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Research Article

Complete Asymptotic Analysis of a Nonlinear Recurrence Relation with Threshold Control

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We consider a three-term nonlinear recurrence relation involving a nonlinear filtering function with a positive threshold λ. We work out a complete asymptotic analysis for all solutions of this equation when the threshold varies from 0^+ to $+\infty$. It is found that all solutions either tends to 0, a limit 1-cycle, or a limit 2-cycle, depending on whether the parameter λ is smaller than, equal to, or greater than a critical value. It is hoped that techniques in this paper may be useful in explaining natural bifurcation phenomena and in the investigation of neural networks in which each neural unit is inherently governed by our nonlinear relation.

1. Introduction

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$. In [1], Zhu and Huang discussed the periodic solutions of the following difference equation:

$$x_n = ax_{n-1} + (1-a) f_\lambda(x_{n-k}), \quad n \in \mathbf{N}, \tag{1.1}$$

where $a \in (0,1)$, k is a positive integer, and $f : \mathbf{R} \to \mathbf{R}$ is a nonlinear signal filtering function of the form

$$f_\lambda(x) = \begin{cases} 1, & x \in (0, \lambda], \\ 0, & x \in (-\infty, 0] \cup (\lambda, \infty), \end{cases} \tag{1.2}$$

in which the *positive* number λ can be regarded as a threshold parameter.

In this paper, we consider the following delay difference equation:

$$x_n = ax_{n-2} + bf_\lambda(x_{n-1}), \quad n \in \mathbf{N}, \tag{1.3}$$

where $a \in (0,1)$ and $b > 0$. Besides the obvious and complementary differences between (1.1) and our equation, a good reason for studying (1.3) is that the study of its behavior is preparatory to better understanding of more general (neural) network models. Another one is that there are only limited materials on basic asymptotic behavior of discrete time dynamical systems with piecewise smooth nonlinearities! (Besides [1], see [2–6]. In particular, in [2], Chen considers the equation

$$x_n = x_{n-1} + g(x_{n-k-1}), \quad n \in \mathbf{N}, \tag{1.4}$$

where k is a nonnegative integer and $g : \mathbf{R} \to \mathbf{R}$ is a McCulloch-Pitts type function

$$g(\xi) = \begin{cases} -1, & \xi \in (\sigma, \infty), \\ 1, & \xi \in (-\infty, \sigma], \end{cases} \tag{1.5}$$

in which $\sigma \in \mathbf{R}$ is a constant which acts as a threshold. In [3], convergence and periodicity of solutions of a discrete time network model of two neurons with Heaviside type nonlinearity are considered, while "polymodal" discrete systems in [4] are discussed in general settings.) Therefore, a complete asymptotic analysis of our equation is essential to further development of polymodal discrete time dynamical systems.

We need to be more precise about the statements to be made later. To this end, we first note that given $x_{-2}, x_{-1} \in \mathbf{R}$, we may compute from (1.3) the numbers $x_0, x_1, x_2, \dots$ in a unique manner. The corresponding sequence $\{x_n\}_{n=-2}^{\infty}$ is called the solution of (1.3) determined by the initial vector (x_{-2}, x_{-1}). For better description of latter results, we consider initial vectors in different regions in the plane. In particular, we set

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 \mid x > 0 \text{ or } y > 0 \right\}, \tag{1.6}$$

which is the complement of nonpositive orthant $(-\infty, 0]^2$ and contains the positive orthant $(0, \infty)^2$. Note that Ω is the union of the disjoint sets

$$U = (0, \infty)^2 \setminus (0, \lambda]^2, \tag{1.7}$$

$$V = (0, \lambda]^2 \cup ((-\infty, 0] \times (0, +\infty)) \cup ((0, +\infty) \times (-\infty, 0]). \tag{1.8}$$

Recall also that a positive integer η is a period of the sequence $\{w_n\}_{n=\alpha}^{\infty}$ if $w_{\eta+n} = w_n$ for all $n \geq \alpha$ and that τ is the least or prime period of $\{w_n\}_{n=\alpha}^{\infty}$ if τ is the least among all periods of $\{w_n\}_{n=\alpha}^{\infty}$. The sequence $\{w_n\}_{n=\alpha}^{\infty}$ is said to be τ-periodic if τ is the least period of $\{w_n\}_{n=\alpha}^{\infty}$. The sequence $w = \{w_n\}_{n=\alpha}^{\infty}$ is said to be asymptotically periodic if there exist real numbers

$w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}$, where ω is a positive integer, such that

$$\lim_{n\to\infty} w_{\omega n+i} = w^{(i)}, \quad i = 0, 1, \ldots, \omega - 1. \tag{1.9}$$

In case $\{w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}, w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)}, \ldots\}$ is an ω-periodic sequence, we say that w is an asymptotically ω-periodic sequence tending to the limit ω-cycle (This term is introduced since the underlying concept is similar to that of the limit cycle in the theory of ordinary differential equations.) $\langle w^{(0)}, w^{(1)}, \ldots, w^{(\omega-1)} \rangle$. In particular, an asymptotically 1-periodic sequence is a convergent sequence and conversely.

Note that (1.3) is equivalent to the following two-dimensional *autonomous* dynamical system:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} v_n \\ au_n + bf_\lambda(v_n) \end{pmatrix}, \quad n \in \mathbf{N}, \tag{1.10}$$

by means of the identification $x_n = u_{n+2}$ for $n \in \{-2, -1, 0, \ldots\}$. Therefore our subsequent results can be interpreted in terms of the dynamics of plane vector sequences defined by (1.10). For the sake of simplicity, such interpretations will be left in the concluding section of this paper.

To obtain complete asymptotic behavior of (1.3), we need to derive results for solutions of (1.3) determined by vectors in the entire plane. The following easy result can help us to concentrate on solutions determined by vectors in Ω.

Theorem 1.1. *A solution $\{x_n\}_{n=-2}^{\infty}$ of* (1.3) *with (x_{-2}, x_{-1}) in the nonpositive orthant $(-\infty, 0]^2$ is nonpositive and tends to* 0.

