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Research Article

Convergence Results on a Second-Order Rational Difference Equation with Quadratic Terms

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We investigate the global behavior of the second-order difference equation $x_{n+1} = x_{n-1}((\alpha x_n + \beta x_{n-1})/(Ax_n + Bx_{n-1}))$, where initial conditions and all coefficients are positive. We find conditions on A, B, α, β under which the even and odd subsequences of a positive solution converge, one to zero and the other to a nonnegative number; as well as conditions where one of the subsequences diverges to infinity and the other either converges to a positive number or diverges to infinity. We also find initial conditions where the solution monotonically converges to zero and where it diverges to infinity.

1. Introduction and Preliminaries

There are a number of studies published on second-order rational difference equations (see, e.g., [1–9]). We investigate the global behavior of the second-order difference equation

$$x_{n+1} = x_{n-1}\left(\frac{\alpha x_n + \beta x_{n-1}}{Ax_n + Bx_{n-1}}\right), \tag{1.1}$$

where the numerator is quadratic and the denominator is linear with $A, B, \alpha, \beta \in (0, \infty)$. Under various hypotheses on the parameters, we establish the existence of different behaviors of even and odd subsequences of solutions of (1.1). Our results are summarized below.

(i) Let $\alpha < A$ and $\beta > B$, then we have the following.

(a) There are infinitely many solutions, $\{x_n\}_{n=-1}^{\infty}$, such that for each, one of its subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other diverges to infinity.

(b) There exist solutions, $\{x_n\}_{n=0}^{\infty}$, which

(1) converge to zero if $A + B > \alpha + \beta$;
(2) diverge to infinity if $A + B < \alpha + \beta$;
(3) are constant if $A + B = \alpha + \beta$.

(i) Let $\alpha = A$ and $\beta > B$. Then for each positive solution $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, diverges to infinity and the other to a positive number that can be arbitrarily large depending on initial values. Further there, are positive initial values for which the corresponding solution, $\{x_n\}_{n=-1}^{\infty}$, increases monotonically to infinity.

(ii) Let $\alpha < A$ and $\beta = B$. Then for each positive solution $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other to a nonnegative number. Further, there are positive initial values for which the corresponding solution, $\{x_n\}_{n=-1}^{\infty}$, decreases monotonically to zero.

We note that the following results address and solve the first five conjectures posed by Sedaghat in [10].

2. Results

In order to establish this first result, we reduce (1.1) to a first-order equation by means of the substitution $r_n = x_n/x_{n-1}$. This transforms (1.1) to

$$r_{n+1} = \frac{\alpha r_n + \beta}{A r_n^2 + B r_n}. \tag{2.1}$$

Theorem 2.1. *Let $\alpha < A$ and $\beta > B$ in* (1.1). *Then one has the following.*

(i) *There are infinitely many solutions, $\{x_n\}_{n=-1}^{\infty}$, such that for each, one of its subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other to infinity.*

(ii) *There exist solutions, $\{x_n\}_{n=-1}^{\infty}$, which*

(a) *converge to zero if $A + B > \alpha + \beta$;*
(b) *diverge to infinity if $A + B < \alpha + \beta$;*
(c) *are constant if $A + B = \alpha + \beta$.*

Proof. Starting with (2.1), let the function $g : (0,\infty) \to (0,\infty)$ be defined as $g(r) = (\alpha r + \beta)/(Ar^2 + Br)$. Note that for $r \in (0,\infty)$, $g(r)$ is a decreasing function since $g'(r) = -(A\alpha r^2 + 2A\beta r + B\beta)/(Ar^2 + Br)^2 < 0$. Also note that $\lim_{r\to 0^+}(g(r) - r) = +\infty$ and $\lim_{r\to+\infty}(g(r) - r) = -\infty$. Hence g has a unique positive fixed point $\overline{r}$.

