Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 517378, 19 pages
doi:10.1155/2010/517378

Research Article

Asymptotical Behaviors of Nonautonomous Discrete Kolmogorov System with Time Lags

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Received 23 November 2009; Accepted 24 April 2010

Academic Editor: Elena Braverman

We discuss a general n-species discrete Kolmogorov system with time lags. We build some new results about the sufficient conditions for permanence, extinction, and balancing survival. When applying these results to some Lotka-Volterra systems, we obtain the criteria on harmless delay for the permanence as well as profitless delay for balancing survival.

1. Introduction

Difference equations are frequently used in modelling the interactions of populations with nonoverlapping generations (see, e.g., May and Oster [1] for the one-species difference equations; [2] for how populations regulate; Hassel [3], Basson and Fogarty [4] and Beddington et al. [5] for predator-prey models). Since one of the most important ecological problems associated with the populations dynamical system is to study the long-term coexistence of all the involved species, such problem in the nondelayed discrete systems had already attracted much attention and thereby many excellent results had appeared ([6–11]). However, recent studies of the natural populations indicated that the interactions of populations, for example, the density-dependent population regulation sometimes takes place over many generations (see [12–29] and the references therein). Turchin [28] evaluated the evidence for delayed density dependence in discrete population dynamics of 14-forest insects, and finded strong evidence for that eight cases exhibited clear evidence for delayed density dependent and lags induce oscillations. And he pointed out that delayed density dependence can arise in natural populations as a result of interactions with other members of the community such as natural enemies, or because high population density may adversely affect the fecundity of the next generation. Furthermore, Turchin and Taylor in [27] proposed the following general delayed discrete populations model:

$$N_t = F\left(N_{t-1}, N_{t-2}, \ldots, N_{t-p}, \varepsilon_t\right), \tag{1.1}$$

where $N_t = (N_t^1, N_t^2, \ldots, N_t^k)$ and N_t^i is the density of species i at time t. Recently, Crone [16] showed that inclusion of effects of parental density on offspring mass fundamentally changes population dynamics models by making recruitment a function of population size in two previous generations. Wikan and Mjølhus [30] showed general delay may have different effects on species. By the above conclusions, it is realistic for us to consider the time-delayed discrete population models.

There have been some excellent works devoted to the delayed discrete models ([11, 17, 22, 31–37]). In 1976, Levin and May [31] showed that, similar to the differential-delay equations, those obey nonoverlapping generations with explicit time lags in the density dependent regulatory mechanisms also lead to stable limit cycle behavior. Ginzburg and Taneyhill [17] developed a two-dimensional model of delayed difference equation which relates the average quality of individuals to patterns of abundance. The delayed density dependence was caused by transmission of quality between generations through maternal effects. They proved that the delayed model can produce patterns of population fluctuations. Later, Crone [32] presented a nondelayed model and revealed that the inclusion of delays changes the shape of population cycles (flip versus Hopf bifurcations) and decreases the range of parameters which were used to predict stable equilibria. In [22], Keeling et al. proposed that delayed density dependence should be one of the reasons for what stabilizes the natural enemy-victim interactions and allows the long-term coexistence of the two species. For more mathematical results, Saito et al. [33, 34], Tang and Xiao [35], Kon [11], and Liu et al. [37] proved that time delays are harmless to the coexistence of two-species Lotka-Volterra difference systems. Tang and Xiao [35] studied the two-species Kolmogorov-type delayed discrete system and obtained the sufficient conditions for its permanence. Recently, Liu et al. [37] focused on a general n-species discrete competitive Lotka-Volterra system with delayed density dependence and delayed interspecific competition which takes the following form:

$$x_i(m+1) = x_i(m) \cdot \exp\left\{ b_i - \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}^{(k)} x_j\left(m - \tau_{ij}^{(k)}\right) \right\}, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Liu et al. [37] showed that under some conditions, the inclusion, exclusion and change of time-delays cannot change the permanence, extinction and balancing survival of species. That is, time-delays maybe harmless for both the permanence and balancing survival of species, in addition to being profitless to the extinction of species. In particular, when $n = 2$, the extinction and permanence of this system were corresponding to some inequalities that only involve the coefficients therein, that is, permanence and extinction in this two-species system are determined only by three elements: growth rate, density dependence and interspecific competition rate.

