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Research Article

Mild Solutions for Fractional Differential Equations with Nonlocal Conditions

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This paper is concerned with the existence and uniqueness of mild solution of the fractional differential equations with nonlocal conditions $d^q x(t)/dt^q = -Ax(t) + f(t, x(t), Gx(t))$, $t \in [0, T]$, and $x(0) + g(x) = x_0$, in a Banach space X, where $0 < q < 1$. General existence and uniqueness theorem, which extends many previous results, are given.

1. Introduction

The fractional differential equations can be used to describe many phenomena arising in engineering, physics, economy, and science, so they have been studied extensively (see, e.g., [1–8] and references therein).

In this paper, we discuss the existence and uniqueness of mild solution for

$$\begin{gathered}\frac{d^q x(t)}{dt^q} = -Ax(t) + f(t, x(t), Gx(t)), \quad t \in [0, T], \\ x(0) + g(x) = x_0,\end{gathered} \tag{1.1}$$

where $0 < q < 1$, $T > 0$, and $-A$ generates an analytic compact semigroup $\{S(t)\}_{t \geq 0}$ of uniformly bounded linear operators on a Banach space X. The term $Gx(t)$ which may be interpreted as a control on the system is defined by

$$Gx(t) := \int_0^t K(t, s)x(s)ds, \tag{1.2}$$

where $K \in C(D, \mathbb{R}^+)$(the set of all positive function continuous on $D := \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$) and

$$G^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty. \tag{1.3}$$

The functions f and g are continuous.

The nonlocal condition $x(0) + g(x) = x_0$ can be applied in physics with better effect than that of the classical initial condition $x(0) = x_0$. There have been many significant developments in the study of nonlocal Cauchy problems (see, e.g., [6, 7, 9–14] and references cited there).

In this paper, motivated by [1–7, 9–15] (especially the estimating approach given by Xiao and Liang [14]), we study the semilinear fractional differential equations with nonlocal condition (1.1) in a Banach space X, assuming that the nonlinear map f is defined on $[0,T] \times X_\alpha \times X_\alpha$ and g is defined on $C([0,T], X_\alpha)$ where $X_\alpha = D(A^\alpha)$, for $0 < \alpha < 1$, the domain of the fractional power of A. New and general existence and uniqueness theorem, which extends many previous results, are given.

2. Preliminaries

In this paper, we set $I = [0,T]$, a compact interval in $\mathbb{R}$. We denote by X a Banach space with norm $\|\cdot\|$. Let $-A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t)\}_{t\ge0}$, that is, there exists $M > 1$ such that $\|S(t)\| \le M$; and without loss of generality, we assume that $0 \in \rho(A)$. So we can define the fractional power A^α for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$, and one has the following known result.

Lemma 2.1 (see [15]). (1) *$X_\alpha = D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$.*

(2) *$S(t) : X \to X_\alpha$ for each $t > 0$ and $\alpha > 0$.*

(3) *For every $u \in D(A^\alpha)$ and $t \ge 0$, $S(t)A^\alpha u = A^\alpha S(t)u$.*

(4) *For every $t > 0$, $A^\alpha S(t)$ is bounded on X and there exists $M_\alpha > 0$ such that*

$$\|A^\alpha S(t)\| \le M_\alpha t^{-\alpha}. \tag{2.1}$$

Definition 2.2. A continuous function $x : I \to X$ satisfying the equation

$$x(t) = S(t)\big(x_0 - g(x)\big) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Gx(s))ds \tag{2.2}$$

for $t \in I$ is called a mild solution of (1.1).

In this paper, we use $\|f\|_p$ to denote the L^p norm of f whenever $f \in L^p(0,T)$ for some p with $1 \le p < \infty$. We denote by C_α the Banach space $C([0,T], X_\alpha)$ endowed with the sup norm given by

$$\|x\|_\infty := \sup_{t\in I} \|x\|_\alpha, \tag{2.3}$$

for $x \in C_\alpha$.

The following well-known theorem will be used later.

Theorem 2.3 (Krasnoselkii, see [16]). *Let Ω be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*

(i) *$Ax + By \in \Omega$ whenever $x, y \in \Omega$.*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in \Omega$ such that $z = Az + Bz$.

3. Main Results

We require the following assumptions.

(H1) The function $f : [0,T] \times X_\alpha \times X_\alpha \to X$ is continuous, and there exists a positive function $\mu(\cdot) : [0,T] \to \mathbb{R}^+$ such that

$$\begin{gathered} \|f(t,x,y)\| \le \mu(t), \text{ the function } s \longmapsto \frac{\mu(s)}{(t-s)^\alpha} \text{ belongs to } L^p([0,t], \mathbb{R}^+), \\ \gamma(t) := \left(\int_0^t \left(\frac{\mu(s)}{(t-s)^\alpha}\right)^p ds\right)^{1/p} \le M_T < \infty, \quad \text{for } t \in [0,T], \end{gathered} \tag{3.1}$$

where $p > 1/q > 1$.

