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Research Article

On an Exponential-Type Fuzzy Difference Equation

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Our goal is to investigate the existence of the positive solutions, the existence of a nonnegative equilibrium, and the convergence of a positive solution to a nonnegative equilibrium of the fuzzy difference equation $x_{n+1} = (1 - \sum_{j=0}^{k-1} x_{n-j})(1 - e^{-Ax_n})$, $k \in \{2, 3, \ldots\}$, $n = 0, 1, \ldots$, where A and the initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0$ belong in a class of fuzzy numbers.

1. Introduction

Fuzzy difference equations are approached by many authors, from a different view.

In [1], the authors developed the stability results for the fuzzy difference equation

$$u_{n+1} = f(n, u_n), \qquad u_{n_0} = u_0, \tag{1.1}$$

in terms of the stability of the trivial solution of the ordinary difference equation

$$z_{n+1} = g(n, z_n), \qquad z_{n_0} = z_0, \tag{1.2}$$

where $f(n, u)$ is continuous in u for each n, and $u_n, f \in E^n$ for each $n \geq n_0$, where $E^n = \{u : \mathbb{R}^n \to [0, 1]\}$ such that u satisfies the following:

(i) u is normal;

(ii) u is fuzzy convex;

(iii) u is upper semicontinuous;

(iv) $[u]_0 = \{x \in \mathbb{R}^n : u(x) > 0\}$ is compact,
and $g(n, r)$ is a continuous and nondecreasing function in r for each n.

In [2], the authors studied the second-order, linear, constant coefficient fuzzy difference equation of the form

$$y(k+2) + ay(k+1) + by(k) = g(k; l_1, l_2, \ldots, l_m) \tag{1.3}$$

for $k = 0, 1, 2, \ldots$, where $y(k)$ is the unknown function of k and a, b are real constants with $b \neq 0$. $g(k; l_1, l_2, \ldots, l_m)$ is a known function of k and m parameters $l_1, l_2, \ldots, l_m$, which is continuous in k. The initial conditions are fuzzy sets.

In [3] the authors considered the associated fuzzy system

$$u_{n+1} = \widehat{f}(u_n), \tag{1.4}$$

of the deterministic system

$$x_{n+1} = f(x_n), \tag{1.5}$$

where $\widehat{f}$ is the Zadeh's extensions of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$. Equations (1.4) and (1.5) have the same real constants coefficient and real equilibriums.

In this paper, we consider the fuzzy difference equation

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right), \quad n = 0, 1, \ldots, k \in \{2, 3, \ldots\}, \tag{1.6}$$

where A and the initial values are in a class of fuzzy numbers (see Preliminaries). This equation is motivated by the corresponding ordinary difference equation which is posed in [4]. Moreover, (1.6) is a special case of an epidemic model (see [5–8]) and was studied in [9] by Zhang and Shi and in [10] by Stević.

In [11] we have, already, investigated the behavior of the solutions of a related system of two parametric ordinary difference equations, of the form

$$y_{n+1} = \left(1 - \sum_{j=0}^{k-1} z_{n-j}\right)\left(1 - e^{-By_n}\right), \quad z_{n+1} = \left(1 - \sum_{j=0}^{k-1} y_{n-j}\right)\left(1 - e^{-Cz_n}\right), \quad n \geq 0, \tag{1.7}$$

where B, C are positive real numbers and the initial values $y_{-k+1}, y_{-k+2}, \ldots, y_0$, $z_{-k+1}, z_{-k+2}, \ldots, z_0$, $k \in \{2, 3, \ldots\}$, are positive real numbers, which satisfy some additional conditions.

We note that, the behavior of the fuzzy difference equation is not always the same with the corresponding ordinary difference equation. For instance, in paper [12] the fuzzy difference equation

$$x_n = \max\left\{\frac{A_0}{x_{n-k}}, \frac{A_1}{x_{n-m}}\right\}, \quad n = 0, 1, \ldots, \tag{1.8}$$

where k, m are positive integers, A_0, A_1, and the initial values x_i, $i \in \{-d, -d+1, \dots, -1\}$, $d = \max\{k, m\}$ are in a class of fuzzy numbers, under some conditions has unbounded solutions, something that does not happen in the case of the corresponding ordinary difference equation (1.8), where k, m are positive integers and A_0, A_1, and the initial values x_i, $i \in \{-d, -d+1, \dots, -1\}$, $d = \max\{k, m\}$ are positive real numbers.

Finally we note that in recent years there has been a considerable interest in the study of the existence of some specific classes of solutions of difference equations such as nontrivial, nonoscillatory, monotone, positive. Various methods have been developed by the experts. For partial review of the theory of difference equations and their applications see, for example, [4, 10, 13–27] and the references therein.

2. Preliminaries

For a set B, we denote by $\overline{B}$ the closure of B.

We denote by E the set of functions A such that,

$$A : \mathbb{R}^+ = (0, \infty) \longrightarrow [0, 1], \tag{2.1}$$

where A satisfies the following conditions:

(i) A is normal, that is, there exists an $x_0 \in \mathbb{R}^+$ such that $A(x_0) = 1$;

(ii) A is fuzzy convex, that is for $x, y \in \mathbb{R}^+$, $0 \leq \lambda \leq 1$;

$$A(\lambda x + (1-\lambda) y) \geq \min\{A(x), A(y)\}; \tag{2.2}$$

(iii) A is upper semicontinuous

(iv) The support of A, supp $A = \overline{\{x : A(x) > 0\}}$ is compact.

Obviously, set E is a class of fuzzy numbers. In this paper, all the fuzzy numbers we use are elements of E. From above (i)–(iv) and Theorems 3.1.5 and 3.1.8 of [28] the a-cuts of the fuzzy number $A \in E$,

$$[A]_a = \{x \in \mathbb{R}^+ : A(x) \geq a\}, \qquad a \in (0, 1] \tag{2.3}$$

are closed intervals. Obviously, supp $A = \overline{\bigcup_{a \in (0,1]} [A]_a}$.

We say that a fuzzy number A is positive if supp$A \subset (0, \infty)$.

To prove our main results, we need the following theorem (see [29]).

Theorem 2.1 (see [29]). *Let $A \in E$, such that $[A]_a = [A_{l,a}, A_{r,a}]$, $a \in (0, 1]$. Then $A_{l,a}, A_{r,a}$ can be regarded as functions on $(0, 1]$ which satisfy*

(i) *$A_{l,a}$ is nondecreasing and left continuous;*

(ii) *$A_{r,a}$ is nonincreasing and left continuous;*

(iii) $A_{l,1} \leq A_{r,1}$.

