

## Research Article

# On Exact Controllability of First-Order Impulsive Differential Equations

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Many dynamical systems have an impulsive dynamical behavior due to abrupt changes at certain instants during the evolution process. The mathematical description of these phenomena leads to impulsive differential equations. In this work, we present some new results concerning the exact controllability of a nonlinear ordinary differential equation with impulses.

## 1. Introduction

Many evolution processes in nature are characterized by the fact that at certain moments in time they experience an abrupt change of state. Such behavior is seen in a range of problems from: mechanics; chemotherapy; population dynamics; optimal control; ecology; industrial robotics; biotechnology; spread of disease; harvesting; physics; medical models. The reader is referred to [1–8] and references therein for some models and applications to the above areas.

The branch of modern, applied analysis known as “impulsive” differential equations furnishes a natural framework to mathematically describe the aforementioned jumping processes. Consequently, the area of impulsive differential equations has been developing at a rapid rate, with the wide applications significantly motivating a deeper theoretical study of the subject [9–11].

Impulsive control systems have been studied by several authors [12–18]. In [15] the problem of controlling a physical object through impacts is studied, called *impulsive manipulation*, which arises in a number of robotic applications. In [16] the authors investigated the optimal harvesting policy for an ecosystem with impulsive harvest. For some recent references on different control strategies, including impulsive control, we refer the reader to [13, 19–26] and the references therein.

Now, let  $T > 0$  and  $0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m < t_{m+1} = T$ ,  $J = [0, T]$ , and  $J' = [0, T] \setminus \{t_1, t_2, \dots, t_m\}$ , and  $\lambda$  is an  $n \times n$  real matrix.

Consider the following impulsive control differential equation:

$$x'(t) + \lambda x(t) = f(t, x(t)) + [Bu](t), \quad t \in [0, T], \quad t \neq t_j, \quad j = 1, 2, \dots, m, \quad (1.1)$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \quad j = 1, 2, \dots, m, \quad (1.2)$$

where  $f : J' \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $B$  is an operator defined on a set of admissible controls  $u$ , and  $I_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $j = 1, 2, \dots, m$ .

As usual,  $x(t_1^-) := \lim_{t \rightarrow t_1^-} x(t)$  and  $x(t_1^+) := \lim_{t \rightarrow t_1^+} x(t)$ .

Our purpose is to the system (1.1)-(1.2) from the initial state  $x_0$  to a desired final state  $x_1$  in the finite time  $T$ .

We say that system (1.1)-(1.2) is exactly controllable in the time  $T > 0$  if for any  $x_0, x_1$  there exists a control  $u$  such that a solution  $x$  of (1.1)-(1.2) satisfies  $x(0) = x_0$ ,  $x(T) = x_1$ . Of course, we specify below the space of solutions and controls.

The main idea of our approach is to transform the controllability problem to the existence of a fixed point of an appropriate nonlinear operator generated by the original problem. This approach is not new and has been used by some authors such as Bhat [27], Chang et al. [28, 29], Sakthivel et al. [30], and Tonkov [31].

## 2. Some General Results on Exact Controllability

Consider the following finite-dimensional linear system:

$$x'(t) = Ax(t) + bu(t), \quad (2.1)$$

where  $A$  is an  $n \times n$  matrix, and  $b \in \mathbb{R}^n$ .

This linear system is completely controllable if for any  $x_0, x_1 \in \mathbb{R}^n$  there exists a  $T > 0$  and a control function  $u(t)$  defined for  $t \in [0, T]$  such that the solution to (2.1) with initial condition  $x(0) = x_0$  satisfies  $x(T) = x_1$ . It is well known [32] that the linear system (2.1) is completely controllable if and only if

$$\text{rank}[b \quad Ab \quad A^2b \quad \dots \quad A^{n-1}b] = n. \quad (2.2)$$

If the system is infinite dimensional, that is,

$$x'(t) = Ax(t) + bu(t), \quad (2.3)$$

where  $A$  is the infinitesimal generator of a strongly continuous semigroup in a Hilbert space  $X$ , and  $b : U \rightarrow X$  a linear bounded operator from a Hilbert space  $U$  into  $X$ , then if the semigroup is compact the linear system (2.3) is not exactly controllable [33–35].

In this paper we study the finite-dimensional nonlinear impulsive control problem (1.1)-(1.2).

### 3. Exact Controllability without Impulses

Consider (1.1) without impulses, that is,

$$x'(t) + \lambda x(t) = f(t, x(t)) + [Bu](t), \quad t \in [0, T], \quad (3.1)$$

with the initial condition

$$x(0) = x_0. \quad (3.2)$$

Here,  $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  continuous and  $B : C[0, T] \rightarrow C[0, T]$ .