These papers, while containing many new and significant results, are far from answering the questions on the effects of time delays upon long time behaviors of discrete system. For example, what will be the long-time behaviors for the general nonautonomous delayed discrete Kolmogorov systems? Following the previous works in delay differential models (see [38–46]), in this paper, we study the n-species nonautonomous discrete Kolmogorov-type system with time delays, which takes the following form:

$$x_i(m+1) = x_i(m) \cdot f_i(m, x_m), \quad i = 1, 2, \dots, n, \tag{1.3}$$

where $x_i(m)$ represents the density of population i at the mth generation; $C = C([-\tau, 0], R^n)$ is the space of continuous mapping $[-\tau, 0]$ to R^n with the uniform norm; and $f = (f_1, \dots, f_n) : R \times C \rightarrow R^n$ is a given function with $f_i(m, x_m) > 0$ with some positive below bounds for all $m \in Z^+$, $x_1(m), \dots, x_n(m) > 0$. We define $R^n_+ = \{(x_1, \dots, x_n) \mid x_i \geq 0,\ i = 1, \dots, n\}$.

Suppose $0 \leq \tau < +\infty$ is a given integer. We denote $C^+ = C([-\tau, 0], R^n_+)$ with the uniform norm $\|\cdot\|$ on $[\tau, 0]$, that is, for $\phi \in C^+$, $\|\phi\| = \sup_{-\tau \leq j \leq 0, j \in Z} |\phi(j)|$, where $\|\cdot\|$ is a given norm on R^n. For any function $x : [-\tau, 0] \rightarrow R^n_+$ with $\tau > 0$ and any $m \in [0, \tau]$, we define $x_m(\cdot) \in C^+$ as $x_m(\theta) = x(m + \theta)$ for $m \in Z^+$, $\theta \in Z$ and $\theta \in [-\tau, 0]$. For the purpose of convenience, we write $x(m, \phi) = x(0, \phi)(m)$.

In this paper, we assume that system (1.3) always satisfies the following positive initial conditions:

$$x_i(\theta) = \phi_i(\theta) \geq 0, \quad \phi_i(0) > 0,\ \theta \in [-\tau, 0],\ i = 1, 2, \dots, n, \tag{1.4}$$

where $\phi_i \ \ (i = 1, \dots, n)$ is continuous. Then we have $x_i(m) > 0$ for all $i = 1, \dots, n$, $m \geq 0$.

Consequently, we get the general discrete Kolmogorov system (1.3) which embodies both the overlapping interactions among its species and the time-varying environments. Our model extends and joint those models in [11, 33–37].

Definition 1.1. Species x_i, $(i = 1, \dots, n)$ is called permanent if there exists a positive interval such that x_i will ultimately enter and stay in this interval. A population system is called a permanent one (uniformly persistent) if all of its species are permanent.

Definition 1.2. Species x_i is called extinct if $\lim_{t \rightarrow +\infty} x_i = 0$. An n-species population system is called r-balancing survival $(1 \leq r < n)$ if $n - r$ species in the system go extinct while the remaining r being permanent.

The above definitions on permanence and balancing survival in difference systems are equivalent to those usually for differential case (see, e.g., [41–43, 45, 47, 47, 48]).

The purpose of this paper is to construct some general results for the long-time behaviors (permanence and balancing survival) of system (1.3) and study the effects of time delays on the asymptotical behaviors. We get the sufficient conditions for the permanence and balancing survival of system (1.3), which directly extend those in [35]. We also apply the main results for (1.3) to the n-species Lotka-Volterra systems of competitive type, which are one of the theoretical interests in population biology since they involve Ricker type (exponential) nonlinearities—one of the standard nonlinearities used in the business. And we obtain the sufficient conditions for system (1.3)'s permanence and balancing survival. These results are applied into the nonautonomous competitive delayed discrete Lotka-Volterra systems and directly generalize some relative results in [33–35, 37]. Moreover, we show the delays do not affect the permanence and balancing survival of the n-species Lotka-Volterra discrete systems. Biologically speaking, that is, time delays are both harmless for permanence and profitless to the balancing survival of the system.