Conversely, for any functions L_a, R_a *defined in* $(0,1]$ *which satisfy* (i)–(iii) *in above and* $\overline{\cup_{a\in(0,1]}[L_a, R_a]}$ *is compact, there exists a unique* $A \in E$ *such that* $[A]_a = [L_a, R_a]$, $a \in (0,1]$.

We need the following arithmetic operations on closed intervals:

(i) $[a,b] + [c,d] = [a+c, b+d]$, a,b,c,d positive real numbers,

(ii) $[a,b] - [c,d] = [a-d, b-c]$, a,b,c,d positive real numbers,

(iii) $[a,b] \cdot [c,d] = [a\cdot c, b\cdot d]$, a,b,c,d positive real numbers.

In this paper, we use the following arithmetic operations on fuzzy numbers based on closed intervals arithmetic (see [30]). Let A, B be positive fuzzy numbers which belong to E with

$$[A]_a = [A_{l,a}, A_{r,a}], \quad [B]_a = [B_{l,a}, B_{r,a}], \quad a \in (0,1]. \tag{2.4}$$

(i) $A + B$ is a positive fuzzy number which belongs to E, with

$$[A+B]_a = [A]_a + [B]_a, \quad a \in (0,1]; \tag{2.5}$$

(ii) $A - B$ is a positive fuzzy number which belongs to E, with

$$[A-B]_a = [A]_a - [B]_a, \quad a \in (0,1] \tag{2.6}$$

if $\operatorname{supp}(A-B) \subset (0,\infty)$;

(iii) AB is a positive fuzzy number which belongs to E, with

$$[AB]_a = [A]_a \cdot [B]_a, \quad a \in (0,1]. \tag{2.7}$$

We note that the subtraction "−" we use, is different than Hukuhara difference (see [31, 32]).

Using Extension Principle (see [28, 30, 33]) for a positive fuzzy number $A \in E$ such that (2.4) holds, we have

$$\left[e^{-A}\right]_a = \left[e^{-A_{r,a}}, e^{-A_{l,a}}\right], \quad a \in (0,1]. \tag{2.8}$$

Let A, B be positive fuzzy numbers which belong to E such that (2.4) holds. We consider the following metric (see [29, 32]):

$$D(A,B) = \sup \max\{|A_{l,a} - B_{l,a}|, |A_{r,a} - B_{r,a}|\}, \tag{2.9}$$

where sup is taken for all $a \in (0,1]$.

We say x_n is a positive solution of (1.6) if x_n is a sequence of positive fuzzy numbers which satisfies (1.6).

We say that a positive fuzzy number x is a positive equilibrium for (1.6) if

$$x = (1 - kx)\left(1 - e^{-Ax}\right), \quad k \in \{2, 3, \ldots\}. \tag{2.10}$$

Let x_n be a sequence of positive fuzzy numbers and x is a positive fuzzy number. Suppose that

$$\begin{gathered} [x_n]_a = [L_{n,a}, R_{n,a}], \quad a \in (0,1],\ n = -k+1, -k+2, \ldots, \\ [\overline{x}]_a = [L_a, R_a], \quad a \in (0,1] \end{gathered} \tag{2.11}$$

are satisfied. We say that x_n nearly converges to $\overline{x}$ with respect to D as $n \to \infty$ if for every $\delta > 0$ there exists a measurable set T, $T \subset (0,1]$ of measure less than δ such that

$$\lim D_T(x_n, \overline{x}) = 0, \quad \text{as } n \longrightarrow \infty, \tag{2.12}$$

where

$$D_T(x_n, \overline{\mathrm{x}}) = \sup_{a \in (0,1]-T} \{\max\{|L_{n,a} - L_a|, |R_{n,a} - R_a|\}\}. \tag{2.13}$$

If $T = \emptyset$, we say that x_n converges to $\overline{x}$ with respect to D as $n \to \infty$.

Let E be the set of positive fuzzy numbers. From Theorem 2.1 we have that $A_{l,a}$, $B_{l,a}$ (resp., $A_{r,a}$, $B_{r,a}$) are increasing (resp., decreasing) functions on $(0,1]$. Therefore, using the condition (iv) of the definition of the fuzzy numbers there exist the Lebesque integrals

$$\int_J |A_{l,a} - B_{l,a}| da, \qquad \int_J |A_{r,a} - B_{r,a}| da, \tag{2.14}$$

where $J = (0,1]$. We define the function $D_1 : E \times E \to R^+$ such that

$$D_1(A, B) = \max\left\{\int_J |A_{l,a} - B_{l,a}| da,\ \int_J |A_{r,a} - B_{r,a}| da\right\}. \tag{2.15}$$

If $D_1(A, B) = 0$ we have that there exists a measurable set T of measure zero such that

$$A_{l,a} = B_{l,a} \quad A_{r,a} = B_{r,a} \quad \forall a \in (0,1] - T. \tag{2.16}$$

We consider however, two fuzzy numbers A, B to be equivalent if there exists a measurable set T of measure zero such that (2.16) hold and if we do not distinguish between equivalent of fuzzy numbers then E becomes a metric space with metric D_1.

We say that a sequence of positive fuzzy numbers x_n converges to a positive fuzzy number x with respect to D_1 as $n \to \infty$ if

$$\lim D_1(x_n, x) = 0, \quad \text{as } n \longrightarrow \infty. \tag{2.17}$$

3. Study of the Fuzzy Difference Equation (1.6)

In order to prove our main results, we need the following Propositions A, B, C, which can be found in [11]. For readers convenience, we cite them below without their proofs.