In what follows,  $Bu = u$ , and hence if  $x \in C^1[0, T]$  is a solution of the initial problem (3.1)-(3.2), then

$$x(t) = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} [f(s, x(s)) + u(s)] ds. \quad (3.3)$$

We can define the following operator defined by the right-hand side of (3.3):

$$\begin{aligned} \mathcal{F} : C[0, T] \times C[0, T] &\longrightarrow C[0, T], \\ [\mathcal{F}(x, u)](t) &= e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} [f(s, x(s)) + u(s)] ds. \end{aligned} \quad (3.4)$$

In what follows  $B$  is the identity operator so that the control space is  $U = \mathbb{R}^n$ . Note that  $\mathcal{F} = \mathcal{F}_{x_0}$  depends on the initial condition  $x_0$  and for any control

$$[\mathcal{F}(x, u)](0) = x_0. \quad (3.5)$$

Now, for  $x_1 \in \mathbb{R}^n$ , suppose that we are able to find a control  $u = \mathcal{G}(x)$  such that

$$[\mathcal{F}(x, \mathcal{G}(x))](0) = x_0, \quad [\mathcal{F}(x, \mathcal{G}(x))](T) = x_1. \quad (3.6)$$

This means that for the control  $u = \mathcal{G}(x)$ , the system transfers the initial state to the desired final state if

$$\mathcal{L} = \mathcal{F}(\cdot, \mathcal{G}(\cdot)) \quad (3.7)$$

has a fixed point. Consequently, if the operator  $\mathcal{L}$ , which of course depends on the initial state  $x_0$ , has a fixed point for any initial state, then the system is exactly controllable.

To clarify the ideas exposed above, suppose that  $Bu = u$  and let us introduce the control

$$u(t) = \frac{1}{T} e^{\lambda(T-t)} \left[ x_1 - e^{-\lambda T} x_0 - \int_0^T e^{-\lambda(T-s)} f(s, x(s)) ds \right]. \quad (3.8)$$

Define  $\mathcal{G} : C[0, T] \rightarrow C[0, T]$  by the right-hand side in (3.8)

$$[\mathcal{G}(x)](t) = \frac{1}{T} e^{\lambda(T-t)} \left[ x_1 - e^{-\lambda T} x_0 - \int_0^T e^{-\lambda(T-s)} f(s, x(s)) ds \right]. \quad (3.9)$$

Thus, for this control,

$$\begin{aligned} [\mathcal{H}(x)](T) &= [\mathcal{F}(x, \mathcal{G}(x))](T) = e^{-\lambda T} x_0 + \int_0^T e^{-\lambda(T-s)} [f(s, x(s)) + u(s)] ds \\ &= e^{-\lambda T} x_0 \\ &\quad + \int_0^T e^{-\lambda(T-s)} \left[ f(s, x(s)) + \frac{1}{T} e^{\lambda(T-t)} \left[ x_1 - e^{-\lambda T} x_0 - \int_0^T e^{-\lambda(T-\tau)} f(\tau, x(\tau)) d\tau \right] \right] ds \\ &= x_1. \end{aligned} \quad (3.10)$$

We have thus the following result.

**Theorem 3.1.** *If for any initial condition  $x_0$  and final condition  $x_1$  the operator*

$$\mathcal{H} : C[0, T] \longrightarrow C[0, T] \quad (3.11)$$

*has a fixed point, then system (3.1) with  $Bu = u$  is exactly controllable.*

#### 4. Exact Controllability with Impulses

As usual, see any of the references on impulsive differential equations, we consider the Banach space

$$\begin{aligned} PC([0, T], \mathbb{R}^n) &= \left\{ x : [0, T] \longrightarrow \mathbb{R}^n : x \in C(J', \mathbb{R}^n), \right. \\ &\quad \left. x \text{ is left continuous at } t_j \text{ and the right - hand limits } x(t_j^+) \text{ exist} \right\} \end{aligned} \quad (4.1)$$

with the norm

$$\|x\|_{PC} = \sup_{t \in [0, T]} \|x(t)\|. \quad (4.2)$$

Let  $J_j = [t_{j-1}, t_j]$ ,  $j = 1, 2, \dots, p$ , and  $x_j$  the restriction of  $x$  to that subinterval  $J_j$ .

