(\widehat{p}_0) \leq \alpha_1$, which is a contradiction. This proves our assertion. We now assume that $x_1(n+1) < x_1(n)$ for all $n \geq n_0$. Then $\lim_{n\to\infty} x_1(n)$ exists, which is denoted by $\overline{x}_1$. We claim that $\overline{x}_1 \leq \exp(B_1 - 1)/a_1$. Suppose to the contrary that $\overline{x}_1 > \exp(B_1 - 1)/a_1$. Taking limits in the first equation in System (F), we set that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(b_1(n) - a_1(n)x_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right) \\ &\leq \lim_{n\to\infty}(b_1(n) - a_1(n)x_1(n)) \leq B_1 - a_1\overline{x}_1 < 0, \end{aligned} \tag{2.10}$$

which is a contradiction. It follows that (2.5) holds, and then we have $x_i(n) \leq \alpha_1$ for all $n \geq n_0$ from (2.5*). Next, we prove that

$$\limsup_{n\to\infty} x_2(n) \leq \alpha_2. \tag{2.11}$$

We first assume that there exists an $k_0 \geq n_0$ such that $k_0 \geq l_0$ and $x_2(k_0+1) \geq x_2(k_0)$. Then

$$-b_2(k_0) - a_2(k_0)x_2(k_0) + c_1(k_0)\sum_{s=-\infty}^{k_0} K_1(k_0 - s)x_1(s) \geq 0. \tag{2.12}$$

Hence,

$$x_2(k_0) \leq \frac{-b_2(k_0) + c_1(k_0)\sum_{s=-\infty}^{k_0} K_1(k_0-s)x_1(s)}{a_2(k_0)} \leq \frac{-b_2 + C_1\alpha_1}{a_2}. \tag{2.13}$$

It follows that

$$\begin{aligned} x_2(k_0+1) &= x_2(k_0)\exp\left\{-b_2(k_0) - a_2(k_0)x_2(k_0) + c_1(k_0)\sum_{s=-\infty}^{k_0} K_1(k_0-s)x_1(s)\right\} \\ &\leq x_2(k_0)\exp\{-b_2 - a_2x_2(k_0) + C_1\alpha_1\} \\ &\leq \frac{\exp(-b_2 + C_1\alpha_1 - 1)}{a_2} := \alpha_2, \end{aligned} \tag{2.14}$$

where we also use the two facts which are used to prove (2.5). Now we claim that

$$x_2(n) \leq \alpha_2 \quad \forall n \geq k_0. \tag{2.11*}$$

Suppose to the contrary that there exists a $q_0 > k_0$ such that $x_2(q_0) > \alpha_2$. Then $q_0 \geq k_0 + 2$. Let $\widehat{q}_0 \geq k_0 + 2$ be the smallest integer such that $x_2(\widehat{q}_0) > \alpha_2$. Then $x_2(\widehat{q}_0 - 1) < x_2(\widehat{q}_0)$. Then the above argument shows that $x_2(\widehat{q}_0) \leq \alpha_2$, which is a contradiction. This prove our claim from (2.11) and (2.11*).

Now, we assume that $x_2(n+1) < x_2(n)$ for all $n \geq n_0$. Then $\lim_{n\to\infty} x_2(n)$ exists, which is denoted by $\overline{x}_2$. We claim that $\overline{x}_2 \leq \exp(-b_2 + C_1\alpha_1 - 1)/a_2$. Suppose to the contrary that $\overline{x}_2 > \exp(-b_2 + C_1\alpha_1 - 1)/a_2$. Taking limits in the first equation in System (F), we set that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(-b_2(n) - a_2(n)x_2(n) + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\right) \\ &\leq -b_2 - a_2\overline{x}_2 + C_1\alpha_1 < 0, \end{aligned} \tag{2.15}$$

which is a contradiction. It follows that (2.11) holds.

We show that

$$\liminf_{n\to\infty} x_1(n) \geq \beta_1. \tag{2.16}$$

According to the above assertion, there exists a $k^* \geq n_0$ such that $x_1(n) \leq \alpha_1 + \epsilon$ and $x_2(n) \leq \alpha_2 + \epsilon$, for all $n \geq k^*$. We assume that there exists an $l_0 \geq k^*$ such that $x_1(l_0+1) \leq x_1(l_0)$. Note that for $n \geq l_0$,

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{b_1(n) - a_1(n)x_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right\} \\ &\geq x_1(n)\exp\{b_1 - C_2\alpha_2 - A_1x_1(n)\}. \end{aligned} \tag{2.17}$$

In particular, with $n = l_0$, we have

$$b_1 - A_1x_1(l_0) - C_2\alpha_2 \leq 0, \tag{2.18}$$

which implies that

$$x_1(l_0) \geq \frac{b_1 - C_2\alpha_2}{A_1}. \tag{2.19}$$

Then,

$$x_1(l_0+1) \geq \frac{b_1 - C_2\alpha_2}{A_1}\exp(b_1 - C_2\alpha_2 - A_1(\alpha_1+\epsilon)) := x_{1\epsilon}. \tag{2.20}$$