Our paper is organized as follows, in the next section we present and prove our main results. In Section 3, we apply the main results into the competitive Lotka-Volterra system and get the corresponding results for its permanence and balancing survival. Discussion follows at the last section.

2. Permanence and Balancing Survival

In ecosystems, the natural resources are limited, so are the species that live in them, therefore, during this paper we always assume that system (1.3) is dissipative, namely system (1.3) is ultimately bounded. Hence there exist a positive constant M and positive integer $N(\phi)$ such that $|x_i(m,\phi)| \le M$ for all $m \ge N(\phi)$.

Definition 2.1. A continuous function $D(x) = (D_1(x),\dots,D_n(x)) : \operatorname{int} R_+^n \to \operatorname{int} R_+^n$ is said to be a boundary function, if for any $M,\delta > 0$, there exist two positive constants $\delta_1,\delta_2 > 0$ such that the following properties hold:

(i) whenever $j \in \{1,\dots,n\}$, $x \in \operatorname{int} R_+^n$ and $|x| \le M$, $D_j(x) \le \delta_1$ implies that $x_j \le \delta$,

(ii) whenever $x \in \operatorname{int} R_+^n$, $|x| \le M$ and $D_j(x) \ge \delta$ for all $j = 1,\dots,n$ imply that $x_j \ge \delta_2$ for all $j = 1,2,\dots,n$.

Definition 2.2. We define $V(x(m,\phi)) = (V_1(x(m,\phi)), V_2(x(m,\phi))),\dots,V_n(x(m,\phi))$, $x \in \operatorname{int} R_+^n$ a vector Liapunov boundary function of system (1.3), if

(i) there is a boundary function $B(x)$ and n continuous functions $G_i : Z \times C^+ \to \operatorname{int} R_+$ such that for all $m \in Z$, $M > 0$ and $i \in \{1,2,\dots,n\}$,

$$0 < \inf_{\|\phi\|\le M} G_i(m,x_m) \le \sup_{\|\phi\|\le M} G_i(m,x_m) < +\infty, \tag{2.1}$$

and for all $m,k \in Z^+$, there is a constant $\alpha_1 > 0$ such that

$$\frac{\inf_{\|\phi\|\le M} G_i(m,x_m)}{\sup_{\|\phi\|\le M} G_i(k,x_k)} > \alpha_1, \tag{2.2}$$

(ii) in addition, for $x_m = x_m(0,\phi)$, the solution of (1.3) with $x(0) = \phi$ and $V_i(m,x) = B(x)\cdot G_i(m,x_m)$, we have

$$V_i(m+1,x) \ge V_i(m,x)\cdot P_i(x_i), \tag{2.3}$$

where $P_i : R^+ \to R$, and there exist some constants $x_i^0 > 0$, $\lambda_1 > 1$ such that $P_i(x_i) > \lambda_1 > 1$ whenever $0 < x_i < x_i^0$.

Theorem 2.3. *System* (1.3) *is permanent if it admits a vector Liapunov boundary function* $V(m,x) = (V_1(m,x), V_2(m,x),\dots,V_n(m,x))$.

Remark 2.4. In [35, Theorem 2.1], Tang and Xiao constructed the sufficient conditions for the permanence of an autonomous two-species Kolmogorov system. Theorem 2.3 directly generalizes their results.

Remark 2.5. Noting Theorem 2.3 does not need any sign conditions on $\partial f_i/\partial x_i$. Thus we can study several population models simultaneously: competitive, predator-prey, mutual, and so forth.

Using the arguments similar to Lemma 1 in [34], we have the following.

Lemma 2.6. *Assume positive initial conditions hold for system* (1.3), *then each of its solution is positive with upper and lower bound.*

Proof of Theorem 2.3. We divide the arguments into the following several steps. □

Step 1. Constructing a bounded subset $\Omega \subset \operatorname{int} R_+^n$.