We next compute the expression $g^2(r) - r$ and simplify, it including canceling the common factor $(Ar + B)r$ from the numerator and denominator, thereby obtaining the following:

$$g^2(r) - r = \frac{a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r}{b_3 r^3 + b_2 r^2 + b_1 r + b_0}, \tag{2.2}$$

where

$$\begin{aligned} a_1 &= \beta(B\alpha - A\beta), & b_0 &= A\beta^2, \\ a_2 &= \alpha(B\alpha - A\beta), & b_1 &= 2A\alpha\beta + B^2\beta, \\ a_3 &= B(A\beta - B\alpha), & b_2 &= A\alpha^2 + AB\beta + B^2\alpha, \\ a_4 &= A(A\beta - B\alpha), & b_3 &= AB\alpha. \end{aligned} \tag{2.3}$$

Note that since $A\beta > B\alpha$, $a_1, a_2 < 0$ and $a_3, a_4 > 0$. Thus the numerator of $g^2(r) - r = 0$ has one and only one sign change. Therefore, by Descartes' rule of signs, the numerator of $g^2(r) - r = 0$ has exactly one positive root, $\bar{r}$.

In addition, we see that $\lim_{r\to+\infty} [g^2(r) - r] = +\infty$ and so, given that $\bar{r}$ is the only positive root of the numerator of $g^2(r) - r = 0$, we have $g^2(r) - r > 0$ for $r > \bar{r}$. Thus, since $g^2(0) = 0$ and g^2 is continuous, we must have $g^2(r) - r < 0$ for $r < \bar{r}$. Therefore,

$$\left[g^2(r) - r\right](r - \bar{r}) > 0 \quad \text{for } r \neq \bar{r}. \tag{2.4}$$

We consider two cases depending on the initial value r_0 for (2.1).

Case 1 ($r_0 \in (0, \bar{r})$). Using induction and the fact that g is a decreasing function so that g^2 is an increasing function, we have

$$0 < \cdots < g^4(r_0) < g^2(r_0) < r_0 < \bar{r} < g(r_0) < g^3(r_0) < g^5(r_0) \cdots . \tag{2.5}$$

Thus, $\lim_{n\to\infty} g^{2n}(r_0) \geq 0$ and $\lim_{n\to\infty} g^{2n+1}(r_0) \leq \infty$. Since $\bar{r}$ is the only positive fixed point of g^2, then we must have $\lim_{n\to\infty} g^{2n}(r_0) = 0$ and $\lim_{n\to\infty} g^{2n+1}(r_0) = \infty$.

Case 2 ($r_0 \in (\bar{r}, \infty)$). The argument is similar to that in Case 1 in showing $\lim_{n\to\infty} g^{2n}(r_0) = \infty$ and $\lim_{n\to\infty} g^{2n+1}(r_0) = 0$. In both cases, the solution, $\{r_n\}_{n=0}^{\infty}$, of (2.1) is divided into even and odd subsequences, $\{r_{2n}\}_{n=0}^{\infty}$ and $\{r_{2n+1}\}_{n=0}^{\infty}$, where one subsequence converges monotonically to zero and the other to infinity.

We now go back to (1.1) by inferring the behavior of x_n from r_n. To do this we first consider $r_0 \neq \bar{r}$. Without loss of generality, we will assume that $0 < r_0 < \bar{r}$ and so $\lim_{n\to\infty} g^{2n}(r_0) = \infty$ and $\lim_{n\to\infty} g^{2n+1}(r_0) = 0$.

Next, observe that

$$\frac{x_{2n+2}}{x_{2n}} = \frac{x_{2n+2}}{x_{2n+1}} \cdot \frac{x_{2n+1}}{x_{2n}} = r_{2n+2} r_{2n+1} = \frac{\alpha r_{2n+1} + \beta}{A r_{2n+1}^2 + B r_{2n+1}} \cdot r_{2n+1} = \frac{\alpha r_{2n+1} + \beta}{A r_{2n+1} + B}. \tag{2.6}$$

From this and our assumption with g^{2n+1}, we have

$$\lim_{n\to\infty} \frac{x_{2n+2}}{x_{2n}} = \lim_{n\to\infty} \frac{\alpha r_{2n+1} + \beta}{A r_{2n+1} + B} = \frac{\beta}{B} > 1. \tag{2.7}$$