We assert that

$$x_1(n) \geq x_{1\epsilon}, \quad \forall n \geq l_0. \tag{2.16*}$$

By way of contradiction, we assume that there exists a $p_0 \geq l_0$ such that $x_1(p_0) < x_{1\epsilon}$. Then $p_0 \geq l_0 + 2$. Let $\widehat{p}_0 + 2$ be the smallest integer such that $x_1(\widehat{p}_0) < x_{1\epsilon}$.Then $x_1(\widehat{p}_0) \leq x_1(\widehat{p}_0 - 1)$. The above argument yields $x_1(\widehat{p}_0) \geq x_{1\epsilon}$, which is a contradiction. This proves our claim. We now assume that $x_1(n+1) < x_1(n)$ for all $n \geq n_0$. Then $\lim_{n\to\infty} x_1(n)$ exists, which is denoted by $\underline{x}_1$. We claim that $\underline{x}_1 \geq (b_1 - C_2\alpha_2)/A_1$. Suppose to the contrary that $\underline{x}_1 < (b_1 - C_2\alpha_2)/A_1$. Taking the limits in the first equation in System (F), we set that

$$0 = \lim_{n\to\infty}\left(b_1(n) - a_1(n)x_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right) \geq b_1 - A_1\underline{x}_1 - C_2\alpha_2 > 0, \tag{2.21}$$

which is a contradiction. It follows that (2.16) holds, and then $\beta_1 \leq x_1(n)$ for all $n \geq n_0$ from (2.16) and (2.16*). Finally, by using the inequality $B_2 < c_1\beta_1$, similar arguments lead to $\liminf_{n\to\infty} x_2 \geq \beta_2$, and then $x_2(n) \geq \beta_2$ for all $n \geq k_0$. This proof is complete. □

Lemma 2.2. *Let K be the closed bounded set in R^2 such that*

$$K = \left\{(x_1, x_2) \in R^2; \beta_i \leq x_i \leq \alpha_i \text{ for each } i = 1,2\right\}. \tag{2.22}$$

Then K is invariant for System (F), that is, one can see that for any $n_0 \in Z$ and any φ_i such that $\varphi_i(s) \in K$, $s \leq 0$ $(i = 1,2)$, every solution of (F) through (n_0, φ_i) remains in K for all $n \geq n_0$ and $i = 1,2$.

Proof. From Lemma 2.1, it is sufficient to prove that this $K \neq \phi$.

To do this, by assumption of almost periodic functions, there exists a sequence $\{n_k\}, n_k \to \infty$ as $k \to \infty$, such that $b_i(n+n_k) \to b_i(n), a_i(n+n_k) \to a_i(n)$, and $c_i(n+n_k) \to c_i(n)$ as $k \to \infty$ uniformly on Z and $i = 1,2$. Let $x(n)$ be a solution of System (F) through (n_0, φ) that remains in K for all $n \geq n_0$, whose existence was ensured by Lemma 2.1. Clearly, the sequence $\{x(n+n_k)\}$ is uniformly bounded on bounded subset of Z. Therefore, we may assume that the sequence $\{x(n+n_k)\}$ converges to a function $y(n) = (y_1(n), y_2(n))$ as $k \to \infty$ uniformly on each bounded subset of Z taking a subset of $\{x(n+n_k)\}$ if necessary. We may assume that $n_k \geq n_0$ for all k. For $n \geq 0$, we have

$$\begin{aligned}
&x_1(n+n_k+1) \\
&\quad = x_1(n+n_k)\exp\left\{b_1(n+n_k) - a_1(n+n_k)x_1(n+n_k) - c_2(n+n_k)\sum_{s=-\infty}^{n+n_k} K_2(n+n_k-s)x_2(s)\right\}, \\
&x_2(n+n_k+1) \\
&\quad = x_2(n+n_k)\exp\left\{-b_2(n+n_k) - a_2(n+n_k)x_2(n+n_k) + c_1(n+n_k)\sum_{s=-\infty}^{n+n_k} K_1(n+n_k-s)x_1(s)\right\}.
\end{aligned} \tag{F_{n_k}}$$

Since $x(n+n_k) \in K$ and $y(n) \in K$ for all $n \in Z$, there exists $r > 0$ such that $|x(n+n_k)| \leq r$ and $|y(n)| \leq r$ for all $n \in Z$. Then, by assumption of delay kernel K_i, for this r and any $\epsilon > 0$, there exists an integer $S = S(\epsilon, r) > 0$ such that

$$\sum_{s=-\infty}^{n-S} |K_i(n+n_k-s)x_i(s)| \leq \epsilon, \qquad \sum_{s=-\infty}^{n-S} \left|K_i(n-s)y_i(s)\right| \leq \epsilon. \tag{2.23}$$

Then, we have

$$\begin{aligned}
&\left|\sum_{s=-\infty}^{n} K_i(n+n_k-s)x_i(s) - \sum_{s=-\infty}^{n} K_i(n-s)y_i(s)\right| \\
&\quad \le \sum_{s=-\infty}^{n-S} |K_i(n+n_k-s)x_i(s)| + \sum_{s=-\infty}^{n-S} \left|K_i(n-s)y_i(s)\right| \\
&\qquad + \sum_{s=n-S}^{n} \left|K_i(n+n_k-s)x_i(s) - K_i(n-s)y_i(s)\right| \\
&\quad \le 2\epsilon + \sum_{s=n-S}^{n} \left|K_i(n+n_k-s)x_i(s) - K_i(n-s)y_i(s)\right|.
\end{aligned} \tag{2.24}$$

Since $x_i(n+n_k-s)$ converges to $y_i(n-s)$ on discrete interval $s \in [n-S, n]$ as $k \to \infty$, there exists an integer $k_0(\epsilon) > k_*$, for some $k_* > 0$, such that

$$\sum_{s=n-S}^{n} \left|K_i(n+n_k-s)x_i(s) - K_i(n-s)y_i(s)\right| \le \epsilon \quad (i = 1, 2) \tag{2.25}$$

when $k \ge k_0(\epsilon)$. Thus, we have

$$\sum_{s=-\infty}^{n} K_i(n+n_k-s)x_i(s) \longrightarrow \sum_{s=-\infty}^{n} K_i(n-s)y_i(s) \tag{2.26}$$

as $k \to \infty$. Letting $k \to \infty$ in (F_{n_k}), we have

$$\begin{aligned}
y_1(n+1) &= y_1(n)\exp\left\{b_1(n) - a_1(n)y_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)y_2(s)\right\}, \\
y_2(n+1) &= y_2(n)\exp\left\{-b_2(n) - a_2(n)y_2(n) + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)y_1(s)\right\},
\end{aligned} \tag{2.27}$$

for all $n \ge n_0$. Then, $y(n) = (y_1(n), y_2(n))$ is a solution of System (F) on Z. It is clear that $y(n) \in K$ for all $n \in Z$. Thus, $K \ne \phi$.