Suppose M is defined as in Definition 2.2 and the vector Liapunov function $V(m, x(m,\phi))$ defined as in Definition 2.2. Let

$$\delta_0 \le \frac{1}{2}\min_{1\le i\le n} x_i^0 \cdot \sup_{1\le i\le n,\ \|\phi\|\le M}\left\{\frac{1}{\|f_i(m, x_m)\|}\right\}. \tag{2.4}$$

By Definition 2.2, there exists a positive constant δ_1 such that for any $i \in \{1, 2, \dots, n\}$ and $x \in \operatorname{int} R_+^n$, the facts $\|x\| \le M$ and $D_i(x) \le \delta_1$ imply that $x_i < \delta_0$. Further, one can choose the sufficient small constant δ_2 with

$$0 < \delta_2 < \delta_1 \cdot \frac{\inf_{\|\phi\|\le M} G_i(m, x(m,\phi))}{\sup_{\|\phi\|\le M} G_i(k, x(k,\phi))} \quad \forall m, k \in Z^+,\ i = 1, 2, \dots, n. \tag{2.5}$$

Define a subset of R_+^n

$$\Omega = \{x = (x_1, \dots, x_n) \in R_+^n : D_i(x) \ge \delta_2,\quad x_i \le M,\ i = 1, \dots, n\}. \tag{2.6}$$

Clearly the definition of $D_i(x)$ yields that $\Omega \subset \operatorname{int} R_+^n$.

In the following we prove that each solution of system (1.3) will eventually enter and stay in Ω, that is, system (1.3) will be permanent.

Step 2. For any $i \in \{i = 1, 2, \dots, n\}, m_0 \ge 0$, $\|x(m,\phi)\| \le M$ for $m \ge m_0$ and $D_i(x(m_0,\phi)) \ge \delta_1$ follows that $D_i(x(m_1,\phi)) \ge \delta_2$ for all $m \ge m_0$.

Suppose this false, then there exist some $m_2 > m_1 \ge m_0$ and some $j \in \{1, \dots, n\}$ such that $D_j(x(m_1,\phi)) \ge \delta_1$ while $D_j(x(m_2,\phi)) < \delta_2$ and $\delta_2 \le D_j(x(m,\phi)) < \delta_1$ for all $m_1 < m < m_2$ unless $m_2 = m_1 + 1$.

Hence we have two cases to consider.

Case 1 ($m_2 = m_1 + 1$). By the definition of δ_1, δ_2, $D_j(x(m_2, \phi)) < \delta_2 \le \delta_1$ implies $x_j(m_2) < \delta_0$. Then by system (1.3),

$$x_j(m_1) = \frac{x_j(m_2)}{f_j(m_1, x_{m_1})} \le \delta_0 \cdot \sup_{1\le i\le n,\ \|\phi\|\le M} \left\{ \frac{1}{\| f_i(m, x_m)\|} \right\} \le \frac{1}{2} \cdot x_j^0. \tag{2.7}$$

Using Definition 2.2 we have $V_j(m_2, x) > V_j(m_1, x)$.

Case 2 ($m_2 > m_1 + 1$). Since $D_j(x(m, \phi)) < \delta_1$ for all $m_1 + 1 \le m \le m_2$, we have $x_j(m) < \delta_0$. Using the analogous arguments to Case (i), we have $x_j(m_1) < (1/2) \cdot x_j^0$. Then we have

$$V_j(m_1, x) < V_j(m + 1, x) < \cdots < V_j(m_2, x). \tag{2.8}$$

Thus we obtain $V_j(m_2, x) > V_j(m_1, x)$. However, since

$$\begin{aligned} V_j(m_2, x) &= D_j(x(m_2, \phi)) \cdot G_j(m_2, x) \le \delta_2 \cdot \sup_{\|\phi\|\le M} G_j(m_2, x) \\ &\le \delta_1 \cdot \frac{\inf_{\|\phi\|\le M} G_i(m, x(m, \phi))}{\sup_{\|\phi\|\le M} G_i(k, x(k, \phi))} \cdot \sup_{\|\phi\|\le M} G_j(m, x) \\ &\le \delta_1 G_j(m_1, x(m, \phi)) \le D_j(x(m_1, \phi)) \cdot G_j(m, x(m_1, \phi)) \\ &= V_j(m_1, x(m_1, \phi)), \end{aligned} \tag{2.9}$$

then we obtain a contradiction, which finishes Step 2.