(H2) The function $g : C_\alpha \to X_\alpha$ is continuous and there exists $b > 0$ such that

$$\|g(x) - g(y)\|_\alpha \le b\ \|x - y\|_\infty, \tag{3.2}$$

for any $x, y \in C_\alpha$.

Theorem 3.1. *Let $-A$ be the infinitesimal generator of an analytic compact semigroup $\{S(t)\}_{t\ge 0}$ with $\|S(t)\| \le M,\ t \ge 0$, and $0 \in \rho(A)$. If the maps f and g satisfy (H1), (H2), respectively, and $Mb < 1$, then* (1.1) *has a mild solution for every $x_0 \in X_\alpha$.*

Proof. Set $\lambda = \sup_{x\in C_\alpha} \|g(x)\|_\alpha$ and choose r such that

$$r \ge M(\|x_0\|_\alpha + \lambda) + \frac{M_\alpha M_T}{\Gamma(q)} M_{p,q} \cdot T^{(q-1)/p}, \tag{3.3}$$

where $M_{p,q} := ((p-1)/(pq-1))^{(p-1)/p}$.

Let $B_r = \{x \in C([0,T], X_\alpha) \mid \|x\|_\infty \le r\}$.
Define

$$(Ax)(t) := \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Gx(s))\,ds,$$
$$(Bx)(t) := S(t)(x_0 - g(x)). \tag{3.4}$$

Let $x, y \in B_r$, then for $t \in [0,T]$ we have the estimates

$$\begin{aligned}
&\|(Ax)(t) + (By)(t)\|_\alpha \\
&\quad \le \|S(t)\|(\|x_0\|_\alpha + \lambda) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A^\alpha S(t-s) f(s, x(s), Gx(s))\|ds \\
&\quad \le M(\|x_0\|_\alpha + \lambda) + \frac{M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{\mu(s)}{(t-s)^\alpha}ds \\
&\quad \le M(\|x_0\|_\alpha + \lambda) + \frac{M_\alpha}{\Gamma(q)}\left(\int_0^t (t-s)^{(q-1)p/(p-1)}ds\right)^{(p-1)/p} \cdot \left(\int_0^t \left(\frac{\mu(s)}{(t-s)^\alpha}\right)^p ds\right)^{1/p} \\
&\quad \le M(\|x_0\|_\alpha + \lambda) + \frac{M_\alpha M_T}{\Gamma(q)} M_{p,q} \cdot T^{q-1/p} \\
&\quad \le r.
\end{aligned} \tag{3.5}$$

Hence we obtain $Ax + By \in B_r$.

Now we show that A is continuous. Let $\{x_n\}$ be a sequence of B_r such that $x_n \to x$ in B_r. Then

$$f(s, x_n(s), Gx_n(s)) \longrightarrow f(s, x(s), Gx(s)), \quad n \longrightarrow \infty, \tag{3.6}$$

since the function f is continuous on $I \times X_\alpha \times X_\alpha$. For $t \in [0,T]$, using (2.1), we have

$$\begin{aligned}
&\|(Ax_n)(t) - (Ax)(t)\|_\alpha \\
&\quad = \frac{1}{\Gamma(q)}\left\|\int_0^t (t-s)^{q-1} S(t-s)\left[f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\right]ds\right\|_\alpha \\
&\quad \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|A^\alpha S(t-s)\left[f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\right]\right\|ds \\
&\quad \le \frac{M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\|(t-s)^{-\alpha}ds.
\end{aligned} \tag{3.7}$$

In view of the fact that

$$\|f(s,x_n(s),Gx_n(s)) - f(s,x(s),Gx(s))\| \le 2\mu(s), \quad s \in [0,T], \tag{3.8}$$

and the function $s \to 2\mu(s)(t-s)^{-\alpha}$ is integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem ensures that

$$\int_0^t (t-s)^{q-1}\|f(s,x_n(s),Gx_n(s)) - f(s,x(s),Gx(s))\|(t-s)^{-\alpha}ds \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Therefore, we can see that

$$\lim_{n\to\infty}\|(Ax_n)(t) - (Ax)(t)\|_\infty = 0, \tag{3.10}$$

which means that A is continuous.

Noting that

$$\begin{aligned}
\|(Ax)(t)\|_\alpha &= \frac{1}{\Gamma(q)}\left\|\int_0^t (t-s)^{q-1}S(t-s)f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A^\alpha S(t-s)f(s,x(s),Gx(s))\|ds \\
&\le \frac{M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{\mu(s)}{(t-s)^\alpha}ds \\
&\le \frac{M_\alpha\, M_T}{\Gamma(q)}M_{p,q}\cdot T^{q-1/p},
\end{aligned} \tag{3.11}$$

we can see that A is uniformly bounded on B_r.