We denote by $\Omega(F)$ the set of all limit functions (G) such that for some sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$, $b_i(n+n_k) \to \overline{b}_i(n)$, $a_i(n+n_k) \to \overline{a}_i(n)$, and $c_i(n+n_k) \to \overline{c}_i(n)$ uniformly on Z as $k \to \infty$. Here, the equation for (G) is

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{\overline{b}_1(n) - \overline{a}_1(n)x_1(n) - \overline{c}_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{-\overline{b}_2(n) - \overline{a}_2(n)x_2(n) + \overline{c}_1(n)\sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\right\}. \end{aligned} \tag{G}$$

Moreover, we denote by $(v, G) \in \Omega(u, F)$ when for the same sequence $\{n_k\}$, $u(n+n_k) \to v(n)$ uniformly on any compact subset in Z as $k \to \infty$. Then a system (G) is called a limiting equation of (F) when $(G) \in \Omega(F)$ and $v(n)$ is a solution of (G) when $(v, G) \in \Omega(u, F)$. □

Lemma 2.3. *If a compact set K in R^2 of Lemma 2.2 is invariant for System (F), then K is invariant also for every limiting equation of System (F).*

Proof. Let (G) be a limiting equation of system (F). Since $(G) \in \Omega(F)$, there exists a sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ and that $b_i(n+n_k) \to \overline{b}_i(n)$, $a_i(n+n_k) \to \overline{a}_i(n)$, and $c_i(n+n_k) \to \overline{c}_i(n)$ uniformly on Z as $k \to \infty$. Let $n_0 \geq 0$, $\phi \in BS$ such that $\phi(s) \in K$ for all $s \leq 0$, and let $y(n)$ be a solution of system (G) through (n_0, ϕ). Let $x^k(n)$ be the solution of System (F) through (n_0+n_k, ϕ). Then $x^k_{n_0+n_k}(s) = \phi(s) \in K$ for all $s \geq 0$ and $x^k(n)$ is defined on $n \geq n_0+n_k$. Since K is invariant for System (F), $x^k(n) \in K$ for all $n \geq n_0+n_k$. If we set $z^k(n) = x^k(n+n_k)$, $k = 1, 2, \ldots$, then $z^k(n)$ is defined on $n \geq n_0$ and is a solution of

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{b_1(n+n_k) - a_1(n+n_k)x_1(n) - c_2(n+n_k)\sum_{s=-\infty}^{n+n_k} K_2(n+n_k-s)x_2(s)\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{-b_2(n+n_k) - a_2(n+n_k)x_2(n) + c_1(n+n_k)\sum_{s=-\infty}^{n+n_k} K_1(n+n_k-s)x_1(s)\right\}, \end{aligned} \tag{2.28}$$

such that $z^k_{n_0}(s) = x^k_{n_0+n_k}(s) = \phi(s) \in K$ for all $s \leq 0$. Since $x^k(n) \in K$ for all $n \geq n_0+n_k$, $z^k(n) \in K$ for all $n \geq n_0$. Since the sequence $\{z^k(n)\}$ is uniformly bounded on $[n_0, \infty)$ and $z^k_{n_0} = \phi$, $\{z^k(n)\}$ can be assumed to converge to the solution $y(n)$ of (G) through (n_0, ϕ) uniformly on any compact set $[n_0, \infty)$, because $y(n)$ is the unique solution through (n_0, ϕ) and the same argument as in the proof of Lemma 2.2. Therefore, $y(n) \in K$ for all $n \geq n_0$ since $z^k(n) \in K$ for all $n \geq n_0$ and K is compact. This shows that K is invariant for limiting (G).

Let K be the compact set in R^m such that $u(n) \in K$ for all $n \in Z$, where $u(n) = \phi^0(n)$ for $n \leq 0$. For any $\theta, \psi \in BS$, we set

$$\rho(\theta, \psi) = \sum_{j=1}^{\infty} \frac{\rho_j(\theta, \psi)}{\left[2^j(1 + \rho_j(\theta, \psi))\right]}, \tag{2.29}$$

where

$$\rho_j(\theta, \psi) = \sup_{-j \leq s \leq 0} \left|\theta(s) - \psi(s)\right|. \tag{2.30}$$

Clearly, $\rho(\theta_n, \theta) \to 0$ as $n \to \infty$ if and only if $\theta_n(s) \to \theta(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $n \to \infty$.