Proposition A (see [11]). *Consider system* (1.7) *where the constants* B, C *are positive real numbers. Let* (y_n, z_n) *be a solution of* (1.7) *with initial values* y_{-j}, z_{-j}, $j = 0, 1, \ldots, k-1$, $k \in \{2, 3, \ldots\}$. *Then the following statements are true.*

(i) *Suppose that*

$$1 - \sum_{j=0}^{k-1} y_{-j} > 0, \qquad 1 - \sum_{j=0}^{k-1} z_{-j} > 0, \tag{3.1}$$

$$0 < B \leq 1, \qquad 0 < C \leq 1, \tag{3.2}$$

$$y_0 = \min\{y_{-j},\ j = 0, 1, \ldots, k-1\} > 0, \qquad z_0 = \min\{z_{-j},\ j = 0, 1, \ldots, k-1\} > 0, \tag{3.3}$$

hold. Then $y_n, z_n > 0$, $n = 1, 2, \ldots$

(ii) *Suppose that*

$$0 < B < k \ln\left(\frac{k}{k-1}\right), \qquad 0 < C < k \ln\left(\frac{k}{k-1}\right), \tag{3.4}$$

$$0 < y_{-j}, z_{-j} < \frac{1}{k}, \qquad j = 0, 1, \ldots, k-1, \tag{3.5}$$

hold. Then

$$0 < y_n < \frac{1}{k}, \qquad 0 < z_n < \frac{1}{k}, \qquad n = 1, 2, \ldots \tag{3.6}$$

Proposition B (see [11]). *Consider the system of algebraic equations*

$$\begin{gathered} y = (1 - kz)\left(1 - e^{-By}\right), \\ z = (1 - ky)\left(1 - e^{-Cz}\right), \qquad y, z \in \left[0, \frac{1}{k}\right),\ k \in \{2, 3, \ldots\}. \end{gathered} \tag{3.7}$$

Then the following statements are true.

(i) *If* (3.2) *holds, the system* (3.7) *has a unique nonnegative solution* $(0, 0)$.

(ii) *Suppose that*

$$0 < B < k \ln\left(\frac{k}{k-1}\right), \quad 1 < C < k \ln\left(\frac{k}{k-1}\right), \quad B < C \tag{3.8}$$

hold; then there are only two nonnegative equilibriums $(\overline{x}, \overline{y})$ *of system* (3.7)*, such that* $\overline{x}\,\overline{y} = 0$*, which are* $(0,0)$*,* $(0, z_1)$*,* $z_1, \in (0, 1/k)$*,* $z_1 = 1 - e^{-Cz_1}$.

Proposition C (see [11]). *Consider system* (1.7)*. Let* (y_n, z_n) *be a solution of* (1.7)*. Then the following statements are true.*

(i) *If* (3.2) *and either* (3.1) *and* (3.3) *or* (3.5) *are satisfied, then for the solution* (y_n, z_n) *of system* (1.7) *we have that*

$$0 < y_n < B^n y_0, \quad 0 < z_n < C^n z_0, \qquad n = 1, 2, \ldots \tag{3.9}$$

holds and obviously (y_n, z_n) *tends to the unique zero equilibrium* $(0,0)$ *of* (1.7) *as* $n \to \infty$.

(ii) *Suppose that* (3.5)*, the first relation of* (3.2) *and the second relation of* (3.8) *are satisfied. Then* (y_n, z_n) *tends to the nonnegative equilibrium* $(0, z_1)$*,* $0 < z_1 < 1/k$ *of* (1.7) *as* $n \to \infty$.

First we study the existence and the uniqueness of the positive solutions of the fuzzy difference equation (1.6).

Proposition 3.1. *Consider the fuzzy difference equation* (1.6)*, where* A *is a positive fuzzy number such that*

$$[A]_a = [A_{l,a}, A_{r,a}] \subset \overline{\bigcup_{a\in(0,1]} [A_{l,a}, A_{r,a}]} \subset [M, N] \subset (0, \infty), \qquad a \in (0,1]. \tag{3.10}$$

Let $x_{-k+1}, x_{-k+2}, \ldots, x_0$ *be fuzzy numbers and* L_{-j}*,* R_{-j}*,* $j = 0, 1, \ldots, k-1$ *positive real numbers such that*

$$\begin{gathered}\left[x_{-j}\right]_a = \left[L_{-j,a}, R_{-j,a}\right] \subset \overline{\bigcup_{a\in(0,1]} \left[L_{-j,a}, R_{-j,a}\right]} \subset \left[L_{-j}, R_{-j}\right] \subset (0, \infty), \\ j = 0, 1, \ldots, k-1, \quad a \in (0,1], \quad k \in \{2, 3, \ldots\}.\end{gathered} \tag{3.11}$$

Then the following statements are true.

(i) *Suppose that*

$$1 - \sum_{j=0}^{k-1} R_{-j} > 0, \tag{3.12}$$

$$M > 0, \qquad N \le 1, \tag{3.13}$$

$$L_{0,a} = \min\left\{L_{-j,a}\right\} > 0, \quad R_{0,a} = \min\left\{R_{-j,a}\right\} > 0, \quad j = 0, 1, \ldots, k-1, \quad a \in (0,1], \tag{3.14}$$

hold. Then there exists a unique positive solution x_n *of the fuzzy difference equation* (1.6) *with initial values* $x_{-k+1}, x_{-k+2}, \ldots, x_0$.

(ii) *Suppose that*

$$M > 0, \qquad N < k \ln\left(\frac{k}{k-1}\right), \tag{3.15}$$

$$\left[L_{-j}, R_{-j}\right] \subset \left(0, \frac{1}{k}\right), \quad j = 0, 1, \ldots, k-1, \tag{3.16}$$

hold. Then there exists a unique positive solution x_n of the fuzzy difference equation (1.6) *with initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0$.*

Proof. We consider the family of systems of parametric ordinary difference equations for $a \in (0,1]$ and $n \geq 0$,

$$L_{n+1,a} = \left(1 - \sum_{j=0}^{k-1} R_{n-j,a}\right)\left(1 - e^{-A_{l,a} L_{n,a}}\right), \qquad R_{n+1,a} = \left(1 - \sum_{j=0}^{k-1} L_{n-j,a}\right)\left(1 - e^{-A_{r,a} R_{n,a}}\right). \tag{3.17}$$

(i) From (3.11) and (3.14), we can consider that

$$L_0 = \min\left\{L_{-j}\right\} > 0, \qquad R_0 = \min\left\{R_{-j}\right\} > 0, \quad j = 0, 1, \ldots, k-1. \tag{3.18}$$

Using relations (3.10)–(3.13), (3.18), and Proposition A, we get that the system of ordinary difference equations

$$L_{n+1} = \left(1 - \sum_{j=0}^{k-1} R_{n-j}\right)\left(1 - e^{-ML_n}\right), \qquad R_{n+1} = \left(1 - \sum_{j=0}^{k-1} L_{n-j}\right)\left(1 - e^{-NR_n}\right), \quad n \geq 0, \tag{3.19}$$

with initial values (L_{-j}, R_{-j}), $j = 0, 1, \ldots, k-1$, has a positive solution (L_n, R_n) and so

$$1 - \sum_{j=0}^{k-1} R_{n-j} > 0, \qquad L_n > 0, \; n \geq 1. \tag{3.20}$$

In addition, from (3.10)–(3.14) and Proposition A, we have that (3.17) has a positive solution $(L_{n,a}, R_{n \cdot a})$, $a \in (0,1]$, with initial values $(L_{-j,a}, R_{-j,a})$, $j = 0, 1, \ldots, k-1$. We prove that $(L_{n,a}, R_{n \cdot a})$, $a \in (0,1]$ determines a sequence of positive fuzzy numbers.