Proof. Let $x_{-2}, x_{-1} \leq 0$. Then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \leq 0, \\ x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} \leq 0, \\ x_2 &= ax_0 + bf_\lambda(x_1) = a^2 x_{-2} \leq 0, \\ x_3 &= ax_1 + bf_\lambda(x_2) = a^2 x_{-1} \leq 0, \end{aligned} \tag{1.11}$$

and by induction, for any $k \in \mathbf{N}$, we have

$$\begin{aligned} x_{2k} &= a^{k+1} x_{-2} \leq 0\ , \\ x_{2k+1} &= a^{k+1} x_{-1} \leq 0. \end{aligned} \tag{1.12}$$

Since $a \in (0, 1)$, we have

$$\lim_{n\to\infty} x_n = 0. \tag{1.13}$$

The proof is complete. □

Note that if we try to solve for an equilibrium solution $\{x\}$ of (1.3), then

$$x = \frac{b}{1-a} f_\lambda(x), \tag{1.14}$$

which has exactly two solutions $x = 0,\ b/(1-a)$ when $\lambda \geq b/(1-a)$ and has the unique solution $x = 0$ when $\lambda \in (0, b/(1-a))$. However, since f_λ is a discontinuous function, the standard theories that employ continuous arguments cannot be applied to our equilibrium solutions $x = 0$ or $b/(1-a)$ to yield a set of complete asymptotic criteria. Fortunately, we may resort to elementary arguments as to be seen below.

To this end, we first note that our equation is autonomous (time invariant), and hence if $\{x_n\}_{n=-2}^{\infty}$ is a solution of (1.3), then for any $k \in \mathbf{N}$, the sequence $\{y_n\}_{n=-2}^{\infty}$, defined by $y_n = x_{n+k}$ for $n = -2, -1, 0, \ldots$, is also a solution. For the sake of convenience, we need to let

$$\begin{aligned} A_0 &= \frac{\lambda - b}{a}, \\ aA_{j+1} + b &= A_j, \quad j \in \mathbf{N}. \end{aligned} \tag{1.15}$$

Then

$$A_j = \frac{\lambda - b\left(1 + a + \cdots + a^j\right)}{a^{j+1}} = \frac{\lambda(1-a) - b}{a^{j+1}(1-a)} + \frac{b}{1-a}, \quad j \in \mathbf{N}, \tag{1.16}$$

$$A_{j+1} - A_j = \frac{\lambda(1-a) - b}{a^{j+2}}, \quad j \in \mathbf{N}, \tag{1.17}$$

$$\lambda = aA_0 + b = a^2 A_1 + ab + b = \cdots = a^{j+1} A_j + a^j b + a^{j-1} b + \cdots + ab + b, \quad j \in N. \tag{1.18}$$

We also let

$$\begin{aligned} B_0 &= -\frac{b}{a}, \\ aB_{j+1} + b &= B_j, \quad j \in \mathbf{N}. \end{aligned} \tag{1.19}$$

Then

$$B_j = \frac{-b\left(1 + a + \cdots + a^j\right)}{a^{j+1}} = \frac{-b + a^{j+1} b}{a^{j+1}(1-a)}, \quad j \in \mathbf{N}, \tag{1.20}$$

$$B_{j+1} - B_j = -\frac{b}{a^{j+2}}, \quad j \in \mathbf{N}, \tag{1.21}$$

$$aB_0 + b = a^2 B_1 + ab + b = \cdots = a^{j+1} B_j + a^j b + \cdots + ab + b, \quad j \in \mathbf{N}, \tag{1.22}$$

$$\lim_{j \to \infty} B_j = -\infty. \tag{1.23}$$

2. The Case $\lambda > b/(1-a)$

Suppose $\lambda > b/(1-a)$. Then

$$\lim_{j\to\infty} A_j = \lim_{j\to\infty}\left\{\frac{\lambda(1-a)-b}{a^{j+1}(1-a)} + \frac{b}{1-a}\right\} = +\infty. \tag{2.1}$$

We first show the following.

Lemma 2.1. *Let* $\lambda > b/(1-a)$. *If* $\{x_n\}_{n=-2}^{\infty}$ *is a solution of* (1.3) *with* $(x_{-2}, x_{-1}) \in \Omega$, *then there exists an integer* $m \in \{-2,-1,0,\ldots\}$ *such that* $0 < x_m, x_{m+1} \le \lambda$.

Proof. From our assumption, we have $a\lambda + b < \lambda$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (1.3) with $(x_{-2}, x_{-1}) \in \Omega$. Then there are eight cases.

Case 1. If $0 < x_{-2}, x_{-1} \le \lambda$, our assertion is true by taking $m = -2$.

Case 2. Suppose $(x_{-2}, x_{-1}) \in (0,\lambda] \times (\lambda, +\infty)$. Then $(\lambda - b)/a > \lambda$. Furthermore, in view of (1.17) and (2.1),

$$(0,\lambda] \times (\lambda,+\infty) = (0,\lambda] \times \left\{\left(\lambda, \frac{\lambda-b}{a}\right] \cup \bigcup_{k=1}^{\infty}(A_{k-1}, A_k]\right\}. \tag{2.2}$$

If $x_{-1} \in (\lambda, (\lambda-b)/a]$, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \in (0,\lambda),\\ 0 < x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le \lambda - b + b = \lambda. \end{aligned} \tag{2.3}$$

This means that our assertion is true by taking $m = 0$. Next, if $x_{-1} \in (A_0, A_1] = ((\lambda-b)/a, (\lambda - b - ab)/a^2]$, then by (1.3) and (1.18),

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \in (0,\lambda),\\ x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b > \lambda,\\ x_2 &= ax_0 + bf_\lambda(x_1) = a^2x_{-2} \in (0,\lambda),\\ 0 < x_3 &= ax_1 + bf_\lambda(x_2) = a^2x_{-1} + ab + b\\ &\le a^2A_1 + ab + b = \lambda. \end{aligned} \tag{2.4}$$

Thus our assertion holds by taking $m = 2$. If $x_{-1} \in (A_p, A_{p+1}]$, where p is an arbitrary positive integer, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \in (0,\lambda), \\ x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b > aA_p + b = A_{p-1} > \lambda, \\ x_2 &= ax_0 + bf_\lambda(x_1) = a^2 x_{-2} \in (0,\lambda). \end{aligned} \tag{2.5}$$

By induction,

$$\begin{aligned} x_{2p} &= a^{p+1}x_{-2} \in (0,\lambda), \\ x_{2p+1} &= a^{p+1}x_{-1} + a^p b + \cdots + ab + b > a^{p+1}A_p + a^p b + \cdots + ab + b \\ &= a^p A_{p-1} + a^{p-1}b + \cdots + ab + b = aA_0 + b = \lambda, \\ x_{2(p+1)} &= ax_{2p} + bf_\lambda\left(x_{2p+1}\right) = a^{p+2}x_{-2} \in (0,\lambda), \\ x_{2(p+1)+1} &= ax_{2p+1} + bf_\lambda\left(x_{2(p+1)}\right) = a^{p+2}x_{-1} + a^{p+1}b + \cdots + ab + b \\ &\le a^{p+2}A_{p+1} + a^{p+1}b + \cdots + ab + b = \lambda. \end{aligned} \tag{2.6}$$

Thus our assertion holds by taking $m = 2(p+1)$.