In what follows, we need the following definitions of stability. □

Definition 2.4. The bounded solution $u(n)$ of System (F) is said to be as follows:

(i) (K, ρ)-totally stable (in short, (K, ρ)-TS) if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta(\epsilon)$ and $h = (h_1, h_2) \in BS([n_0, \infty))$ which satisfies $|h|_{[n_0,\infty)} < \delta(\epsilon)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp\left\{ b_1(n) - a_1(n)x_1(n) - c_2(n) \sum_{s=-\infty}^{n} K_2(n-s)x_2(s) \right\} + h_1(n), \\ x_2(n+1) &= x_2(n) \exp\left\{ -b_2(n) - a_2(n)x_2(n) + c_1(n) \sum_{s=-\infty}^{n} K_1(n-s)x_1(s) \right\} + h_2(n), \end{aligned} \tag{$F+h$}$$

through (n_0, ϕ) such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$. In the case where $h(n) \equiv 0$, this gives the definition of the (K, ρ)-US of $u(n)$;

(ii) (K, ρ)-attracting in $\Omega(F)$ (in short, (K, ρ)-A in $\Omega(F)$) if there exists a $\delta_0 > 0$ such that if $n_0 \geq 0$ and any $(v, G) \in \Omega(u, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta_0$, then $\rho(x_n, v_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of (limiting equation of (2.5)); (G) through (n_0, ψ) such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$;

(iii) (K, ρ)-weakly uniformly asymptotically stable in $\Omega(F)$ (in short, (K, ρ)-WUAS in $\Omega(F)$) if it is (K, ρ)-US in $\Omega(F)$, that is, if for any $\varepsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $n_0 \geq 0$ and any $(v, G) \in \Omega(u, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta(\epsilon)$, then $\rho(x_n, v_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of (G) through (n_0, ψ) such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$, and (K, ρ)-A in $\Omega(F)$.

Proposition 2.5. *Under the assumption (i), (i), and (iii), if the solution $u(n)$ of System (F) is (K,ρ)-WUAS in $\Omega(F)$, then the solution $u(n)$ of System (F) is (K,ρ)-TS.*

Proof. Suppose that $u(n)$ is not (K,ρ)-TS. Then there exist a small $\epsilon > 0$, sequences $\{\epsilon_k\}, 0 < \epsilon_k < \epsilon$ and $\epsilon_k \to 0$ as $k \to \infty$, sequences $\{s_k\}$, $\{n_k\}$, $\{h_k\}$, $\{x^k\}$ such that $s_k \to \infty$ as $k \to \infty$, $0 < s_k + 1 < n_k$, $h_k : Z \to R^2$ is bounded function satisfying $|h_k(n)| < \epsilon_k$ for $n \geq s_k$ and such that

$$\rho\left(u_{s_k}, x^k_{s_k}\right) < \epsilon_k, \quad \rho\left(u_{n_k}, x^k_{n_k}\right) \geq \epsilon, \quad \rho\left(u_n, x^k_n\right) < \epsilon \quad [s_k, n_k), \tag{2.31}$$

where $x^k(n)$ is a solution of

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{b_1(n) - a_1(n)x_1(n) - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\right\} + h_{k1}(n), \\ x_2(n+1) &= x_2(n)\exp\left\{-b_2(n) - a_2(n)x_2(n) + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\right\} + h_{k2}(n), \end{aligned} \tag{$F+h_k$}$$

such that $x^k_{s_k}(s) \in K$ for all $s \leq 0$. We can assume that $\epsilon < \delta_0$ where δ_0 is the number for (K,ρ)-A in $\Omega(F)$ of Definition 2.4. Moreover, by (2.31), we can chose sequence $\{\tau_k\}$ such that $s_k < \tau_k < n_k$,

$$\rho\left(u_{\tau_k}, x^k_{\tau_k}\right) = \frac{\delta(\epsilon/2)}{2}, \tag{2.32}$$

$$\frac{\delta(\epsilon/2)}{2} \leq \rho\left(u_n, x^k_n\right) \leq \epsilon \quad \text{for } n \in [\tau_k, n_k], \tag{2.33}$$

where $\delta(\cdot)$ is the number for (K,ρ)-US in $\Omega(F)$. We may assume that $u(n+\tau_k) \to v(n)$ as $k \to \infty$ on each bounded subset of Z for a function v, and for the sequence $\{\tau_k\}$, $\tau_k \to \infty$ as $k \to \infty$, taking a subsequence if necessary, there exists a $(v,G) \in \Omega(u,F)$. Moreover, we may assume that $x^k(n+\tau_k) \to z(n)$ as $k \to \infty$ uniformly on any bounded subset of Z for function z, since the sequence $\{x^k(n+\tau_k)\}$ is uniformly bounded on Z. Because, if we set $y^k(n) = x^k(n+\tau_k)$, then $y^k(n)$ is defined on $n \geq n_0 + \tau_k$ and $y^k(n)$ is a solution of

$$\begin{aligned} &x_1(n+1) \\ &= x_1(n)\exp\left\{b_1(n+\tau_k) - a_1(n+\tau_k)x_1(n) - c_2(n+\tau_k)\sum_{s=-\infty}^{n} K_2(n+n_k-s)x_2(s)\right\} + h_{k1}(n+\tau_k), \end{aligned}$$

$$x_2(n+1)$$
$$= x_2(n)\exp\left\{-b_2(n+\tau_k)-a_2(n+\tau_k)x_2(n)+c_1(n+\tau_k)\sum_{s=-\infty}^{n}K_1(n+n_k-s)x_1(s)\right\}+h_{k2}(n+\tau_k), \tag{2.34}$$

such that $y_0^k(s) = x_{\tau_k}^k(s) \in K$ for all $s \le 0$. Then we may show that taking a subsequence if necessary, $y^k(n)$ converges to a solution $z(n)$ of (G) such that $z_0(s) \in K$ for $s \le 0$, by the same argument for Σ-calculations with condition of K_i as in the proof of Lemma 2.2. Then, the same argument as in the proof of Lemma 2.2 shows that $z \in K$. Now, suppose that $n_k - \tau_k \to \infty$ as $k \to \infty$. Letting $k \to \infty$ in (2.33), we have $\delta(\epsilon/2)/2 \le \rho(v_n, z_n) \le \epsilon$ on $n \ge 0$. Since $\epsilon < \delta_0$ and $u(n)$ is (K,ρ)-A in $\Omega(F)$, we have $\delta(\epsilon/2)/2 \le \rho(v_n, z_n) \to 0$ as $n \to \infty$, which is a contradiction. Thus $n_k - \tau_k \nrightarrow \infty$ as $k \to \infty$. Taking a subsequence again if necessary, we can assum that $n_k - \tau_k \to r < \infty$ as $k \to \infty$. Letting $k \to \infty$ in (2.32), we have $\rho(v_0, z_0) = \delta(\epsilon/2)/2 < \delta(\epsilon/2)$ and hence $\rho(v_n, z_n) < \epsilon/2$ for all $n \ge 0$, because u is (K,ρ)-US in $\Omega(F)$. On the other hand, from (2.31), we have $\rho(v_n, z_n) \ge \epsilon$, which is a contradiction. This shows that $u(n)$ is (K,ρ)-TS.