Hence, for $0 < \epsilon < \beta/B - 1$, there exists $N \geq 0$ such that

$$1 < \frac{\beta}{B} - \epsilon < \frac{x_{2n+2}}{x_{2n}} < \frac{\beta}{B} + \epsilon \tag{2.8}$$

for all $n \geq N$. We then have

$$\begin{aligned} x_{2(N+1)} &> \left(\frac{\beta}{B} - \epsilon\right)^1 x_{2N} \\ x_{2(N+2)} &> \left(\frac{\beta}{B} - \epsilon\right)^1 x_{2(N+1)} > \left(\frac{\beta}{B} - \epsilon\right)^2 x_{2N} \\ x_{2(N+3)} &> \left(\frac{\beta}{B} - \epsilon\right)^1 x_{2(N+2)} > \left(\frac{\beta}{B} - \epsilon\right)^3 x_{2N} \end{aligned} \tag{2.9}$$

and by induction, for $m \geq 1$,

$$x_{2(N+m)} > \left(\frac{\beta}{B} - \epsilon\right)^m x_{2N}. \tag{2.10}$$

This, in turn, implies that

$$\lim_{n \to \infty} x_{2n+2} = \infty. \tag{2.11}$$

The argument is similar in showing that $\lim_{n \to \infty} x_{2n+1} = 0$, since

$$\frac{x_{2n+1}}{x_{2n-1}} = \frac{x_{2n+1}}{x_{2n}} \cdot \frac{x_{2n}}{x_{2n-1}} = r_{2n+1} r_{2n} = \frac{\alpha r_{2n} + \beta}{A r_{2n}^2 + B r_{2n}} \cdot r_{2n} = \frac{\alpha r_{2n} + \beta}{A r_{2n} + B}. \tag{2.12}$$

Hence, result (i) is true.

Now consider $r_0 = \overline{r}$. Then $r_n = \overline{r}$ for all $n \geq 1$, and so $x_n / x_{n-1} = \overline{r}$ for all $n \geq 1$. Induction then gives us $x_n = \overline{r}^{n+1} x_{-1}$ for all $n \geq 1$. We thus have one of the following:

(1) If $\overline{r} < 1$ $(A + B > \alpha + \beta)$, then $\lim_{n \to \infty} x_n = 0$.

(2) If $\overline{r} > 1$ $(A + B < \alpha + \beta)$, then $\lim_{n \to \infty} x_n = \infty$.

(3) If $\overline{r} = 1$ $(A + B = \alpha + \beta)$, then $\{x_n\}_{n=-1}^{\infty}$ is a constant solution $x_{-1} = x_0 = x_1 = \cdots$.

Thus the result (ii) is true and this completes the proof. □

For the next couple of results we rewrite (1.1) in the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{2.13}$$

Note that if either $\alpha \leq A$ and $\beta < B$, or $\alpha < A$ and $\beta \leq B$, then f satisfies the following properties:

(P1) $f \in C[[0,\infty)^2 - \{0,0\}, [0,\infty)]$, with $f(u,v)$ undefined when $u = v = 0$.

(P2) $f \in C[[0,\infty) \times (0,\infty), (0,\infty)]$

(P3) $f(u,v) < v$ if $u, v \in (0,\infty)$.

If we consider the addition restriction that $\alpha < A$ and $\beta = B$, we also obtain

(P4) if $f(u,v) = v$, then $u = 0,\ v > 0$, or $u > 0,\ v = 0$.

Lemma 2.2. *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of* (1.1) *with $\alpha < A$ and $\beta = B$. Then there exist $L_o \geq 0$ and $L_e \geq 0$ such that the following statements are true:*

(1) $x_{2n-1} \downarrow L_o$ *as* $n \to \infty$,

(2) $x_{2n} \downarrow L_e$ *as* $n \to \infty$,

(3) $L_o = L_e = 0$, *and* $f(L_o, L_e)$ *and* $f(L_e, L_o)$ *are undefined; or if either L_o or L_e is not zero, then* $(L_o, L_e, L_o, L_e, \ldots)$ *is a solution of* (1.1).

(4) $L_o \cdot L_e = 0$.

Proof. Statements 1 and 2 follow from the fact that

$$0 < x_{2n+1} = f(x_{2n}, x_{2n-1}) < x_{2n-1}, \qquad 0 < x_{2n+2} = f(x_{2n+1}, x_{2n}) < x_{2n} \tag{2.14}$$

by properties (P2) and (P3). Statement 3 follows from the fact that either $L_o = L_e = 0$, and so $f(L_o, L_e)$ and $f(L_e, L_o)$ are undefined by property (P1); or $L_o \neq L_e$ and

$$\begin{aligned} L_o &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, x_{2n-1}) = f(L_e, L_o) \\ L_e &= \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} f(x_{2n+1}, x_{2n}) = f(L_o, L_e), \end{aligned} \tag{2.15}$$

where Statements 1 and 2 and the continuity of f (Property (P1)) hold. Finally, Statement 4 follows immediately from Statement 3 and Property (P4). □

In the first three results, we characterize the convergence of the odd and even subsequences of solutions of (1.1).