Case 3. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (\lambda, +\infty)$. We assert that there is a nonnegative integer μ such that $x_n > \lambda$ for $n = -2, -1, \ldots, \mu - 1$ and $x_\mu \in (0, \lambda]$. Otherwise we have $x_n \in (\lambda, +\infty)$ for $n \in \mathbf{N}$. It follows that

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} > \lambda, \\ x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} > \lambda, \\ x_2 &= ax_0 + bf_\lambda(x_1) = a^2 x_{-2} > \lambda, \\ x_3 &= ax_1 + bf_\lambda(x_2) = a^2 x_{-1} > \lambda. \end{aligned} \tag{2.7}$$

By induction, for any $k \in \mathbf{N}$, we have

$$\begin{aligned} x_{2k} &= a^{k+1}x_{-2} > \lambda, \\ x_{2k+1} &= a^{k+1}x_{-1} > \lambda, \end{aligned} \tag{2.8}$$

which implies

$$\lim_{k\to\infty} x_{2k} = 0 = \lim_{k\to\infty} x_{2k+1}. \tag{2.9}$$

This is contrary to the fact that $x_n \in (\lambda, +\infty)$ for $n \in \mathbf{N}$.

Now that there exists an integer $\mu \in \mathbf{N}$ such that $x_{-2}, x_{-1}, \ldots, x_{\mu-1} \in (\lambda, +\infty)$ and $x_\mu \in (0, \lambda]$, it then follows

$$0 < x_{\mu+1} = ax_{\mu-1} + bf_\lambda(x_\mu) = ax_{\mu-1} + b. \tag{2.10}$$

If $x_{\mu+1} \leq \lambda$, then our assertion holds by taking $m = \mu$. If $x_{\mu+1} > \lambda$, then $x_{\mu-1} > (\lambda - b)/a$. Thus

$$\begin{gathered} 0 < x_{\mu+2} = ax_\mu + bf_\lambda(x_{\mu+1}) = ax_\mu \leq a\lambda < \lambda, \\ 0 < x_{\mu+3} = ax_{\mu+1} + bf_\lambda(x_{\mu+2}) = ax_{\mu+1} + b = a^2 x_{\mu-1} + ab + b. \end{gathered} \tag{2.11}$$

If $0 < x_{\mu+3} \leq \lambda$, then our assertion holds by taking $m = \mu + 2$. If $x_{\mu+3} > \lambda$, we have $x_{\mu-1} > (\lambda - b - ab)/a^2$. Hence

$$\begin{gathered} 0 < x_{\mu+4} = ax_{\mu+2} + bf_\lambda(x_{\mu+3}) = ax_{\mu+2} = a^2 x_\mu < \lambda, \\ 0 < x_{\mu+5} = ax_{\mu+3} + bf_\lambda(x_{\mu+4}) = ax_{\mu+3} + b = a^3 x_{\mu-1} + a^2 b + ab + b. \end{gathered} \tag{2.12}$$

Repeating the procedure, we have

$$\begin{gathered} 0 < x_{\mu+2k} = a^k x_\mu < \lambda, \\ 0 < x_{\mu+(2k+1)} = a^{k+1} x_{\mu-1} + a^k b + \cdots + ab + b. \end{gathered} \tag{2.13}$$

If $0 < x_{\mu+(2k+1)} \leq \lambda$, then our assertion holds by taking $m = \mu + (2k+1)$. Otherwise,

$$x_{\mu-1} > \frac{\lambda - b - ab - \cdots - a^k b}{a^{k+1}} \tag{2.14}$$

for all $k \in \mathbf{N}$. But this is contrary to (2.1). Thus we conclude that $0 < x_{\mu+(2k+1)} \leq \lambda$ for some k. Our assertion then holds by taking $m = \mu + 2k$.

Case 4. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$. As in Case 2,

$$(\lambda, +\infty) \times (0, \lambda] = \left\{ \left(\lambda, \frac{\lambda - b}{a} \right] \cup \bigcup_{k=1}^{\infty} (A_{k-1}, A_k] \right\} \times (0, \lambda]. \tag{2.15}$$

If $x_{-2} \in (\lambda, (\lambda - b)/a]$, then by (1.3),

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \leq \lambda. \tag{2.16}$$

Thus our assertion holds taking $m=-1$. If $x_{-2}\in(A_0,A_1]$, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2}+bf_\lambda(x_{-1}) = ax_{-2}+b>\lambda,\\ 0<x_1 &= ax_{-1}+bf_\lambda(x_0) = ax_{-1}+b\le a\lambda+b<\lambda,\\ 0<x_2 &= ax_0+bf_\lambda(x_1) = a^2x_{-2}+ab+b\le a^2A_1+ab+b=\lambda. \end{aligned} \tag{2.17}$$

Thus our assertion holds by taking $m=1$. If $x_{-2}\in(A_p,A_{p+1}]$, where p is an arbitrary positive integer, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2}+bf_\lambda(x_{-1}) = ax_{-2}+b>aA_p+b=A_{p-1}>\lambda,\\ 0<x_1 &= ax_{-1}+bf_\lambda(x_0)=ax_{-1}<\lambda,\\ x_2 &= ax_0+bf_\lambda(x_1)=a^2x_{-2}+ab+b>A_{p-2}>\lambda,\\ &\vdots\\ x_{2p} &= a^{p+1}x_{-2}+a^pb+\cdots+ab+b>a^{p+1}A_p+a^pb+\cdots+ab+b=\lambda,\\ 0<x_{2p+1} &= a^{p+1}x_{-1}<\lambda,\\ 0<x_{2(p+1)} &= ax_{2p}+bf_\lambda\left(x_{2p+1}\right)=a^{p+2}x_{-2}+a^{p+1}b+\cdots+ab+b\\ &\le a^{p+2}A_{p+1}+a^{p+1}b+\cdots+ab+b=\lambda. \end{aligned} \tag{2.18}$$

Thus our assertion holds by taking $m=2p+1$.