Now we will see that the existence of a strictly positive almost periodic solution of System (F) can be obtained under conditions (i), (ii), and (iii). □

Theorem 2.6. *one assumes conditions (i), (ii), and (iii). Then System (F) has a unique almost periodic solution $p(n)$ in compact set K.*

Proof. For System (F), we first introduce the change of variables:

$$u_i(n) = \exp\{v_i(n)\}, \quad x_i(n) = \exp\{y_i(n)\}, \quad i = 1,2. \tag{2.35}$$

Then, System (F) can be written as

$$\begin{aligned} y_1(n+1) - y_1(n) &= b_1(n) - a_1(n)\exp\{y_1(n)\} - c_2(n)\sum_{s=-\infty}^{n}K_2(n-s)\exp\{y_2(s)\}, \\ y_2(n+1) - y_2(n) &= -b_2(n) - a_2(n)\exp\{y_2(n)\} + c_1(n)\sum_{s=-\infty}^{n}K_1(n-s)\exp\{y_1(s)\}, \end{aligned} \tag{$\widehat{F}$}$$

We now consider Liapunov functional:

$$V(v(n), y(n)) = \sum_{i=1}^{2}\left\{\left|v_i(n) - y_i(n)\right| + \sum_{s=0}^{\infty}K_i(s)\sum_{l=n-s}^{n-1}c_i(s+l)\left|\exp\{v_i(l)\} - \exp\{y_i(l)\}\right|\right\}, \tag{2.36}$$

where $y(n)$ and $v(n)$ are solutions of $(\widehat{F})$ which remains in K. Calculating the differences, we have

$$
\begin{aligned}
&\Delta V(v(n), y(n)) \\
&\le |v_1(n+1) - v_1(n)| - \left|y_1(n+1) - y_1(n)\right| \\
&\quad + \sum_{s=0}^{\infty} K_1(s)\left(c_1(s+n)\left|\exp\{v_1(n)\} - \exp\{y_1(n)\}\right| - c_1(n)\left|\exp\{v_1(n-s)\} - \exp\{y_1(n-s)\}\right|\right) \\
&\quad + |v_2(n+1) - v_2(n)| - \left|y_2(n+1) - y_2(n)\right| \\
&\quad + \sum_{s=0}^{\infty} K_2(s)\left(c_2(s+n)\left|\exp\{v_2(n)\} - \exp\{y_2(n)\}\right| - c_2(n)\left|\exp\{v_2(n-s)\} - \exp\{y_2(n-s)\}\right|\right) \\
&= \left|b_1(n) - a_1(n)\exp\{v_1(n)\} - c_2(n)\sum_{s=0}^{\infty} K_2(n-s)\exp\{v_2(s)\}\right| \\
&\quad - \left|b_1(n) - a_1(n)\exp\{y_1(n)\} - c_2(n)\sum_{s=0}^{\infty} K_2(n-s)\exp\{y_2(s)\}\right| \\
&\quad + \sum_{s=0}^{\infty} K_1(s)\left(c_1(s+n)\left|\exp\{v_1(n)\} - \exp\{y_1(n)\}\right|\right. \\
&\qquad\qquad \left. - c_1(n)\left|\exp\{v_1(n-s)\} - \exp\{y_1(n-s)\}\right|\right) \\
&\quad + \left|-b_2(n) - a_2(n)\exp\{v_2(n)\} + c_1(n)\sum_{s=0}^{\infty} K_1(n-s)\exp\{v_1(s)\}\right| \\
&\quad - \left|-b_2(n) - a_2(n)\exp\{y_2(n)\} + c_1(n)\sum_{s=0}^{\infty} K_1(n-s)\exp\{y_1(s)\}\right| \\
&\quad + \sum_{s=0}^{\infty} K_2(s)\left(c_2(s+n)\left|\exp\{v_2(n)\} - \exp\{y_2(n)\}\right|\right. \\
&\qquad\qquad \left. - c_2(n)\left|\exp\{v_2(n-s)\} - \exp\{y_2(n-s)\}\right|\right) \\
&\le \left|b_1(n) - a_1(n)\exp\{v_1(n)\} - c_2(n)\exp\{v_2(n-s)\}\right| \\
&\quad - \left|b_1(n) - a_1(n)\exp\{y_1(n)\} - c_2(n)\exp\{y_2(n-s)\}\right| \\
&\quad + c_1(s+n)\left|\exp\{v_1(n)\} - \exp\{y_1(n)\}\right| - c_1(n)\left|\exp\{v_1(n-s)\} - \exp\{y_1(n-s)\}\right| \\
&\quad + \left|-b_2(n) - a_2(n)\exp\{v_2(n)\} + c_1(n)\exp\{v_1(n-s)\}\right| \\
&\quad - \left|-b_2(n) - a_2(n)\exp\{y_2(n)\} + c_1(n)\exp\{y_1(n-s)\}\right| \\
&\quad + c_2(s+n)\left|\exp\{v_2(n)\} - \exp\{y_2(n)\}\right| - c_2(n)\left|\exp\{v_2(n-s)\} - \exp\{y_2(n-s)\}\right| \\
&\le -a_1(n)\left|\exp\{v_1(n)\} - \exp\{y_1(n)\}\right| + c_1(s+n)\left|\exp\{v_1(n)\} - \exp\{y_1(n)\}\right| \\
&\le -a_2(n)\left|\exp\{v_2(n)\} - \exp\{y_2(n)\}\right| + c_2(s+n)\left|\exp\{v_2(n)\} - \exp\{y_2(n)\}\right|.
\end{aligned}
\tag{2.37}
$$