Since x_{-j}, $j = 0, 1, \ldots, k-1$ and A are positive fuzzy numbers, from Theorem 2.1 we have that $R_{-j,a}, L_{-j,a}$, $j = 0, 1, \ldots, k-1$, and $A_{l,a}, A_{r,a}$, $a \in (0,1]$, are left continues and so from (3.17), we get that $L_{1,a}, R_{1,a}$, $a \in (0,1]$ are left continuous as well.

In addition, for any $a_1, a_2 \in (0,1]$, $a_1 \leq a_2$, we have

$$\begin{gathered} 0 < A_{l,a_1} \leq A_{l,a_2} \leq A_{r,a_2} \leq A_{r,a_1} \\ 0 < L_{-j,a_1} \leq L_{-j,a_2} \leq R_{-j,a_2} \leq R_{-j,a_1}, \quad j = 0, 1, \ldots, k-1, \end{gathered} \tag{3.21}$$

and so from (3.10)–(3.13), and (3.17)

$$L_{1,a_1} \leq L_{1,a_2} \leq R_{1,a_2} \leq R_{1,a_1}. \tag{3.22}$$

Moreover, from (3.10)–(3.13), (3.17), and (3.19), we get

$$0 < L_1 < L_{1,a} \leq R_{1,a} < R_1, \quad a \in (0,1]. \tag{3.23}$$

Therefore, from Theorem 2.1 relations (3.22), (3.23), and since $L_{1,a}$, $R_{1,a}$ are left continuous, we have that $L_{1,a}$, $R_{1,a}$ determine a positive fuzzy number x_1 such that

$$[x_1]_a = [L_{1,a}, R_{1,a}] \subset \overline{\bigcup_{a\in(0,1]} [L_{1,a}, R_{1,a}]} \subset [L_1, R_1], \quad a \in (0,1]. \tag{3.24}$$

Since $L_{-j,a}$, $R_{-j,a}$, $j = -1, 0, 1, \ldots, k-1$ are left continuous from (3.17) and working inductively, we get that $L_{n,a}, R_{n,a}$, $n = 2, 3, \ldots$, $a \in (0,1]$ are also left continuous. In addition, using (3.10), (3.11), (3.13), (3.17), (3.20), (3.21), (3.22), and working inductively, we get for any $a_1, a_2 \in (0,1]$, $a_1 \leq a_2$ and $n = 2, 3, \ldots$

$$L_{n,a_1} \leq L_{n,a_2} \leq R_{n,a_2} \leq R_{n,a_1}. \tag{3.25}$$

Finally, using (3.10), (3.11), (3.13), (3.17), (3.19), (3.20), (3.23), and working inductively, we get for $n = 2, 3, \ldots$

$$0 < L_n < L_{n,a} \leq R_{n,a} < R_n, \quad a \in (0,1], \tag{3.26}$$

where (L_n, R_n) is the solution of (3.19).

Therefore, since $L_{n,a}, R_{n,a}$, $n = 1, 2, \ldots$, $a \in (0,1]$ are left continuous and (3.22), (3.23), (3.25), (3.26) are satisfied, from Theorem 2.1, we get that the positive solution $(L_{n,a}, R_{n,a})$, $n = 1, 2, \ldots, a \in (0,1]$, of (3.17), with initial values $L_{-j,a}, R_{-j,a}$, $j = 0, 1, \ldots, k-1$, $a \in (0,1]$, $k \in \{2, 3, \ldots\}$ satisfying (3.11), (3.12), (3.14), determines a sequence of positive fuzzy numbers x_n, such that

$$[x_n]_a = [L_{n,a}, R_{n,a}] \subset \overline{\bigcup_{a\in(0,1]} [L_{n,a}, R_{n,a}]} \subset [L_n, R_n], \; n \geq 1, \; a \in (0,1]. \tag{3.27}$$

We claim that x_n is a solution of (1.6) with initial values x_{-j}, $j = 0, 1, \ldots, k-1$, such that (3.11), (3.12), and (3.14) hold. From (3.17) and (3.27) we have for all $a \in (0,1]$

$$\begin{aligned}
[x_{n+1}]_a &= [L_{n+1,a}, R_{n+1,a}] \\
&= \left[\left(1 - \sum_{j=0}^{k-1} R_{n-j,a}\right)\left(1 - e^{-A_{l,a}L_{n,a}}\right), \left(1 - \sum_{j=0}^{k-1} L_{n-j,a}\right)\left(1 - e^{-A_{r,a}R_{n,a}}\right)\right].
\end{aligned} \tag{3.28}$$

In addition, from (3.10), (3.23), and (3.26), we get

$$1 - e^{-A_{l,a}L_{n,a}} > 0, \quad a \in (0,1], \quad n \geq 1 \tag{3.29}$$

and so from (3.17), (3.23), and (3.26)

$$1 - \sum_{j=0}^{k-1} R_{n-j,a} > 0, \quad n \geq 1. \tag{3.30}$$

Therefore, using (3.28) and arithmetic multiplication on closed intervals

$$[x_{n+1}]_a = \left[1 - \sum_{j=0}^{k-1} R_{n-j,a},\ 1 - \sum_{j=0}^{k-1} L_{n-j,a}\right]\left[1 - e^{-A_{l,a}L_{n,a}},\ 1 - e^{-A_{r,a}R_{n,a}}\right]. \tag{3.31}$$

Using arithmetic operations on positive fuzzy numbers and (2.8) we have

$$[x_{n+1}]_a = \left(1 - \sum_{j=0}^{k-1} [x_{n-j}]_a\right)\left(1 - e^{-[Ax_n]_a}\right) = \left[\left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right)\right]_a \tag{3.32}$$

and thus, our claim is true.