Case 5. Suppose $(x_{-2},x_{-1})\in(-\infty,0]\times(0,\lambda]$. Then by (1.21) and (1.23),

$$(-\infty,0]\times(0,\lambda]=\left\{\left(\bigcup_{j=1}^{\infty}(B_j,B_{j-1}]\right)\cup\left(-\frac{b}{a},0\right]\right\}\times(0,\lambda]. \tag{2.19}$$

If $x_{-2}\in(-b/a,0]$, then by (1.3),

$$0<x_0=\mathrm{a}x_{-2}+bf_\lambda(x_{-1})=ax_{-2}+b\le b<\lambda. \tag{2.20}$$

Thus our assertion holds by $m=-1$. If $x_{-2}\in(B_1,B_0]=((-b-ab)/a^2,-b/a]$, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2}+bf_\lambda(x_{-1})=ax_{-2}+b\le 0,\\ 0<x_1 &= ax_{-1}+bf_\lambda(x_0)=ax_{-1}<\lambda,\\ 0=-b-ab+ab+b<x_2 &= ax_0+bf_\lambda(x_1)\\ &= a^2x_{-2}+ab+b\le a^2B_0+ab+b=b<\lambda. \end{aligned} \tag{2.21}$$

Thus our assertion holds by taking $m = 1$. If $x_{-2} \in (B_{p+1}, B_p]$, where p is an arbitrary positive integer, then by (1.3), we have

$$
\begin{gathered}
B_p = aB_{p+1} + b < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \le aB_p + b = B_{p-1}, \\
0 < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} < \lambda.
\end{gathered}
\tag{2.22}
$$

That is, $(x_0, x_1) \in (B_p, B_{p-1}] \times (0, \lambda)$. Therefore we may conclude our assertion by induction.
Case 6. Suppose $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (\lambda, +\infty)$. Since

$$
\begin{gathered}
\frac{\lambda}{a^k} < \frac{\lambda}{a^{k+1}}, k \in \mathbf{N}, \\
\lim_{k\to\infty} \frac{\lambda}{a^{k+1}} = +\infty,
\end{gathered}
\tag{2.23}
$$

we see that

$$
(-\infty, 0] \times (\lambda, +\infty) = (-\infty, 0] \times \bigcup_{k=0}^{\infty} \left(\frac{\lambda}{a^k}, \frac{\lambda}{a^{k+1}} \right]. \tag{2.24}
$$

If $x_{-1} \in (\lambda, \lambda/a]$, then by (1.3),

$$
\begin{gathered}
x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \le 0, \\
0 < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} \le \lambda.
\end{gathered}
\tag{2.25}
$$

That is, $(x_0, x_1) \in (-\infty, 0] \times (0, \lambda]$. We may thus apply the conclusion of Case 5 and the time invariance property of (1.3) to deduce our assertion. If $x_{-1} \in (\lambda/a^{p+1}, \lambda/a^{p+2}]$, where p is an arbitrary nonnegative integer, then by (1.3), we have

$$
\begin{gathered}
x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \le 0, \\
\frac{\lambda}{a^p} < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} \le \frac{\lambda}{a^{p+1}}.
\end{gathered}
\tag{2.26}
$$

That is, $(x_0, x_1) \in (-\infty, 0] \times (\lambda/a^p, \lambda/a^{p+1}]$. We may thus use induction to conclude our assertion.
Case 7. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (-\infty, 0]$. As in Case 5,

$$
(0, \lambda] \times (-\infty, 0] = (0, \lambda] \times \left\{ \left(\bigcup_{k=1}^{\infty} (B_k, B_{k-1}] \right) \cup \left(-\frac{b}{a}, 0 \right] \right\}. \tag{2.27}
$$

If $x_{-1} \in (-b/a, 0]$, then by (1.3),

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda,$$
$$0 < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le b < \lambda. \tag{2.28}$$

Thus our assertion holds by taking $m = 0$. If $x_{-1} \in (B_1, B_0] = ((-b - ab)/a^2, -b/a]$, then by (1.3),

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda,$$
$$-\frac{b}{a} = \frac{-b - ab}{a} + b < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le 0. \tag{2.29}$$

That is, $(x_0, x_1) \in (0, \lambda) \times (-b/a, 0]$. Thus our assertion holds by taking $m = 2$.

If $x_{-1} \in (B_{p+1}, B_p]$. where p is an arbitrary positive integer, then by (1.3), we have

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda\ ,$$
$$B_p < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le B_{p-1}. \tag{2.30}$$

That is, $(x_0, x_1) \in (0, \lambda) \times (B_p, B_{p-1}]$. Thus our assertion follows from induction.

Case 8. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (-\infty, 0]$. Then

$$(\lambda, +\infty) \times (-\infty, 0] = \left(\bigcup_{k=0}^{\infty} \left(\frac{\lambda}{a^k}, \frac{\lambda}{a^{k+1}} \right] \right) \times (-\infty, 0]. \tag{2.31}$$

If $x_{-2} \in (\lambda, \lambda/a]$, then by (1.3),

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \le \lambda. \tag{2.32}$$

That is, $(x_{-1}, x_0) \in (-\infty, 0] \times (0, \lambda]$. We may now apply the assertion in Case 5 to conclude our proof. If $x_{-2} \in (\lambda/a^{p+1}, \lambda/a^{p+2}]$, where p is an arbitrary nonnegative integer, then by (1.3), we have

$$\frac{\lambda}{a^p} < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} \le \frac{\lambda}{a^{p+1}}\ ,$$
$$x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} \le 0. \tag{2.33}$$

That is, $(x_0, x_1) \in (\lambda/a^p, \lambda/a^{p+1}] \times (-\infty, 0]$. We may thus complete our proof by induction. □