From the mean value theorem, we have

$$
\left|\exp\{v_i(n)\} - \exp\{y_i(n)\}\right| = \exp\{\theta_i(n)\}\left|v_i(n) - y_i(n)\right|, \quad i = 1, 2, \tag{2.38}
$$

where $\theta_i(n)$ lies between $v_i(n)$ and $y_i(n)$ $(i = 1, 2)$. Then, by (iii), we have

$$\Delta V(v(n), y(n)) \leq -mD \sum_{i=1}^{2} |v_i(n) - y_i(n)|, \tag{2.39}$$

where set $D = \max\{\exp\{\beta_1\}, \exp\{\beta_2\}\}$, and let solutions $x_i(n)$ of System (F) be such that $x_i(n) \geq \beta_i$ for $n \geq n_0$ $(i = 1, 2)$. Thus $\sum_{i=1}^{2} |v_i(n) - y_i(n)| \to 0$ as $n \to \infty$, and hence $\rho(v_n, y_n) \to 0$ as $n \to \infty$. Moreover, we can show that $v(n)$ is (K, ρ)-US in Ω of $(\widehat{F})$, by the same argument as in [5]. By using similar Liapunov functional to (2.36), we can show that $v(n)$ is (K, ρ)-A in Ω of $(\widehat{F})$. Therefore, $v(n)$ is (K, ρ)-WUAS in $\Omega(\widehat{F})$. Thus, from Proposition 2.5, $v(n)$ is (K, ρ)-TS, because K is invariant. By the equivalence between $(\widehat{F})$ and (F), solution $u(n)$ of System (F) is (K, ρ)-TS. Therefore, it follows from Theorem 4.4 in [5] and [9] that System (F) has an almost periodic solution $p(n)$ such that $\beta_i \leq p_i(n) \leq \alpha_i$, $(i = 1, 2)$, for all $n \in Z$. □

3. Competitive System

We will consider the l-species almost periodic competitive Lotka-Volterra system:

$$x_i(n+1) = x_i(n) \exp\left\{ b_i(n) - a_{ii} x_i(n) - \sum_{j=1, j \neq i}^{l} a_{ij}(n) \sum_{s=-\infty}^{n} K_{ij}(n-s) x_j(s) \right\}, \quad i = 1, 2, \ldots, l, \tag{H}$$

where $b_i(n), a_{ij}(n)$ are positive almost periodic sequences on Z; $a_{ij}(n)$ are strictly positive, and, moreover,

$$\begin{gathered} a_{ij} = \inf_{n \in Z} a_{ij}(n), \quad A_{ij} = \sup_{n \in Z} a_{ij}(n), \quad b_i = \inf_{n \in Z} b_i(n), \quad B_i = \sup_{n \in Z} b_i(n), \\ K_{ij} : Z^+ = [0, \infty) \longrightarrow R^+ \quad (i, j = 1, 2, \ldots, l), \end{gathered} \tag{3.1}$$

which can be seen as the discretization of the differential equation in [3]. We set

$$\begin{gathered} \alpha_i = \exp \frac{\{B_i - 1\}}{a_{ii}}, \\ \beta_i = \min\left\{ \frac{\exp\left\{ b_i - A_{ii}\alpha_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j \right\} \left(b_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j \right)}{A_{ii}}, \frac{\left\{ b_i - \sum_{j=1, j\neq i}^{l} \alpha_j \right\}}{A_{ii}} \right\}. \end{gathered} \tag{3.2}$$

Now, we make the following assumptions:

(iv) $K_{ij}(s) \geq 0$, and $\sum_{s=0}^{\infty} K_{ij}(s) = 1$, $\sum_{s=0}^{\infty} s K_{ij}(s) < \infty$ $(i = 1, 2, \ldots, l)$;

(v) $b_i > \sum_{j=1, j\neq i} A_{ij}\alpha_j$ for $i = 1, 2, \ldots, l$;

(vi) there exists a positive constant m such that

$$a_{ii} > \sum_{j=1, j\neq i}^{l} A_{ij} + m \quad (i = 1, 2, \ldots, l). \tag{3.3}$$

Then, we have $0 < \beta_i < \alpha_i$ for each $i = 1, 2, \ldots, l$. Under the assumptions (iv) and (v), it follows that for any $(n_0, \phi) \in Z^+ \times BS$, there is a unique solution $u(n) = (u_1(n), u_2(n), \ldots, u_l(n))$ of (H) through (n_0, ϕ), if it remains bounded.

Then, we can show the similar lemmas to Lemma 2.1.