Step 3. By Step 2 and the definition that $\Omega = \bigcap_{i=1}^{n} \{x = (x_1, \ldots, x_n) \in R_+^n : \ D_i(x) \ge \delta_2,\ x_i \le M,\ i = 1, 2, \ldots, n\}$, we obtain that if there exists an integer $m_0 \ge 0$ such that $D_i(x(m_0, \phi)) \ge \delta_1$ for all $i = 1, 2, \ldots, n$, then $x(m, \phi) \in \Omega$ for all $m \ge m_0$.

Step 4. We claim that there exist $\delta_3 = \delta_3(\phi) > 0$ and some integer $m_* \ge 0$ such that $x_i(m, \phi) > \delta_3$ for $m \ge m_*$, $i = 1, 2, \ldots, n$.

Let $m_* \ge 0$ be sufficiently large such that $|x(m, \phi)| \le M$ for $m \ge m_*$, let $\delta_1' = \min_{1\le i\le n}\{D_i(x(m_*, \phi))\}$, that is, $D_i(x(m_*, \phi)) \ge \delta_1'$, $i = 1, 2, \ldots, n$. Following Step 2, we can find a $\delta_2' \in (0, \delta_1')$ such that $D_i(x(m, \phi)) > \delta_2'$ for all $m \ge m_*$ and $i = 1, 2, \ldots, n$. By Property (ii) of Definition 2.2, we see that there is a desired $\delta_3 > 0$ such that $x_i(m, \phi) \ge \delta_3$ for $m \ge m_*, i = 1, 2, \ldots, n$. Here δ_3 is dependent on ϕ.

Using Definition 2.2, we have $V_i(m, x_i), (i = 1, \ldots, n)$ are bounded for all $m \ge m_*, i = 1, 2, \ldots, n$.

Step 5. We claim that solution $x(m, \phi)$ enters and stays in Ω for sufficiently large m. We have two cases to consider.

Case A. There exists an $m_0 \geq 0$ such that $D_i(x(m_0, \phi)) \geq \delta_1$ for all $i = 1, \ldots, n,$; for this case, Step 3 directly implies the claim.

Case B. $x(m, \phi)$ remains in $S \setminus \Omega$ for all large integer m, where

$$\begin{gathered} S = \{x = (x_1, \ldots, x_n) \in R_+^n : x_i \leq M,\ i = 1, \ldots, n\}, \\ \Omega_1 = \{x = (x_1, \ldots, x_n) \in R_+^n : D_i(x) \geq \delta_1, x_i \leq M,\ i = 1, \ldots, n\}. \end{gathered} \tag{2.10}$$

In this case, we first claim that there exists a sufficiently large $m = m(i)$ for each i such that $D_i(x(m, \phi)) \geq \delta_1$. If it is not true, that is, $D_i(x(m, \phi)) < \delta_1$ for all m and for each $1 \leq i \leq n$. The definition of $D_i(x(m, \phi))$ implies that $x(m, \phi) < \delta_0$ for all large m, and by the choice of δ_0 and the definition of $V_i(m+1, x)$, we have $V_i(m+1, x) \geq \xi \cdot V_i(m, x)$, where ξ is some constant with $\xi > 1$. Hence $V_i(m, x) \to \infty$ as $m \to \infty$. A contradiction to the boundedness of $V_i(m, x)$ (see Step 4). Then for each i, there exists a sufficiently large integer $m(i)$ such that $D_i(x(m, \phi)) \geq \delta_1$. By Step 2, $D_i(x(m, \phi)) \geq \delta_2$ for all $m \geq m(i)$.