Finally, suppose that there exists another solution $\overline{x}_n = [\overline{L}_{n,a}, \overline{R}_{n,a}]_a$ of the fuzzy difference equation (1.6) with initial values x_{-j}, $j = 0, 1, \ldots, k-1$, such that (3.10)–(3.14) hold. Then using the uniqueness of the solutions of the system (3.17) and arithmetic operations on positive fuzzy numbers and (2.8), we can easily prove that

$$\begin{aligned}
[\overline{x}_{n+1}]_a &= \left[\left(1 - \sum_{j=0}^{k-1} \overline{x}_{n-j}\right)\left(1 - e^{-A\overline{x}_n}\right)\right]_a \\
&= \left[\left(1 - \sum_{j=0}^{k-1} \overline{R}_{n-j,a}\right)\left(1 - e^{-A_{l,a}\overline{L}_{n,a}}\right), \left(1 - \sum_{j=0}^{k-1} \overline{L}_{n-j,a}\right)\left(1 - e^{-A_{r,a}\overline{R}_{n,a}}\right)\right] \\
&= [L_{n+1,a}, R_{n+1,a}] = [x_{n+1}]_a, \quad n \geq 1,\ a \in (0,1],
\end{aligned} \tag{3.33}$$

and so we have that x_n is the unique positive solution of the fuzzy difference equation (1.6) with initial values x_{-j}, $j = 0, 1, \ldots, k-1$, such that (3.11), (3.12), and (3.14) hold. This completes the proof of statement (i).

(ii) From (3.10), (3.15), (3.16), and Proposition A, we get that system (3.19) with initial values (L_{-j}, R_{-j}), $j = 0, 1, \ldots, k-1$ has a positive solution (L_n, R_n) such that (3.20) and

$$[L_n, R_n] \subset \left(0, \frac{1}{k}\right), \quad n \geq 1, \; k \in \{2, 3, \ldots\}, \tag{3.34}$$

hold.

From (3.10), (3.11), (3.15), (3.16), and Proposition A, we have that (3.17) has a positive solution $(L_{n,a}, R_{n \cdot a})$, $a \in (0,1]$, with initial values $(L_{-j,a}, R_{-j,a})$, $j = 0, 1, \ldots, k-1$, such that

$$0 < L_{n,a}, \quad R_{n,a} < \frac{1}{k}, \quad n \geq 1, \; a \in (0,1]. \tag{3.35}$$

We prove that $(L_{n,a}, R_{n \cdot a})$, $a \in (0,1]$ determines a sequence of positive fuzzy numbers.

From (3.10), (3.11), (3.15)–(3.17), (3.19), and (3.20), we get that (3.23) holds. Moreover, arguing as in statement (i), we can easily prove that $L_{1,a}, R_{1,a}$ determine a positive fuzzy number x_1 such that (3.24) holds.

As in statement (i), using (3.10), (3.11), (3.15)–(3.17), (3.24), Theorem 2.1 and working inductively, we get that the positive solution $(L_{n,a}, R_{n,a})$, $n = 1, 2, \ldots, a \in (0,1]$, of (3.17), determines a sequence of positive fuzzy numbers x_n, such that (3.27) holds.

Finally, arguing as in statement (i) we have that x_n is the unique positive solution of the fuzzy difference equation (1.6) with initial values x_{-j}, $j = 0, 1, \ldots, k-1$, such that (3.10), (3.11), (3.15) and (3.16) hold. This completes the proof of the proposition. □

In the next proposition we study the existence of nonnegative equilibriums of the fuzzy difference equation (1.6).

Proposition 3.2. *Consider the fuzzy difference equation* (1.6) *where A is a positive fuzzy number such that* (3.10) *holds and the initial values x_{-j}, $j = 0, 1, \ldots, k-1$ are positive fuzzy numbers. Then the following statements are true.*

(i) *If*

$$A_{l,a} > 0, \quad A_{r,a} \leq 1, \quad a \in (0,1], \tag{3.36}$$

then zero is the unique nonnegative equilibrium of the fuzzy difference equation (1.6).

(ii) *If*

$$A_{l,a} > 0, \quad 1 < A_{r,a} < k \ln\left(\frac{k}{k-1}\right), \quad a \in (0,1], \quad k = 2, 3, \ldots, \tag{3.37}$$

then zero and $\overline{x}$ where

$$[\overline{x}]_a = [0, R_a], \quad a \in (0,1], \tag{3.38}$$

$$0 < R_a = 1 - e^{-A_{r,a} R_a} < \frac{1}{k}, \quad a \in (0,1], \tag{3.39}$$

are the only nonnegative equilibriums of the fuzzy difference equation (1.6)*, such that* (2.11) *and* $L_a R_a = 0$ *hold.*

Proof. We consider the fuzzy equation

$$x = (1 - kx)\left(1 - e^{-Ax}\right), \quad k \in \{2, 3, \ldots\}, \tag{3.40}$$

where A is a positive fuzzy number such that (3.10) holds. Suppose that $\overline{x}$, is a solution of (3.40) such that

$$[\overline{x}]_a = [x_{l,a}, x_{r,a}], \quad 0 \leq x_{l,a}, x_{r,a} < \frac{1}{k}, \quad a \in (0,1], \quad k \in \{2, 3, \ldots\}. \tag{3.41}$$

Then using arithmetic operations on fuzzy numbers and (2.8), (3.10), we can easily prove that $(x_{l,a}, x_{r,a})$ satisfies the family of parametric algebraic systems

$$\begin{gathered} x_{l,a} = (1 - kx_{r,a})\left(1 - e^{-A_{l,a}x_{l,a}}\right), \\ x_{r,a} = (1 - kx_{l,a})\left(1 - e^{-A_{r,a}x_{r,a}}\right), \quad a \in (0,1]. \end{gathered} \tag{3.42}$$

(i) If (3.36) holds then from (3.10), (3.41), (3.42), and statement (i) of Proposition B, we get that

$$x_{l,a} = x_{r,a} = 0, \quad \text{for any } a \in (0,1]. \tag{3.43}$$

This completes the proof of statement (i).

(ii) If (3.37) and (3.41) hold then from (3.10) and statement (ii) of Proposition B, we get that system (3.42) has only two solutions, which are

$$(x_{l,a}, x_{r,a}) = (0, 0), \quad a \in (0,1], \tag{3.44}$$

$$(x_{l,a}, x_{r,a}) = (0, R_a), \quad a \in (0,1], \tag{3.45}$$

where R_a, $a \in (0,1]$, is the unique function which satisfies (3.39).