Lemma 3.1. *If $x(n) = (x_1(n), x_2(n), \ldots, x_l(n))$ is a solution of (H) through (n_0, ϕ) such that $\beta_i \leq \phi(s) \leq \alpha_i$ $(i = 1, 2, \ldots, l)$ for all $s \leq 0$, then one has $\beta_i \leq x_i(n) \leq \alpha_i$ $(i = 1, 2, \ldots, l)$ for all $n \geq n_0$.*

Proof. First, we claim that

$$\limsup_{n\to\infty} x_i(n) \leq B_i, \quad i = 1, 2, \ldots, l. \tag{3.4}$$

Clearly, $x_i(n) > 0$ for $n \geq n_0$. To prove this, we first assume that there exists an $l_0 \geq n_0$ such that $x_i(l_0 + 1) \geq x_i(l_0)$. Then, it follows from the first equation of (H) that

$$b_i(l_0) - a_{ii}(l_0)x_i(l_0) - \sum_{j=1, j\neq i}^{l} a_{ij}(l_0) \sum_{s=-\infty}^{l_0} K_{ij}(l_0 - s)x_j(s) \geq 0. \tag{3.5}$$

Hence

$$x_i(l_0) \leq \frac{b_i(l_0) - \sum_{j=1, j\neq i}^{l} a_{ij}(l_0) \sum_{s=-\infty}^{l_0} K_{ij}(l_0 - s)x_j(s)}{a_{ij}(l_0)} \leq \frac{b_i(l_0)}{a_{ii}(l_0)} \leq \frac{B_i}{a_{ii}}. \tag{3.6}$$

It follows that

$$\begin{aligned} x_i(l_0 + 1) &= x_i(l_0) \exp\left\{ b_i(l_0) - a_{ii}(l_0)x_i(l_0) - \sum_{j=1, j\neq i}^{l} a_{ij}(l_0) \sum_{s=-\infty}^{l_0} K_{ij}(l_0 - s)x_j(s) \right\} \\ &\leq x_i(l_0) \exp\{B_i - a_{ii}x_i(l_0)\} \leq \frac{\exp\{B_i - 1\}}{a_{ii}} := \alpha_i. \end{aligned} \tag{3.7}$$

Now we claim that

$$x_i(n) \leq B_i, \quad \text{for } n \geq l_0. \tag{3.8}$$

By way of contradiction, we assume that there exists a $p_0 > l_0$ such that $x_i(p_0) > \alpha_i$. Then, $p_0 \geq l_0 + 2$. Let $\widehat{p}_0 \geq l_0 + 2$ be the smallest integer such that $x_i(\widehat{p}_0) > \alpha_i$. Then, $x_i(\widehat{p}_0 - 1) \leq x_i(\widehat{p}_0)$. The above argument shows that $x_i(\widehat{p}_0) \le \alpha_i$, which is a contradiction. This proves our assertion. We now assume that $x_i(n+1) < x_i(n)$ for all $n \ge n_0$. Then $\lim_{n\to\infty} x_i(n)$ exists, which is denoted by $\overline{x}_i$. We claim that $\overline{x}_i \le \exp(B_i - 1)/a_{ii}$. Suppose to the contrary that $\overline{x}_i > \exp(B_i - 1)/a_{ii}$. Taking limits in the first equation in System (H), we set that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(b_i(n) - a_{ii}(n)x_i(n) - \sum_{j=1, j\neq i}^{l} a_{ij}(n) \sum_{s=-\infty}^{n} K_{ij}(n-s)x_j(s) \right) \\ &\le \lim_{n\to\infty}(b_i(n) - a_{ii}(n)x_i(n)) \le B_i - a_{ii}\overline{x}_i < 0, \end{aligned} \tag{3.9}$$

which is a contradiction. It follows that (3.4) holds.

We first show that

$$\liminf_{n\to\infty} x_i(n) \ge \beta_i. \tag{3.10}$$

According to above assertion, there exists a $k^* \ge n_0$ such that $x_i(n) \le \alpha_i + \varepsilon$, for all $n \ge k^*$. We assume that there exists an $l_0 \ge k^*$ such that $x_i(l_0 + 1) \le x_i(l_0)$. Note that for $n \ge l_0$,

$$\begin{aligned} x_i(n+1) &= x_i(n)\exp\left\{ b_i(n) - a_{ii}(n)x_i(n) - \sum_{j=1, j\neq i}^{l} a_{ij}(n) \sum_{s=-\infty}^{n} K_{ij}(n-s)x_j(s) \right\} \\ &\ge x_i(n)\exp\left\{ b_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j - A_{ii}x_i(n) \right\}. \end{aligned} \tag{3.11}$$

In particular, with $n = l_0$, we have

$$b_i - A_{ii}x_i(l_0) - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j \le 0, \tag{3.12}$$

which implies that

$$x_i(l_0) \ge \frac{b_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j}{A_{ii}}. \tag{3.13}$$

Then,

$$x_i(l_0 + 1) \ge \frac{b_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j}{A_{ii}} \exp\left(b_i - \sum_{j=1, j\neq i}^{l} A_{ij}\alpha_j - A_{ii}(\alpha_i + \epsilon) \right) := x_{i\epsilon}. \tag{3.14}$$

We assert that

$$x_i(n) \ge x_{i\epsilon}, \quad \forall n \ge l_0. \tag{3.15}$$

By way of contradiction, we assume that there exists a $p_0 \geq l_0$ such that $x_i(p_0) < x_{i\varepsilon}$. Then $p_0 \geq l_0+2$. Let $\widehat{p}_0+2$ be the smallest integer such that $x_i(\widehat{p}_0) < x_{i\varepsilon}$.Then $x_i(\widehat{p}_0) \leq x_i(\widehat{p}_0-1)$. The above argument yields $x_i(\widehat{p}_0) \geq x_{i\varepsilon}$, which is a contradiction. This proves our claim. We now assume that $x_i(n+1) < x_i(n)$ for all $n \geq n_0$. Then $\lim_{n\to\infty} x_i(n)$ exists, which is denoted by $\underline{x}_i$. We claim that $\underline{x}_i \geq (b_i - \sum_{j=1,j\neq i}^{l} A_{ij}\alpha_j)/A_{ii}$. Suppose to the contrary that $\underline{x}_i < (b_i - \sum_{j=1,j\neq i}^{l} A_{ij}\alpha_j)/A_{ii}$. Taking limits in the first equation in System (F), we set that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(b_i(n) - a_{ii}(n)x_i(n) - \sum_{j=1,j\neq i}^{l} a_{ij}(n)\sum_{s=-\infty}^{l} K_{ij}(n-s)x_j(s)\right) \\ &\geq b_i - A_{ii}\underline{x}_i - \sum_{j=1,j\neq i}^{l} A_{ij}\alpha_j > 0, \end{aligned} \tag{3.16}$$

which is a contradiction. It follows that (3.10) holds. This proof is complete. □

By the same arguments of Lemmas 2.2, 2.3 and Proposition 2.5, we obtain Lemmas 3.2, 3.3 and Proposition 3.4. So, we will omit to these proofs.