The space

$$PC^1([0, T], \mathbb{R}^n) = \left\{ x \in PC([0, T], \mathbb{R}^n) : x_j \in C^1(J_j, \mathbb{R}^n), \text{ and the limits } x'(t_j^-), x'(t_j^+) \text{ exist} \right\} \quad (4.3)$$

with the norm

$$\|u\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\} \tag{4.4}$$

is a Banach space.

Now consider the impulsive control differential equation

$$x'(t) + \lambda x(t) = f(t, x(t)) + u(t), \quad t \in [0, T], \quad t \neq t_j, \quad j = 1, 2, \dots, m, \tag{4.5}$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \quad j = 1, 2, \dots, m, \tag{4.6}$$

where  $f : J' \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and there exist the limits

$$\begin{aligned} f(t_j^+, x) &= \lim_{t \rightarrow t_j^+} f(t, x), & f(t_j^-, x) &= \lim_{t \rightarrow t_j^-} f(t, x), \\ f(t_j, x) &= f(t_j^-, x), \end{aligned} \tag{4.7}$$

$u \in PC[0, T]$  and  $I_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous

By a solution of (4.5)-(4.6), and for continuous controls  $u(\cdot)$ , we mean a function  $x \in PC^1([0, T], \mathbb{R}^n)$  satisfying (4.5) for every  $t \in J'$  and the impulses (4.6). In the case that the control  $u$  is, for example, in the space  $L^2$  locally, the solution must satisfy (4.5) for almost every  $t \in J_j$  for each  $j = 0, 1, \dots, p$  and the impulses indicated in (4.6).

**Lemma 4.1.** *If  $x \in PC^1([0, T], \mathbb{R}^n)$  is a solution of (4.5)-(4.6), then  $x$  satisfies*

$$x(t) = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} [f(s, x(s)) + u(s)] ds + \sum_{0 < t_j < t} e^{-\lambda(T-t_j)} I_j(x(t_j)). \tag{4.8}$$

Reciprocally, if  $x \in PC([0, T], \mathbb{R}^n)$  satisfies (4.8), then  $x$  is a solution of (4.5)-(4.6).

See [2] for the proof.

Now, define the operators  $\mathcal{F} : PC([0, T], \mathbb{R}^n) \times PC([0, T], \mathbb{R}^n) \rightarrow PC([0, T], \mathbb{R}^n)$  by

$$[\mathcal{F}(x, u)](t) = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} [f(s, x(s)) + u(s)] ds + \sum_{0 < t_j < t} e^{-\lambda(T-t_j)} I_j(x(t_j)), \tag{4.9}$$

and  $\mathcal{G} : PC([0, T], \mathbb{R}^n) \rightarrow PC([0, T], \mathbb{R}^n)$  by

$$[\mathcal{G}(x)](t) = \frac{1}{T} e^{\lambda(T-t)} \left[ x_1 - e^{-\lambda T} x_0 - \int_0^T e^{-\lambda(T-s)} f(s, x(s)) ds - \sum_{j=1}^m e^{-\lambda(T-t_j)} I_j(x(t_j)) \right]. \tag{4.10}$$

As in the nonimpulsive case, this control  $u = \mathcal{G}x$  steers the system for the initial state  $x_0$  to the final state  $x_1$  in the finite time  $T > 0$ .

Consequently we have the following result.

**Theorem 4.2.** *If  $\mathcal{L} = \mathcal{F}(\cdot, \mathcal{G}(\cdot)) : PC[0, T] \rightarrow PC[0, T]$  has a fixed point for any initial condition  $x_0$  and final condition  $x_1$ , then the impulsive system (4.5)-(4.6) is exactly controllable.*

## 5. Main Results

Schauder fixed point theorem states that any continuous mapping of a nonempty convex subset of a normed space into a compact set of that normed space has a fixed point [36, Theorem 4.1.1]. One of the most useful consequences is Schaefer's theorem [36, Theorem 4.3.2].

**Theorem 5.1.** *Let  $E$  be a normed space with  $\mathcal{L} : E \rightarrow E$  a compact mapping. If the set*

$$S := \{x \in E : x = \mu \mathcal{L}(x) \text{ for some } \mu \in [0, 1)\} \quad (5.1)$$

*is bounded then  $\mathcal{L}$  has at least one fixed-point.*

The operators  $\mathcal{F}$  and  $\mathcal{G}$  are continuous and compact [2, 37]. Consequently,  $\mathcal{L}$  is also continuous and compact and we can apply Schaefer's theorem.