Using (3.41) and (3.44) we have that zero is a solution of the fuzzy equation (3.40).

To continue, we have to prove that $[0, R_a]$, $a \in (0,1]$, determines a fuzzy number, where R_a, satisfies (3.39). From (3.39), we get

$$e^{A_{r,a}} = \frac{1}{(1 - R_a)^{1/R_a}}. \tag{3.46}$$

We consider the function

$$K(x) = \frac{1}{(1-x)^{1/x}}, \quad 0 < x < \frac{1}{k}; \tag{3.47}$$

then

$$K'(x) = \frac{1}{x^2(1-x)^{1/x}}\left(\ln(1-x) + \frac{x}{1-x}\right). \tag{3.48}$$

We can easily prove that

$$G(x) = \ln(1-x) + \frac{x}{1-x} \tag{3.49}$$

is an increasing and positive function for $0 < x < 1$ and so using (3.48), we get that $K(x)$ is an increasing function for $0 < x < 1/k$. Since $A_{r,a}$ is a positive, decreasing function with respect to a, $a \in (0,1]$, we get that

$$e^{A_{r,a_2}} \leq e^{A_{r,a_1}}, \quad \text{for } a_1, a_2 \in (0,1], \quad \text{with } a_1 \leq a_2, \tag{3.50}$$

and so from (3.46)

$$\frac{1}{(1-R_{a_2})^{1/R_{a_2}}} \leq \frac{1}{(1-R_{a_1})^{1/R_{a_1}}}, \tag{3.51}$$

which means that

$$R_{a_2} \leq R_{a_1}, \quad \text{for } a_1, a_2 \in (0,1], \quad \text{with } a_1 \leq a_2, \tag{3.52}$$

since $K(x)$ is an increasing function. From (3.52) it is obvious that R_a is a decreasing function with respect to a, $a \in (0,1]$.

In addition, since $K(x)$ is a continuous and increasing function, we have that $K^{-1}(x)$ is also a continuous and increasing function. Moreover, $A_{r,a}$ is a left continuous function with respect to a, $a \in (0,1]$.

Therefore,

$$K^{-1}\left(e^{A_{r,a}}\right) = K^{-1}(K(R_a)) = R_a \tag{3.53}$$

is a left continues function with respect to a, $a \in (0,1]$.

Finally, from (3.39) we have that

$$\overline{\bigcup_{a\in(0,1]} [0, R_a]} \subset \left[0, \frac{1}{k}\right], \quad k \in \{2, 3, \ldots\}. \tag{3.54}$$

From Theorem 2.1, (3.39), (3.52), (3.54), and since R_a is a left continuous function with respect to a, $a \in (0,1]$, we have that $[0, R_a]$, $a \in (0,1]$ determines a fuzzy number $\overline{x}$ such that (3.38) holds. Therefore, from (3.45) $\overline{x}$ is a solution of the fuzzy equation (3.40). This completes the proof of the proposition. □

In the last proposition we study the asymptotic behavior of the positive solutions of the fuzzy difference equation (1.6).

Proposition 3.3. *Consider the fuzzy difference equation* (1.6) *where A is a positive fuzzy number such that* (3.10) *holds. Let x_{-j}, $j = 0, 1, \dots, k-1$ be the initial values such that* (3.11) *holds. Then the following statements are true.*

(i) *Suppose that*

$$M > 0, \qquad N < 1 \tag{3.55}$$

and either (3.12) *and* (3.14) *or* (3.16) *are satisfied. Then every positive solution of the fuzzy difference equation* (1.6) *tends to the zero equilibrium as $n \to \infty$.*

(ii) *Suppose that*

$$0 < M < A_{l,a} \leq 1 < A_{r,a} < N < k \ln\left(\frac{k}{k-1}\right), \qquad a \in (0,1], \tag{3.56}$$

and (3.16) *are satisfied. Then every positive solution of the fuzzy difference equation* (1.6) *nearly converges to the nonnegative equilibrium $\overline{x}$ with respect to D as $n \to \infty$ and converges to $\overline{x}$ with respect to D_1 as $n \to \infty$, where $\overline{x}$ was defined by* (3.38) *and* (3.39).

Proof. (i) Since (3.55) and either (3.12) and (3.14) or (3.16) are satisfied, from Proposition 3.1 the fuzzy difference equation (1.6) has unique positive solution x_n, such that (3.27) holds.

In addition, (3.10) and (3.55) imply that (3.36) holds. So, from statement (i) of Proposition 3.2, zero is the unique nonnegative equilibrium of the fuzzy difference equation (1.6).

From the analogous relation of (3.9) of Proposition C and using (3.10), (3.11), we get

$$0 < R_{n,a} < A_{r,a}^{n} R_{0,a} < N^{n} R_0, \qquad \text{for any } a \in (0,1],\ n = 1, 2, \dots, \tag{3.57}$$

and since

$$0 < \lim D(x_n, 0) = \lim \sup\{\max\{|L_{n,a} - 0|, |\mathrm{R}_{n,a} - 0|\}\} = \lim \sup\{R_{n,a}\}, \tag{3.58}$$

where $n \to \infty$ and sup is taken for all $a \in (0,1]$, from (3.55) and (3.57), we get

$$\lim D(x_n, 0) = 0, \qquad n \longrightarrow \infty. \tag{3.59}$$

This completes the proof of statement (i).

(ii) Since from (3.56), we have that (3.15) and (3.37) are fulfilled, we get from (3.16) and statement (ii) of Propositions 3.1 and 3.2 that the fuzzy difference equation (1.6) has unique positive solution x_n such that (3.27) holds, and a nonnegative equilibrium $\overline{x}$, such that (3.38) and (3.39) hold. Since $(L_{n,a}, R_{n,a})$ is a positive solution of system (3.17), from (3.11), (3.16), (3.56), and Proposition C we have that

$$\lim_{n\to\infty} L_{n,a} = 0, \qquad \lim_{n\to\infty} R_{n,a} = R_a, \qquad a \in (0,1]. \tag{3.60}$$

Using (3.60) and arguing as in Proposition 2 of [34], we can prove that the positive solution x_n of (1.6) nearly converges to $\overline{x}$ with respect to D as $n \to \infty$ and converges to $\overline{x}$ with respect to D_1 as $n \to \infty$. Thus, the proof of the proposition is completed. □

To illustrate our results we give some examples in which the conditions of our propositions hold.