Lemma 3.2. *Let K be the closed bounded set in R^l such that*

$$K = \left\{(x_1, x_2, \dots, x_l) \in R^l; \beta_i \leq x_i \leq \alpha_i \text{ for each } i = 1, 2, \dots, l\right\}. \tag{3.17}$$

Then K is invariant for System (H), that is, one can see that for any $n_0 \in Z$ and any φ such that $\varphi(s) \in K$, $s \leq 0$, every solution of (H) through (n_0, φ) remains in K for all $n \geq n_0$ and $i = 1, 2, \dots, l$.

Lemma 3.3. *If a compact set K in R^l of Lemma 3.2 is invariant for System (H), then K is invariant also for every limiting equation of System (H).*

Proposition 3.4. *Under the assumption (iv), (v), and (vi), if the solution $u(n)$ of System (H) is (K,ρ)-WUAS in $\Omega(H)$, then the solution $u(n)$ of System (H) is (K,ρ)-TS.*

By making changes of the variables $x_i(n) = \exp\{y_i(n)\}$ and defining the Liapunov functional V by

$$V(v(n), y(n)) = \sum_{i=1}^{l}\left\{\left|v_i(n) - y_i(n)\right| + \sum_{s=0}^{\infty} K_{ij}(s)\sum_{l=n-s}^{n-1} c_i(s+l)\left|\exp\{v_i(l)\} - \exp\{y_i(l)\}\right|\right\}, \tag{3.18}$$

where $y(n)$ and $v(n)$ are solutions of changing equation $(\widehat{H})$ for (H) which remains in K, the arguments similar to the Theorem 2.6 lead to the following results.

Theorem 3.5. *one assumes conditions (iv), (v), and (vi). Then System (H) has a unique almost periodic solution $p(n)$ in compact set K.*

From Theorem 3.5, one obtains the following result, which was proved by Gopalsamy in [10] when System (H) is continuous case.

Corollary 3.6. *Under the assumption (iv), (v), and (vi), suppose that $b_i(n)$ and $a_{ij}(n)$ are positive ω-periodic sequences ($\omega \in Z$) for all $i, j = 1, 2, \ldots, l$. Then System (H) has a unique ω-periodic solution in K.*

4. Examples

For simplicity, we consider the following prey-predator system with finite delay:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{1.5 + \frac{\sin n}{2} - 0.5x_1(n) - \frac{1+\sin n}{16}\sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{s+1} x_2(n-s)\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{-\frac{1+0.2\sin n}{50} - 0.5x_2(n) + \frac{1}{4}\sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{s+1} x_1(n-s)\right\}. \end{aligned} \tag{E_1}$$

Then, we have

$$\begin{gathered} b_1 = 1, \quad B_1 = 2, \quad a_1 = A_1 = 0.5, \quad c_1 = C_1 = \frac{1}{4}, \quad K_1(s) = \left(\frac{1}{2}\right)^{s+1}, \\ b_2 = 0.016, \quad B_2 = 0.024, \quad a_2 = A_2 = 0.5, \quad c_2 = 0, \quad C_2 = \frac{1}{8}, \quad K_2(s) = \left(\frac{1}{2}\right)^{s+1} \end{gathered} \tag{4.1}$$

for System (F). Thus,

$$\alpha_1 \approx 5.436, \quad \alpha_2 \approx 2.818, \quad \beta_1 \approx 0.163. \tag{4.2}$$

It is easy to verify that System (E_1) satisfies all the assumptions in our Theorem 2.6. Thus, System (E_1) has an almost periodic solution.

We next consider the following competitive system with finite delay:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left\{2 + \frac{\sin\sqrt{2}n}{2} - x_1(n) - \frac{1}{16}\sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{s+1} x_2(n-s)\right\}, \\ x_2(n+1) &= x_2(n)\exp\left\{2 + \cos 2n - x_2(n) - \frac{1}{4}\sum_{s=-\infty}^{n}\left(\frac{1}{2}\right)^{s+1} x_1(n-s)\right\}. \end{aligned} \tag{E_2}$$

Then, we have

$$\begin{gathered} b_1 = 1.5, \quad B_1 = 2.5, \quad a_{11} = A_{11} = 1, \quad a_{12} = A_{12} = \frac{1}{8}, \quad K_{12}(s) = \left(\frac{1}{2}\right)^{s+1}, \\ b_2 = 1, \quad B_2 = 3, \quad a_{22} = A_{22} = 1, \quad a_{21} = A_{21} = \frac{1}{4}, \quad K_{21}(s) = \left(\frac{1}{2}\right)^{s+1} \end{gathered} \tag{4.3}$$

for System (E_2). Thus,

$$\alpha_1 \approx 4.077, \qquad \alpha_2 \approx 7.389. \tag{4.4}$$

It is easy to verify that System (E_2) satisfies all the assumptions in Theorem 3.5. Thus, System (E_2) has an almost periodic solution.

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