Next, we prove that $(Ax)(t)$ is equicontinuous. Let $0 < t_2 < t_1 < T$, and let $\varepsilon > 0$ be small enough, then we have

$$\begin{aligned}
\|(Ax)(t_1) - (Ax)(t_2)\|_\alpha &\le \frac{1}{\Gamma(q)}\left\|\int_0^{t_2}\left[(t_1-s)^{q-1} - (t_2-s)^{q-1}\right]S(t_2-s)f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\quad + \frac{1}{\Gamma(q)}\left\|\int_{t_2}^{t_1}(t_1-s)^{q-1}S(t_1-s)f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\quad + \frac{1}{\Gamma(q)}\left\|\int_0^{t_2}(t_1-s)^{q-1}[S(t_1-s) - S(t_2-s)]f(s,x(s),Gx(s))ds\right\|_\alpha \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.12}$$

Using (2.1) and (H1), we have

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(q)}\left\|\int_0^{t_2}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right]S(t_2-s)f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_2}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|\left\|A^\alpha S(t_2-s)f(s,x(s),Gx(s))\right\|ds \\
&\le \frac{M_\alpha}{\Gamma(q)}\int_0^{t_2}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|\frac{\mu(s)}{(t_2-s)^\alpha}ds \\
&\le \frac{M_\alpha}{\Gamma(q)}\int_0^{t_2-\varepsilon}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\frac{\mu(s)}{(t_2-s)^\alpha}ds \\
&\quad+\frac{M_\alpha}{\Gamma(q)}\int_{t_2-\varepsilon}^{t_2}(t_2-s)^{q-1}\frac{\mu(s)}{(t_2-s)^\alpha}ds \\
&= I_1'+I_1''.
\end{aligned} \tag{3.13}$$

It follows from the assumption of $\mu(s)$ that I_1' tends to 0 as $t_2 \to t_1$. For I_1'', using the Hölder inequality, we can see that I_1'' tends to 0 as $t_2 \to t_1$ and $\varepsilon \to 0$.

For I_2, using (2.1), (H1), and the Hölder inequality, we have

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(q)}\left\|\int_{t_2}^{t_1}(t_1-s)^{q-1}S(t_1-s)f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\le \frac{1}{\Gamma(q)}\int_{t_2}^{t_1}(t_1-s)^{q-1}\left\|A^\alpha S(t_1-s)f(s,x(s),Gx(s))\right\|ds \\
&\le \frac{M_\alpha}{\Gamma(q)}\int_{t_2}^{t_1}(t_1-s)^{q-1}\frac{\mu(s)}{(t_1-s)^\alpha}ds \longrightarrow 0 \quad \text{as } t_2 \longrightarrow t_1.
\end{aligned} \tag{3.14}$$

Moreover,

$$\begin{aligned}
I_3 &\le \frac{1}{\Gamma(q)}\left\|\int_0^{t_2-\varepsilon}(t_1-s)^{q-1}\left[S(t_1-s)-S(t_2-s)\right]f(s,x(s),Gx(s))ds\right\|_\alpha \\
&\quad+\frac{1}{\Gamma(q)}\left\|\int_{t_2-\varepsilon}^{t_2}(t_1-s)^{q-1}\left[S(t_1-s)-S(t_2-s)\right]f(s,x(s),Gx(s))ds\right\|_\alpha
\end{aligned}$$

$$\le \frac{1}{\Gamma(q)}\int_0^{t_2-\varepsilon}(t_1-s)^{q-1}\left\|S\left(\frac{t_1-t_2}{2}+\frac{t_1-s}{2}\right)-S\left(\frac{t_2-s}{2}\right)\right\|$$

$$\cdot\left\|A^{\alpha}S\left(\frac{t_2-s}{2}\right)f(s,x(s),Gx(s))\right\|ds$$

$$+\frac{M_\alpha}{\Gamma(q)}\int_{t_2-\varepsilon}^{t_2}(t_1-s)^{q-1}\left[\frac{\mu(s)}{(t_1-s)^{\alpha}}+\frac{\mu(s)}{(t_2-s)^{\alpha}}\right]ds$$

$$\le \frac{2^{\alpha}M_\alpha}{\Gamma(q)}\int_0^{t_2-\varepsilon}(t_1-s)^{q-1}\left\|S\left(\frac{t_1-t_2}{2}+\frac{t_1-s}{2}\right)-S\left(\frac{t_2-s}{2}\right)\right\|\cdot\frac{\mu(s)}{(t_2-s)^{\alpha}}ds$$

$$+\frac{M_\alpha}{\Gamma(q)}\int_{t_2-\varepsilon}^{t_2}(t_1-s)^{q-1}\left[\frac{\mu(s)}{(t_1-s)^{\alpha}}+\frac{\mu(s)}{(t_2-s)^{\alpha}}\right]ds$$