Theorem 2.2. *Suppose $\lambda > b/(1 - a)$, then a solution $x = \{x_n\}_{n=-2}^{\infty}$ of* (1.3) *with $(x_{-2}, x_{-1}) \in \Omega$ will tend to $b/(1 - a)$.*

Proof. In view of Lemma 2.1, we may assume without loss of generality that $0 < x_{-2}, x_{-1} \le \lambda$. From our assumption, we have $a\lambda + b < \lambda$. Furthermore, by (1.3),

$$\begin{aligned}
0 < x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \le a\lambda + b < \lambda, \\
0 < x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le a\lambda + b < \lambda, \\
0 < x_2 &= ax_0 + bf_\lambda(x_1) = a^2x_{-2} + ab + b \le a^2\lambda + ab + b = a(a\lambda + b) + b < a\lambda + b < \lambda, \\
0 < x_3 &= ax_1 + bf_\lambda(x_2) = a^2x_{-1} + ab + b \le a^2\lambda + ab + b = a(a\lambda + b) + b < a\lambda + b < \lambda, \\
0 < x_4 &= ax_2 + bf_\lambda(x_3) = a^3x_{-2} + a^2b + ab + b \le a^3\lambda + a^2b + ab + b \\
&= a^2(a\lambda + b) + ab + b < a^2\lambda + ab + b < \lambda, \\
0 < x_5 &= ax_3 + bf_\lambda(x_4) = a^3x_{-1} + a^2b + ab + b \le a^3\lambda + a^2b + ab + b \\
&= a^2(a\lambda + b) + ab + b < a^2\lambda + ab + b < \lambda.
\end{aligned} \tag{2.34}$$

By induction, for any $k \in \mathbf{N}$, we have

$$\begin{aligned}
0 < x_{2k} &= a^{k+1}x_{-2} + a^kb + a^{k-1}b + \cdots + ab + b \\
&\le a^{k+1}\lambda + a^kb + a^{k-1}b + \cdots + ab + b = a^k(a\lambda + b) + a^{k-1}b + \cdots + ab + b \\
&< a^k\lambda + a^{k-1}b + \cdots + ab + b = a^{k-1}(a\lambda + b) + a^{k-2}b + \cdots + ab + b \\
&< a^{k-1}\lambda + a^{k-2}b + \cdots + ab + b < \cdots < a^2\lambda + ab + b < \lambda,
\end{aligned} \tag{2.35}$$

and similarly

$$0 < x_{2k+1} = a^{k+1}x_{-1} + a^kb + a^{k-1}b + \cdots + ab + b < \lambda. \tag{2.36}$$

Thus $x_{2k}, x_{2k+1} \in (0, \lambda]$ for any $k \in \mathbf{N}$ and

$$\begin{aligned}
\lim_{k\to\infty} x_{2k} &= \lim_{k\to\infty}\left\{a^{k+1}x_{-2} + b \times \frac{1 - a^{k+1}}{1 - a}\right\} = \frac{b}{1 - a}, \\
\lim_{k\to\infty} x_{2k+1} &= \lim_{k\to\infty}\left\{a^{k+1}x_{-1} + b \times \frac{1 - a^{k+1}}{1 - a}\right\} = \frac{b}{1 - a}.
\end{aligned} \tag{2.37}$$

The proof is complete. □

3. The Case $\lambda \in (0, b/(1 - a))$

We first show that following result.

Lemma 3.1. *Let $0 < \lambda < b/(1-a)$. If $x = \{x_n\}_{n=-2}^{\infty}$ is a solution of* (1.3) *with $(x_{-2}, x_{-1}) \in \Omega$, there exists an integer $m \in \{-2, -1, 0, \dots\}$ such that $0 < x_m \leq \lambda$ and $x_{m+1} > \lambda$ (or $x_m > \lambda$ and $0 < x_{m+1} \leq \lambda$).*

Proof. From our assumption, we have $a\lambda + b > \lambda$. Let $\{x_n\}_{n=-2}^{\infty}$ be the solution of (1.3) determined by $(x_{-2}, x_{-1}) \in \Omega$. Then there are eight cases to show that there exists an integer $m \in \{-2, -1, 0, \dots\}$ such that $0 < x_m \leq \lambda$ and $x_{m+1} > \lambda$.
Case 1. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$. Then our assertion is true by taking $m = -2$.
Case 2. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$. By (1.3)

$$x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b > a\lambda + b > \lambda. \tag{3.1}$$

This means that our assertion is true by taking $m = -1$.
Case 3. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (0, \lambda]$. If $x_n \in (0, \lambda]$ for any $n \in \mathbf{N}$, then by (1.3),

$$\begin{aligned} x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b, \\ x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b, \\ x_2 &= ax_0 + bf_\lambda(x_1) = a^2x_{-2} + ab + b, \\ x_3 &= ax_1 + bf_\lambda(x_2) = a^2x_{-1} + ab + b. \end{aligned} \tag{3.2}$$

By induction, for any $k \in \mathbf{N}$, we have

$$\begin{aligned} x_{2k} &= a^{k+1}x_{-2} + a^k b + \cdots + ab + b = a^{k+1}x_{-2} + b \times \frac{1-a^{k+1}}{1-a}, \\ x_{2k+1} &= a^{k+1}x_{-1} + a^k b + \cdots + ab + b = a^{k+1}x_{-1} + b \times \frac{1-a^{k+1}}{1-a}. \end{aligned} \tag{3.3}$$

Hence

$$\lim_{k\to\infty} x_{2k} = \frac{b}{1-a} = \lim_{k\to\infty} x_{2k+1}. \tag{3.4}$$

But this is contrary to our assumption that $0 < \lambda < b/(1-a)$. Hence there exists an integer $\mu \in \{-1, 0, 1, \dots\}$ such that $x_{-2}, x_{-1}, \dots, x_\mu \in (0, \lambda]$ and $x_{\mu+1} \in (\lambda, +\infty)$. Thus our assertion holds by taking $m = \mu$.
Case 4. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (\lambda, +\infty)$. As in Case 3 of Lemma 2.1, we may show that if $x_n \in (\lambda, +\infty)$ for all $n \in \mathbf{N}$, then it follows that

$$\lim_{n\to\infty} x_n = 0. \tag{3.5}$$

But this is contrary to the fact that $x_n \in (\lambda, +\infty)$ for $n \in \mathbf{N}$. Hence there exists an integer $\mu \in \mathbf{N}$ such that $x_{-2}, x_{-1}, \ldots, x_{\mu-1} \in (\lambda, +\infty)$ and $x_\mu \in (0, \lambda]$, it then follows

$$x_{\mu+1} = ax_{\mu-1} + bf_\lambda(x_\mu) = ax_{\mu-1} + b > a\lambda + b > \lambda. \tag{3.6}$$

This means that our assertion is true by taking $m = \mu$.