Example 3.4. Consider the fuzzy equation (1.6) for $k = 2$

$$x_{n+1} = (1 - x_n - x_{n-1})\left(1 - e^{-Ax_n}\right), \qquad n = 0, 1, \ldots, \tag{3.61}$$

where A is a fuzzy number such that

$$A(x) = \begin{cases} 10x - 5, & 0.5 \le x \le 0.6, \\ -5x + 4, & 0.6 \le x \le 0.8. \end{cases} \tag{3.62}$$

We take the initial values x_{-1}, x_0 such that

$$\begin{gathered} x_{-1}(x) = \begin{cases} 10x - 2, & 0.2 \le x \le 0.3 \\ -\dfrac{10}{3}x + 2, & 0.3 \le x \le 0.6, \end{cases} \\ x_0(x) = \begin{cases} 10x - 1, & 0.1 \le x \le 0.2, \\ -\dfrac{20}{3}x + \dfrac{7}{3}, & 0.2 \le x \le 0.35. \end{cases} \end{gathered} \tag{3.63}$$

From (3.62), we get

$$[A]_a = \left[\frac{a+5}{10}, \frac{4-a}{5}\right], \qquad a \in (0,1] \tag{3.64}$$

and so

$$\overline{\bigcup_{a\in(0,1]} [A]_a} \subset [0.2, 0.8]. \tag{3.65}$$

Moreover from (3.63) we take

$$[x_{-1}]_a = \left[\frac{a+2}{10}, \frac{6-3a}{10}\right], \qquad [x_0]_a = \left[\frac{a+1}{10}, \frac{7-3a}{20}\right] \tag{3.66}$$

and so

$$\overline{\bigcup_{a\in(0,1]} [x_{-1}]_a} \subset [0.2,\ 0.6], \qquad \overline{\bigcup_{a\in(0,1]} [x_0]_a} \subset [0.1, 0.35]. \tag{3.67}$$

Therefore the conditions (3.10)–(3.14) are satisfied. So from statement (i) of Proposition 3.1 the solution x_n of (3.61) with initial values x_{-1}, x_0 is positive and unique. In addition it is obvious that (3.36) are satisfied. Then from the statement (i) of Proposition 3.2 we have that zero is the unique nonnegative equilibrium of (3.61). Finally from Proposition 3.3 the unique positive solution x_n of (3.61) with initial values x_{-1}, x_0 tends to the zero equilibrium of (3.61) as $n \to \infty$.

Example 3.5. Consider the fuzzy equation (3.61) where A is a fuzzy number such that

$$A(x) = \begin{cases} 5x-4, & 0.8 \le x \le 1, \\ -\dfrac{10}{3}x + \dfrac{13}{3}, & 1 \le x \le 1.3. \end{cases} \tag{3.68}$$

We take the initial values x_{-1}, x_0 such that

$$x_{-1}(x) = \begin{cases} 10x-1, & 0.1 \le x \le 0.2, \\ -5x+2, & 0.2 \le x \le 0.4, \end{cases}$$
$$x_0(x) = \begin{cases} \dfrac{20}{3}x - 1, & 0.15 \le x \le 0.3, \\ -\dfrac{20}{3}x + 3, & 0.3 \le x \le 0.45. \end{cases} \tag{3.69}$$

From (3.68), we get

$$[A]_a = \left[\frac{a+4}{5}, \frac{13-3a}{10}\right], \quad a \in (0,1] \tag{3.70}$$

and so

$$\overline{\bigcup_{a\in(0,1]} [A]_a} \subset [0.8, 1.3]. \tag{3.71}$$

Moreover from (3.69) we take

$$[x_{-1}]_a = \left[\frac{a+1}{10}, \frac{2-a}{5}\right], \qquad [x_0]_a = \left[\frac{3(a+1)}{20}, \frac{3(3-a)}{20}\right] \tag{3.72}$$

and so

$$\overline{\bigcup_{a\in(0,1]} [x_{-1}]_a} \subset [0.1, 0.4], \qquad \overline{\bigcup_{a\in(0,1]} [x_0]_a} \subset \left[\frac{3}{20}, \frac{9}{20}\right]. \tag{3.73}$$

Therefore the conditions (3.15), (3.16) are satisfied. So from statement (ii) of Proposition 3.1 the solution x_n of (3.61) with initial values x_{-1}, x_0 is positive and unique.

Example 3.6. We consider equation (3.61) where the fuzzy number A is given as follows

$$A(x) = \begin{cases} 20x - 23, & 1.15 \le x \le 1.2, \\ -10x + 13, & 1.2 \le x \le 1.3. \end{cases} \tag{3.74}$$

Then from (3.74), we get

$$[A]_a = \left[\frac{a+23}{20}, \frac{13-a}{10}\right], \qquad a \in (0,1]. \tag{3.75}$$

Then it is obvious that (3.37) are satisfied. Then from the statement (ii) of Proposition 3.2 we have that zero and $\overline{x}$ where $[\overline{x}]_a = [0, R_a]$, $a \in (0,1]$, $0 < R_a = 1 - e^{(13-a)/10R_a} < 1/2$, $a \in (0,1]$ are the only nonnegative equilibriums of the fuzzy difference equation (3.61), such that (2.11) and $L_a R_a = 0$ hold.

Example 3.7. We consider the fuzzy difference equation (3.61) where A is given by (3.62). Let x_{-1}, x_0 be the fuzzy numbers given by (3.69). Then since (3.15), (3.16), and (3.36) hold from Propositions 3.1, 3.2 and 3.3 the unique positive solution x_n of (3.61) with initial values x_{-1}, x_0 tends to the zero equilibrium of (3.61) as $n \to \infty$.

Example 3.8. Consider the fuzzy difference equation (3.61) where the fuzzy number A is given by

$$A(x) = \begin{cases} 10x - 9, & 0.9 \le x \le 1, \\ 1, & 1 \le x \le 1.2, \\ -10x + 13, & 1.2 \le x \le 1.3. \end{cases} \tag{3.76}$$

Then from (3.76), we get

$$[A]_a = \left[\frac{a+9}{10}, \frac{13-a}{10}\right], \quad a \in (0,1]. \tag{3.77}$$

Then it is obvious that relations (3.37) are satisfied. So from the statement (ii) of Proposition 3.2 we have that zero and $\overline{x}$ where $[\overline{x}]_a = [0, R_a]$, $a \in (0,1]$, $0 < R_a = 1 - e^{((13-a)/10)R_a} < 1/2$, $a \in (0,1]$ are the only nonnegative equilibriums of the fuzzy difference equation (3.61), such that (2.11) and $L_a R_a = 0$ hold. Let x_{-1}, x_0 be the fuzzy numbers defined in (3.69). Then from the statement (ii) of Proposition 3.1 and statement (ii) of Proposition 3.3 we have that the unique positive solution x_n of (3.61) with initial values x_{-1}, x_0 nearly converges to the nonnegative equilibrium $\overline{x}$ with respect to D as $n \to \infty$ and converges to $\overline{x}$ with respect to D_1 as $n \to \infty$.