$$= I_3'+I_3''. \tag{3.15}$$

Using the compactness of $S(t)$ in X implies the continuity of $t\mapsto\|S(t)\|$ for $t\in[0,T]$; integrating with $s\mapsto\mu(s)/(t_2-s)^{\alpha}\in L^1_{\text{loc}}([0,t_2],\mathbb{R}^+)$, we see that I_3' tends to 0, as $t_2\to t_1$. For I_3'', from the assumption of $\mu(s)$ and the Hölder inequality, it is easy to see that I_3'' tends to 0 as $t_2\to t_1$ and $\varepsilon\to 0$.

Thus, $\|(Ax)(t_1)-(Ax)(t_2)\|_\alpha\to 0$, as $t_2\to t_1$, which does not depend on x.

So, $A(B_r)$ is relatively compact. By the Arzela-Ascoli Theorem, A is compact.

Now, let us prove that B is a contraction mapping. For $x,y\in C([0,T],X_\alpha)$ and $t\in[0,T]$, we have

$$\|(Bx)(t)-(By)(t)\|_\alpha\le\|S(t)\|\|g(x)-g(y)\|_\alpha\le Mb\|x-y\|_\infty<\|x-y\|_\infty. \tag{3.16}$$

So, we obtain

$$\|(Bx)(t)-(By)(t)\|_\infty<\|x-y\|_\infty. \tag{3.17}$$

We now conclude the result of the theorem by Krasnoselkii's theorem. □

Now we assume the following.

(H3) There exists a positive function $\mu_1(\cdot):[0,T]\to\mathbb{R}^+$ such that

$$\|f(t,x(t),Gx(t))-f(t,y(t),Gy(t))\|\le\mu_1(t)\left(\|x-y\|_\alpha+\|Gx-Gy\|_\alpha\right), \tag{3.18}$$

the function $s\mapsto\mu_1(s)/(t-s)^{\alpha}$ belongs to $L^1([0,t],\mathbb{R}^+)$ and

$$\gamma'(t):=\left(\int_0^t\left(\frac{\mu_1(s)}{(t-s)^{\alpha}}\right)^p ds\right)^{1/p}\le M_T'<\infty,\quad\text{for } t\in[0,T]. \tag{3.19}$$

(H4) The function $L_{\alpha,q} : I \to \mathbb{R}^+,\ 0 < \alpha,\ q < 1$ satisfies

$$L_{\alpha,q}(t) = Mb + \frac{M_\alpha M'_T}{\Gamma(q)} M_{p,q} \cdot t^{q-1/p}(1 + G^*) \leq \omega < 1, \quad t \in [0,T]. \tag{3.20}$$

Theorem 3.2. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ with $\|S(t)\| \leq M,\ t \geq 0$ and $0 \in \rho(A)$. If $x_0 \in X_\alpha$ and (H2)–(H4) hold, then* (1.1) *has a unique mild solution $x \in C_\alpha$.*

Proof. Define the mapping $\mathcal{F} : C([0,T], X_\alpha) \to C([0,T], X_\alpha)$ by

$$(\mathcal{F}x)(t) = S(t)\big(x_0 - g(x)\big) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Gx(s)) ds. \tag{3.21}$$

Obviously, $\mathcal{F}$ is well defined on $C([0,T], X_\alpha)$.

Now take $x, y \in C([0,T], X_\alpha)$, then we have

$$\begin{aligned}
&\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|_\alpha \\
&\quad \leq \|S(t)(g(x) - g(y))\|_\alpha \\
&\qquad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|S(t-s)[f(s,x(s),Gx(s)) - f(s,y(s),Gy(s))]\|_\alpha ds \\
&\quad \leq M\|g(x) - g(y)\|_\alpha \\
&\qquad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|A^\alpha S(t-s)\big[f(s,x(s),Gx(s)) - f\big(s,y(s),Gy(s)\big)\big]\| ds \\
&\quad \leq Mb\|x-y\|_\infty + \frac{M_\alpha}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{\mu_1(s)}{(t-s)^\alpha} (\|x-y\|_\alpha + \|Gx - Gy\|_\alpha) ds \\
&\quad \leq Mb\|x-y\|_\infty + \frac{M_\alpha M'_T}{\Gamma(q)} M_{p,q} \cdot t^{q-1/p}(1 + G^*)\|x-y\|_\alpha \\
&\quad \leq L_{\alpha,q}(t)\|x-y\|_\infty.
\end{aligned} \tag{3.22}$$

Therefore, we obtain

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|_\infty \leq \omega\|x-y\|_\infty < \|x-y\|_\infty, \tag{3.23}$$

and the result follows from the contraction mapping principle. □

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