Case 5. Suppose $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (0, \lambda]$. Then by (1.21) and (1.23),

$$(-\infty, 0] \times (0, \lambda] = \left\{ \left(\bigcup_{j=1}^{\infty} (B_j, B_{j-1}] \right) \cup \left(-\frac{b}{a}, 0 \right] \right\} \times (0, \lambda]. \tag{3.7}$$

If $x_{-2} \in (-b/a, 0]$, then by (1.3),

$$x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b > -b + b = 0. \tag{3.8}$$

When $\lambda \geq b$, we have

$$0 < x_0 = ax_{-2} + b \leq b \leq \lambda. \tag{3.9}$$

That is, $(x_{-1}, x_0) \in (0, \lambda] \times (0, \lambda]$. We may thus apply the conclusion of Case 3 to deduce our assertion.

Suppose $\lambda < b$. If $-b/a < x_{-2} \leq (\lambda - b)/a < 0$, then we have

$$0 < x_0 = ax_{-2} + b \leq \lambda. \tag{3.10}$$

We may apply the conclusion of Case 3 to deduce our assertion. If $(\lambda - b)/a < x_{-2} \leq 0$, we have

$$x_0 = ax_{-2} + b > \lambda. \tag{3.11}$$

Thus our assertion holds by taking $m = -1$. If $x_{-2} \in (B_1, B_0] = ((-b - ab)/a^2, -b/a]$, then by (1.3),

$$\begin{gathered} B_0 = aB_1 + b < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \leq 0, \\ 0 < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} < \lambda. \end{gathered} \tag{3.12}$$

That is, $(x_0, x_1) \in (-b/a, 0] \times (0, \lambda]$. In view of the above discussions, our assertion is true. If $x_{-2} \in (B_{p+1}, B_p]$, where p is an arbitrary positive integer, then by (1.3), we have

$$B_p = aB_{p+1} + b < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \le aB_p + b = B_{p-1},$$
$$0 < x_1 = ax_{-1} + bf_\lambda(x_0) = ax_{-1} < \lambda. \tag{3.13}$$

That is, $(x_0, x_1) \in (B_p, B_{p-1}] \times (0, \lambda]$. Therefore we may conclude our assertion by induction.

Case 6. Suppose

$$(x_{-2}, x_{-1}) \in (-\infty, 0] \times (\lambda, +\infty) = (-\infty, 0] \times \bigcup_{k=0}^{\infty} \left(\frac{\lambda}{a^k}, \frac{\lambda}{a^{k+1}} \right]. \tag{3.14}$$

As in Case 6 of Lemma 2.1, if $x_{-1} \in (\lambda, \lambda/a]$, then by (1.3), we have $(x_0, x_1) \in (-\infty, 0] \times (0, \lambda]$. We may thus apply the conclusion of Case 5 to deduce our assertion. If $x_{-1} \in (\lambda/a^{p+1}, \lambda/a^{p+2}]$, where p is an arbitrary nonnegative integer, then by (1.3), we have $(x_0, x_1) \in (-\infty, 0] \times (\lambda/a^p, \lambda/a^{p+1}]$. We may thus use induction to conclude our assertion.

Case 7. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (-\infty, 0]$. By (1.3), we have

$$0 < x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda. \tag{3.15}$$

That is, $(x_{-1}, x_0) \in (-\infty, 0] \times (0, \lambda]$. We may thus apply the conclusion of Case 5 to deduce our assertion.

Case 8. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (-\infty, 0]$. Then

$$(\lambda, +\infty) \times (-\infty, 0] = \bigcup_{k=0}^{\infty} \left(\frac{\lambda}{a^k}, \frac{\lambda}{a^{k+1}} \right] \times (-\infty, 0]. \tag{3.16}$$

As in Case 8 of Lemma 2.1, if $x_{-2} \in (\lambda, \lambda/a]$, then by (1.3), we have $(x_{-1}, x_0) \in (-\infty, 0] \times (0, \lambda]$. We may now apply the assertion in Case 5 to conclude our proof. If $x_{-2} \in (\lambda/a^{p+1}, \lambda/a^{p+2}]$, where p is an arbitrary nonnegative integer, then by (1.3), we have $(x_0, x_1) \in (\lambda/a^p, \lambda/a^{p+1}] \times (-\infty, 0]$. We may thus complete our proof by induction. □

Theorem 3.2. *Let* $0 < \lambda < b/(1-a)$. *Then any solution* $\{x_n\}_{n=-2}^{\infty}$ *of* (1.3) *with* $(x_{-2}, x_{-1}) \in \Omega$ *is asymptotically* 2*-periodic with limit* 2*-cycle* $\langle 0, b/(1-a) \rangle$.

Proof. In view of Lemma 3.1, we may assume without loss of generality that $0 < x_{-2} \le \lambda$ and $x_{-1} > \lambda$. Then by (1.3),

$$\begin{aligned}
0 < x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda,\\
x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b > a\lambda + b > \lambda,\\
0 < x_2 &= ax_0 + bf_\lambda(x_1) = a^2x_{-2} < \lambda,\\
x_3 &= ax_1 + bf_\lambda(x_2) = a^2x_{-1} + ab + b > a^2\lambda + ab + b\\
&= a(a\lambda + b) + b > a\lambda + b > \lambda,\\
0 < x_4 &= ax_2 + bf_\lambda(x_3) = a^3x_{-2} < \lambda,\\
x_5 &= ax_3 + bf_\lambda(x_4) = a^3x_{-1} + a^2b + ab + b > a^3\lambda + a^2b + ab + b\\
&= a^2(a\lambda + b) + ab + b > a^2\lambda + ab + b > \lambda.
\end{aligned} \tag{3.17}$$