**Theorem 5.2.** *Suppose that  $f$  has a sublinear growth, that is, there exist constants  $a, b \geq 0$  and  $0 < \alpha < 1$  such that for every  $t \in J, x \in \mathbb{R}^n$*

$$|f(t, x)| \leq a + b|x|^\alpha. \quad (5.2)$$

*Assume that the impulses have sublinear growth. For every  $j = 1, 2, \dots, m$ , there exist  $a_j, b_j \geq 0$  and  $0 < \alpha_j < 1$  such that for every  $x \in \mathbb{R}^n$  one has*

$$|I_j(x)| \leq a_j + b_j|x|^{\alpha_j}, \quad (5.3)$$

*then the operator  $\mathcal{L}$  has a fixed point for any  $x_0, x_1$  and the impulsive system (4.5)-(4.6) is exactly controllable.*

*Proof.* Let  $x \in PC([0, T], \mathbb{R}^n)$ . Using (5.2) and (5.3), it is evident that for any  $t \in [0, T]$

$$|[\mathcal{G}(x)](t)| \leq c_0 + c'_0 \|x\|^\alpha + \sum_{j=1}^m c_j \|x\|^{\alpha_j}, \quad (5.4)$$

where  $c_0, c'_0, c_j, j = 1, 2, \dots, m$  are constants.

Also, there exist constants  $d_0, d'_0, d''_0, d_j, j = 1, 2, \dots, m$  such that for any  $t \in [0, T]$ ,

$$|[\mathcal{F}(x, u)](t)| \leq d_0 + d'_0 \|x\|^\alpha + d''_0 \|u\| + \sum_{j=1}^m d_j \|x\|^{\alpha_j}. \quad (5.5)$$

Combining these two last inequalities we get

$$|[\mathcal{L}(x)](t)| \leq e_0 + e'_0 \|x\|^\alpha + \sum_{j=1}^m e_j \|x\|^{\alpha_j}, \quad t \in [0, T], \tag{5.6}$$

for some constants  $e_0, e'_0, e_j, j = 1, 2, \dots, m$ . Consequently, for any  $x \in PC([0, T], \mathbb{R}^n)$  we have

$$\|\mathcal{L}(x)\| \leq e_0 + e'_0 \|x\|^\alpha + \sum_{j=1}^m e_j \|x\|^{\alpha_j}. \tag{5.7}$$

If  $x \in PC([0, T], \mathbb{R}^n)$  is a solution of the equation  $x = \mu \mathcal{L}(x)$ ,  $\mu \in (0, 1)$ , then

$$\|x\| = \|\mu \mathcal{L}(x)\| \leq \|\mathcal{L}(x)\| \leq e_0 + e'_0 \|x\|^\alpha + \sum_{j=1}^m e_j \|x\|^{\alpha_j}. \tag{5.8}$$

For  $\theta > 0$ , define  $\Psi(\theta) = \theta - e_0 - e'_0 \theta^\alpha + \sum_{j=1}^m e_j \theta^{\alpha_j}$ . Noting that  $\alpha, \alpha_j \in [0, 1)$ , we see that  $\lim_{\theta \rightarrow +\infty} \Psi(\theta) = +\infty$ . Taking  $\theta = \|x\|$  we deduce that the set  $\{\theta > 0 : \Psi(\theta) \leq 0\}$  is a bounded set.

Hence all the possible solutions of the equation  $x = \mu \mathcal{L}(x), \mu \in (0, 1)$ , are bounded "a priori." By Schaeffer's theorem,  $\mathcal{L}$  has a fixed point, which is equivalent to the exact controllability of the impulsive system (4.5)-(4.6).

As a consequence we have the following. □

**Theorem 5.3.** *Assume that  $f$  is bounded and the impulses  $I_j, j = 1, 2, \dots, m$  are also bounded. then the operator  $\mathcal{L}$  has a fixed point for any  $x_0, x_1$  and the impulsive system (4.5)-(4.6) is exactly controllable.*

When the nonlinearity  $f$  is bounded, (4.5) is not exactly controllable in general. Even in the linear case  $f(t, x) = Ax$  the equation is not exactly controllable in general; see condition (2.2). However, by adding adequate impulses we can control the equation and hence the system becomes exactly controllable.

*Example 5.4.* Let  $M$  be an  $n \times n$  real matrix. Consider the system

$$x' = Mx - \lambda x + bu \tag{5.9}$$

such that  $A = M - \lambda I$  does not satisfy (2.2). Then, (5.9) is not completely controllable. However, by adding the impulse

$$I_1(x(t_1)) = \sqrt{\|x(t_1)\|} \tag{5.10}$$

for some  $t_1 \in (0, T)$ , the impulsive system (5.9)-(5.10) is completely controllable.

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