Theorem 2.3. *Let $\alpha < A$ and $\beta = B$ in* (1.1). *Then for each positive solution, $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other to a nonnegative number.*

Proof. Consider (1.1) with $\alpha < A, \beta = B$, and $f(u,v) = v((\alpha u + \beta v)/(Au + Bv))$. Then it follows from Lemma 2.2 that for each positive solution of (1.1), $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other to a nonnegative number. □

Theorem 2.4. *Let $\alpha = A$ and $\beta > B$ in* (1.1). *Then for each positive solution $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, diverges to infinity and the other to a positive number or diverges to infinity.*

Proof. Consider (1.1) with $\alpha = A$ and $\beta > B$. Using the transformation $y_n = 1/x_n$, convert (1.1) to the equation

$$y_{n+1} = y_{n-1}\left(\frac{By_n + Ay_{n-1}}{\beta y_n + \alpha y_{n-1}}\right). \tag{2.16}$$

Then $f(u,v) = v((Av+Bu)/(\alpha v+\beta u))$, and so it follows from Lemma 2.2 that for each positive solution of (2.16), $\{y_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{y_{2n}\}_{n=0}^{\infty}$, $\{y_{2n-1}\}_{n=0}^{\infty}$, converges to zero and the other to a nonnegative number. Hence, for each positive solution of (1.1), $\{x_n\}_{n=-1}^{\infty}$, one of the subsequences, $\{x_{2n}\}_{n=0}^{\infty}$, $\{x_{2n-1}\}_{n=0}^{\infty}$, diverges to infinity and the other to a positive number or diverges to infinity. □

In the following results, we show the existence of monotonic solutions for (1.1). As with Theorem 2.1 we use the substitution $r_n = x_n/x_{n-1}$.

Theorem 2.5. *Let $\alpha < A$ and $\beta = B$ in (1.1). Then there are positive initial values for which the corresponding solutions, $\{x_n\}_{n=-1}^{\infty}$, decrease monotonically to zero.*

Proof. Note that an equilibrium equation for (2.1) satisfies,

$$Ar^3 + Br^2 - \alpha r - \beta = 0. \tag{2.17}$$

Set $p(r) = Ar^3 + Br^2 - \alpha r - \beta$. Given Descartes' rule of signs, we have that there exists a unique positive equilibrium, $\overline{r} < 1$, where $p(0) < 0$ and $p(1) > 0$. Recall that $r_n = x_n/x_{n-1}$, and let $r_n = \overline{r}$ for all $n \geq 0$. Then $x_n/x_{n-1} = \overline{r}$ for all $n \geq 0$. It follows from induction that $x_n = \overline{r}^{n+1}x_{-1}$ for all $n \geq 0$. Since $\overline{r} < 1$, $\{x_n\}_{n=-1}^{\infty}$, with $x_0 = \overline{r}x_{-1}$, decreases monotonically to zero. □

Theorem 2.6. *Let $\alpha = A$ and $\beta > B$ in (1.1). Then there are positive initial values for which the corresponding solution, $\{x_n\}_{n=-1}^{\infty}$, increases monotonically to infinity.*

Proof. As in the previous proof, an equilibrium equation for (2.1) satisfies (2.17). Setting $p(r) = Ar^3 + Br^2 - \alpha r - \beta$, we obtain from Descartes' rule of signs, a unique positive equilibrium, $\overline{r} > 1$, where $p(0) < 0$ and $\lim_{r\to\infty} p(r) > 0$. Recall that $r_n = x_n/x_{n-1}$, and let $r_n = \overline{r}$ for all $n \geq 0$. Then $x_n/x_{n-1} = \overline{r}$ for all $n \geq 0$. It follows from induction that $x_n = \overline{r}^{n+1}x_{-1}$ for all $n \geq 0$. Since $\overline{r} > 1$, $\{x_n\}_{n=-1}^{\infty}$, with $x_0 = \overline{r}x_{-1}$, increases monotonically to infinity. □

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