By induction, for any $k \in \mathbf{N}$, we have

$$\begin{aligned}
0 < x_{2k} &= a^{k+1}x_{-2} < \lambda,\\
x_{2k+1} &= a^{k+1}x_{-1} + a^k\mathrm{b} + \cdots + ab + b > a^{k+1}\lambda + a^kb + \cdots + ab + b\\
&= a^k(a\lambda + b) + a^{k-1}b + \cdots + ab + b > a^k\lambda + a^{k-1}b + \cdots + ab + b\\
&> \cdots > a^2\lambda + ab + b > \lambda.
\end{aligned} \tag{3.18}$$

Thus $x_{2k} \in (0, \lambda]$ and $x_{2k+1} \in (\lambda, +\infty)$ for any $k \in \mathbf{N}$. Then

$$\begin{aligned}
\lim_{k\to\infty} x_{2k} &= \lim_{k\to\infty} a^{k+1}x_{-2} = 0,\\
\lim_{k\to\infty} x_{2k+1} &= \lim_{k\to\infty}\left\{a^{\mathrm{k}+1}x_{-1} + b \times \frac{1 - a^{k+1}}{1 - a}\right\} = \frac{b}{1 - a}.
\end{aligned} \tag{3.19}$$

□

4. The Case $\lambda = b/(1 - a)$

Suppose $\lambda = b/(1-a)$. Then $\lambda = a\lambda + b > b$. We need to consider solutions with initial vectors in U or V defined by (1.7) and (1.8), respectively.

Lemma 4.1. *Let $\lambda = b/(1 - a)$. If $\{x_n\}_{n=-2}^{\infty}$ is a solution of* (1.3) *with $(x_{-2}, x_{-1}) \in V$, then there exists an integer $m \in \mathbf{N}$ such that $0 < x_m, x_{m+1} \le \lambda$.*

The proof is the same as the discussions in Cases 5 through Case 8 in the proof of Lemma 2.1, and hence is skipped.

Theorem 4.2. *Suppose $\lambda = b/(1 - a)$, then a solution $x = \{x_n\}_{n=-2}^{\infty}$ of* (1.3) *with $(x_{-2}, x_{-1}) \in V$ will tend to $b/(1 - a)$.*

Proof. In view of Lemma 4.1, we may assume without loss of generality that $0 < x_{-2}, x_{-1} \le \lambda$. By (1.3),

$$
\begin{aligned}
0 < x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b \le a\lambda + b = \lambda, \\
0 < x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b \le a\lambda + b = \lambda, \\
0 < x_2 &= ax_0 + bf_\lambda(x_1) = a^2x_{-2} + ab + b \le a^2\lambda + ab + b = a(a\lambda + b) + b = a\lambda + b = \lambda, \\
0 < x_3 &= ax_1 + bf_\lambda(x_2) = a^2x_{-1} + ab + b \le a^2\lambda + ab + b = a(a\lambda + b) + b = a\lambda + b = \lambda, \\
0 < x_4 &= ax_2 + bf_\lambda(x_3) = a^3x_{-2} + a^2b + ab + b \le a^3\lambda + a^2b + ab + b \\
&= a^2(a\lambda + b) + ab + b = a^2\lambda + ab + b = \lambda, \\
0 < x_5 &= ax_3 + bf_\lambda(x_4) = a^3x_{-1} + a^2b + ab + b \le a^3\lambda + a^2b + ab + b \\
&= a^2(a\lambda + b) + ab + b = a^2\lambda + ab + b = \lambda.
\end{aligned}
\tag{4.1}
$$

By induction, for any $k \in \mathbf{N}$, we have

$$
\begin{aligned}
0 < x_{2k} &= a^{k+1}x_{-2} + a^kb + a^{k-1}b + \cdots + ab + b \\
&\le a^{k+1}\lambda + a^kb + a^{k-1}b + \cdots + ab + b = a^k(a\lambda + b) + a^{k-1}b + \cdots + ab + b \\
&= a^k\lambda + a^{k-1}b + \cdots + ab + b = a^{k-1}(a\lambda + b) + a^{k-2}b + \cdots + ab + b \\
&= a^{k-1}\lambda + a^{k-2}b + \cdots + ab + b = \cdots = a^2\lambda + ab + b = \lambda,
\end{aligned}
\tag{4.2}
$$

and similarly

$$
0 < x_{2k+1} = a^{k+1}x_{-1} + a^kb + a^{k-1}b + \cdots + ab + b \le \lambda. \tag{4.3}
$$

Thus $x_{2k}, x_{2k+1} \in (0, \lambda]$ for any $k \in \mathbf{N}$. Thus (2.37) hold so that

$$
\lim_{n\to\infty} x_n = \frac{b}{1-a}. \tag{4.4}
$$

The proof is complete. □

Theorem 4.3. *Suppose $\lambda = b/(1-a)$, then any solution $\{x_n\}_{n=-2}^{\infty}$ of (1.3) with $(x_{-2}, x_{-1}) \in U$ is asymptotically 2-periodic with limit 2-cycle $\langle 0, b/(1-a)\rangle$.*

Proof. We first discuss the case, where $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$. By (1.3),

$$\begin{aligned}
0 < x_0 &= ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} < \lambda, \\
x_1 &= ax_{-1} + bf_\lambda(x_0) = ax_{-1} + b > a\lambda + b = \lambda, \\
0 < x_2 &= ax_0 + bf_\lambda(x_1) = a^2 x_{-2} < \lambda, \\
x_3 &= ax_1 + bf_\lambda(x_2) = a^2 x_{-1} + ab + b > a^2\lambda + ab + b \\
&= a(a\lambda + b) + b = a\lambda + b = \lambda, \\
0 < x_4 &= ax_2 + bf_\lambda(x_3) = a^3 x_{-2} < \lambda, \\
x_5 &= ax_3 + bf_\lambda(x_4) = a^3 x_{-1} + a^2 b + ab + b > a^3\lambda + a^2 b + ab + b \\
&= a^2(a\lambda + b) + ab + b = a^2\lambda + ab + b = \lambda.
\end{aligned} \tag{4.5}$$