4. Conclusions

In this paper, we considered the fuzzy difference equation (1.6), where A and the initial values $x_{-k+1}, \ldots, x_0$ are positive fuzzy numbers. The corresponding ordinary difference equation (1.6) is a special case of an epidemic model. The combine of difference equations and Fuzzy Logic is an extra motivation for studying this equation. A mathematical modelling of a real world phenomenon, very often, leads to a difference equation and on the other hand, Fuzzy Logic can handle uncertainness, imprecision or vagueness related to the experimental input-output data.

The main results of this paper are the following. Firstly, under some conditions on A and initial values we found positive solutions and nonnegative equilibriums and then we studied the convergence of the positive solutions to the nonnegative equilibrium of the fuzzy difference equations (1.6). We note that, in order to study the fuzzy difference equation (1.6), we used the results concerning the behavior of the solutions of the related system of two parametric ordinary difference equations (1.7) (see [11]).

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References

[1] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fuzzy difference equations," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 957–968, 2002.

[2] J. J. Buckley, T. Feuring, and Y. Hayashi, "Fuzzy difference equations: the initial value problem," *Journal of Advanced Computational Intelligence*, vol. 5, no. 6, pp. 315–325, 2001.

[3] R. C. Bassanezi, L. C. de Barros, and P. A. Tonelli, "Attractors and asymptotic stability for fuzzy dynamical systems," *Fuzzy Sets and Systems*, vol. 113, no. 3, pp. 473–483, 2000.

[4] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[5] J. R. Beddington, C. A. Free, and J. H. Lawton, "Dynamic complexity in predator prey models framed in difference equations," *Nature*, vol. 255, no. 5503, pp. 58–60, 1975.

[6] K. L. Cooke, D. F. Calef, and E. V. Level, "Stability or chaos in discrete epidemic models," in *Nonlinear Systems and Applications*, pp. 73–93, Academic Press, New York, NY, USA, 1977.

[7] E.-G. Gu, "The nonlinear analysis on a discrete host-parasitoid model with pesticidal interference," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 6, pp. 2720–2727, 2009.

[8] B. E. Kendall, "Cycles, chaos, and noise in predator-prey dynamics," *Chaos, Solitons and Fractals*, vol. 12, no. 2, pp. 321–332, 2001.

[9] D. C. Zhang and B. Shi, "Oscillation and global asymptotic stability in a discrete epidemic model,"

Journal of Mathematical Analysis and Applications, vol. 278, no. 1, pp. 194–202, 2003.

[10] S. Stević, "On a discrete epidemic model," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 87519, 10 pages, 2007.

[11] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, "On a system of two exponential type difference equations," *Communications on Applied Nonlinear Analysis*, vol. 17, no. 2, pp. 1–13, 2010.

[12] G. Stefanidou and G. Papaschinopoulos, "The periodic nature of the positive solutions of a nonlinear fuzzy max-difference equation," *Information Sciences*, vol. 176, no. 24, pp. 3694–3710, 2006.

[13] L. Berg, "On the asymptotics of nonlinear difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.

[14] L. Berg, "Inclusion theorems for non-linear difference equations with applications," *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.

[15] L. Berg, "On the asymptotics of the difference equation $x_{n-3} = x_n(1 + x_{n-1}x_{n-2})$," *Journal of Difference Equations and Applications*, vol. 14, no. 1, pp. 105–108, 2008.

[16] L. Gutnik and S. Stević, "On the behaviour of the solutions of a second-order difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 27562, 14 pages, 2007.

[17] B. Iričanin and S. Stević, "Eventually constant solutions of a rational difference equation," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 854–856, 2009.

[18] C. M. Kent, "Convergence of solutions in a nonhyperbolic case," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 7, pp. 4651–4665, 2001.

[19] S. Stević, "On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$," *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 405–414, 2002.

[20] S. Stević, "Global stability and asymptotics of some classes of rational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.

[21] S. Stević, "On monotone solutions of some classes of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 53890, 9 pages, 2006.

[22] S. Stević, "On positive solutions of a $(k+1)$-th order difference equation," *Applied Mathematics Letters*, vol. 19, no. 5, pp. 427–431, 2006.

[23] S. Stević, "Asymptotics of some classes of higher-order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.

[24] S. Stević, "Existence of nontrivial solutions of a rational difference equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.

[25] S. Stević, "Nontrivial solutions of higher-order rational difference equations," *Matematicheskie Zametki*, vol. 84, no. 5, pp. 772–780, 2008.

[26] T. Sun, "On non-oscillatory solutions of the recursive sequence $x_{n+1} = (p + x_{n-k})/x_n$," *Journal of Difference Equations and Applications*, vol. 11, no. 6, pp. 483–485, 2005.

[27] H. D. Voulov, "Existence of monotone solutions of some difference equations with unstable equilibrium," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 2, pp. 555–564, 2002.

[28] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Raton, Fla, USA, 1997.

[29] C. Wu and B. Zhang, "Embedding problem of noncompact fuzzy number space $E^{\sim}$. I," *Fuzzy Sets and Systems*, vol. 105, no. 1, pp. 165–169, 1999.

[30] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, USA, 1995.

[31] B. Bede and G. B. Tanali, *Perspectives of Fuzzy Initial Value Problems*, Research Reports Series, University of Texas at El Paso, 2006.

[32] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Application*, World Scientific, River Edge, NJ, USA, 1994.

[33] H. Román-Flores, L. C. Barros, and R. C. Bassanezi, "A note on Zadeh's extensions," *Fuzzy Sets and Systems*, vol. 117, no. 3, pp. 327–331, 2001.

[34] G. Papaschinopoulos and G. Stefanidou, "Boundedness and asymptotic behavior of the solutions of a fuzzy difference equation," *Fuzzy Sets and Systems*, vol. 140, no. 3, pp. 523–539, 2003.