By induction, for any $k \in \mathbf{N}$, we have

$$\begin{gathered}
0 < x_{2k} = a^{k+1} x_{-2} < \lambda, \\
x_{2k+1} = a^{k+1} x_{-1} + a^k b + \cdots + ab + b > a^{k+1}\lambda + a^k b + \cdots + ab + b = \lambda.
\end{gathered} \tag{4.6}$$

Thus $x_{2k} \in (0, \lambda]$ and $x_{2k+1} \in (\lambda, +\infty)$ for any $k \in \mathbf{N}$. Then

$$\begin{gathered}
\lim_{k\to\infty} x_{2k} = 0, \\
\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \left\{ a^{k+1} x_{-1} + b \times \frac{1 - a^{k+1}}{1 - a} \right\} = \frac{b}{1 - a}.
\end{gathered} \tag{4.7}$$

If $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$, then by (1.3),

$$x_0 = ax_{-2} + bf_\lambda(x_{-1}) = ax_{-2} + b > a\lambda + b = \lambda. \tag{4.8}$$

That is, $(x_{-1}, x_0) \in (0, \lambda] \times (\lambda, +\infty)$. We may thus apply the previous conclusion to deduce our assertion.

If $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (\lambda, +\infty)$, then similar to the discussions of Case 3 of Lemma 2.1, there exists an integer $\mu \in \mathbf{N}$ such that $x_{-2}, x_{-1}, \ldots, x_{\mu-1} \in (\lambda, +\infty)$ and $x_\mu \in (0, \lambda]$. That is, $(x_{\mu-1}, x_\mu) \in (\lambda, +\infty) \times (0, \lambda]$. In view of the previous case, our assertion holds. The proof is complete. □

5. Concluding Remarks

The results in the previous sections can be stated in terms of the two-dimensional dynamical system (1.10). Indeed, a solution of (1.10) is a vector sequence of the form $\{(u_n, v_n)^\dagger\}_{n=0}^\infty$ that renders (1.10) into an identity for each $n \in \mathbf{N}$. It is uniquely determined by $(u_0, v_0)^\dagger$.

Let us say that a solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) eventually falls into a plane region Ψ if $(u_n, v_n)^\dagger \in \Psi$ for all large n; that it is eventually falls into two disjoint plane regions Ψ_1 and Ψ_2 alternately if there is some $m \in \mathbf{N}$ such that $(u_{m+2i}, v_{m+2i})^\dagger \in \Psi_1$ and $(u_{m+2i+1}, v_{m+2i+1})^\dagger \in \Psi_2$ for all $i \in \mathbf{N}$; and that it approaches a limit 2-cycle $\langle(\alpha_1, \beta_1)^\dagger, (\alpha_2, \beta_2)^\dagger\rangle$ if there is some $m \in \mathbf{N}$ such that $(u_{m+2i}, v_{m+2i})^\dagger \to (\alpha_1, \beta_1)^\dagger$ and $(u_{m+2i+1}, v_{m+2i+1})^\dagger \to (\alpha_2, \beta_2)^\dagger$ as $i \to +\infty$. Then we may restate the previous theorems as follows.

(i) The vectors $(0,0)^\dagger, (0, b/(1-a))^\dagger, (b/(1-a), b/(1-a))^\dagger$, and $(b/(1-a), 0)^\dagger$ form the corners of a square in the plane.

(ii) A solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) with $(u_0, v_0)^\dagger$ in the nonpositive orthant $(-\infty, 0]^2$ (is nonpositive and) tends to $(0,0)^\dagger$.

(iii) Suppose $\lambda > b/(1-a)$, then a solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) with $(u_0, v_0)^\dagger$ in Ω will (eventually falls into $(0, \lambda]^2$ and) tend to $(b/(1-a), b/(1-a))^\dagger$.

(iv) Suppose $0 < \lambda < b/(1-a)$, then a solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) with $(u_0, v_0)^\dagger$ in Ω will (eventually falls into $(0, \lambda] \times (\lambda, +\infty)$ and $(\lambda, +\infty) \times (0, \lambda]$ alternately and) approach the limit 2-cycle $\langle(0, b/(1-a))^\dagger, (b/(1-a), 0)^\dagger\rangle$.

(v) Suppose $\lambda = b/(1-a)$, then a solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) with $(u_0, v_0)^\dagger$ in V will (eventually falls into $(0, \lambda]^2$) tend to $(b/(1-a), b/(1-a))^\dagger$.

(vi) Suppose $\lambda = b/(1-a)$, Then a solution $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ of (1.10) with $(u_0, v_0)^\dagger$ in U will (eventually falls into $(0, \lambda] \times (\lambda, +\infty)$ and $(\lambda, +\infty) \times (0, \lambda]$ alternately) approach the limit 2-cycle $\langle(0, b/(1-a))^\dagger, (\mathrm{b}/(1-a), 0)^\dagger\rangle$.

Since we have obtained a complete set of asymptotic criteria, we may deduce (bifurcation) results such as the following.

If $0 < \lambda < b/(1-a)$, then all solutions $\{(u_n, v_n)^\dagger\}_{n=0}^{\infty}$ originated from the positive orthant approach the limit 2-cycle $\langle(0, b/(1-a))^\dagger, (b/(1-a), 0)^\dagger\rangle$; if $\lambda > b/(1-a)$, then all solutions originated from the positive orthant tend to $(b/(1-a), b/(1-a))^\dagger$; if $\lambda = b/(1-a)$, then all solutions originated from the positive orthant tend to $(b/(1-a), b/(1-a))^\dagger$ if $(u_0, v_0)^\dagger \in (0, \lambda]^2$ and approach the limit cycle $\langle(0, b/(1-a))^\dagger, (b/(1-a), 0)^\dagger\rangle$ otherwise.

Roughly the above statements show that when the threshold parameter λ is a relatively small positive parameter, all solutions from the positive orthant tend to a limit 2-cycle; when it reaches the critical value $b/(1-a)$, some of these solutions (those from $(0, b/(1-a)]^2$) switch away and tend to a limit 1-cycle, and when λ drifts beyond the critical value, all solutions tend to the limit 1-cycle. Such an observation seems to appear in many natural processes and hence our model may be used to explain such phenomena. It is also expected that when a group of neural units interact with each other in a network where each unit is governed by evolutionary laws of the form (1.3), complex but manageable analytical results can be obtained. These will be left to other studies in the future.

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