Selecting $m_1 = \max_{1 \leq i \leq n}\{m(i)\}$, then we have $D_i(x(m, \phi)) \geq \delta_2$ for all $m \geq m_1,\ i = 1, \ldots, n$, that is, $x \in \Omega$ for all $m \geq m_1$, proving Theorem 2.3.

Definition 2.7. $E(x) = (E_1(x), \ldots, E_n(x))$ is called an r-boundary function $(1 \leq r \leq n)$, if for any $k > 0$ and $\sigma > 0$, there exists $\sigma_1, \sigma_2 > 0$ such that the following properties hold true:

(i) whenever $j \in \{1, \ldots, r\}$, $x \in \operatorname{int} R_+^n$, $|x| \neq k$ and $E_j(x) \leq \sigma_1$, then $x_j \leq \sigma$,

(ii) whenever $x \in \operatorname{int} R_+^n$ and $E_i(x) \leq \sigma$ for all $i = 1, \ldots, r$, then $x_i \geq \sigma_2$ for all $i = 1, \ldots, r$.

Definition 2.8. The vector function $U(m) = (U_1(m), \ldots, U_n(m))$ is called a vector r-balancing survival function for system (1.3) if the following properties hold:

(1) $U_i(m) = E_i(x(m, \phi)) \cdot H_i(m, x(m, \phi)),\ i = 1, \ldots, n$, where $E(x(m, \phi)) = (E_1, \ldots, E_n)$ is an r-boundary function $H(m, x(m, \phi)) = (H_1, \ldots, H_n)$ is a continuous function with $H_i : Z \times C^+ \to \operatorname{int} R_+$ such that for M and all $m \in Z^+$ and any $i \in \{1, \ldots, n\}$,

$$0 < \inf_{\|\phi\| \leq M} H_i(m, x(m, \phi)) \leq \sup_{\|\phi\| \leq M} H_i(m, x(m, \phi)) < +\infty, \tag{2.11}$$

and for all $m, s \in Z^+$, there exists $\alpha_2 > 0$ such that $\inf_{\|\phi\| \leq M} H_i(m, x(m, \phi)) / \sup_{\|\phi\| \leq M} H_i(s, x(s, \phi)) > \alpha_2$.

(2) $U_i(m+1, x_{m+1}) \geq U_i(m, x_m) \cdot Q_i(m, x_i) \cdot F_i(m, x_{r+1}, \ldots, x_n)$ for all $i = 1, \ldots, r$ while $U_j(m+1, x_{m+1}) \leq U_j(m, x_m) \cdot Q_j(m, x_1, \ldots, x_j) \cdot F_j(m, x_{j+1}, \ldots, x_n)$ for all $j = r+1, \ldots, n$.

Here $x(m, \phi)$ is a solution of system (1.3) with $x(0) = \phi$, $Q_i(m, x_i)$ admits some $x_i^r > 0$, $\lambda_2 > 1$ and $\alpha_3 < 1$ such that $Q_i(m, x_i) > \lambda_2 > 1$ for all $0 < x_i < x_i^r$, $(i = 1, \ldots, r)$ while $Q_j(m, x_1, \ldots, x_j) < \alpha_3 < 1$ for sufficiently large m, $(j = r+1, \ldots, n)$; $F_i(m, x_{j+1}, \ldots, x_n) \to 1$ as $\max_{j+1 \leq k \leq n}\{|x_k|\} \to 0$.

We have the following theorem.

Theorem 2.9. *Assume there exists a vector r-balancing survival Liapunov function $U(m, x_m)$ for system* (1.3)*, then system* (1.3) *is r-balancing survival, that is, the species $r+1, \ldots, n$ are extinct while the rest r populations are permanent.*

Proof. First we prove the extinction of species $r+1, \ldots, n$. Let $U_n(m+1, x)$ be the vector Liapunov boundary function for system (1.3). By Definition 2.7, for the $m > N_0$, we have the following inequality:

$$U_n(m+1, x) \leq U_n(m, x) \cdot \exp\{-\alpha^{\rho}\}, \tag{2.12}$$

then we get $\lim_{m\to\infty} U_n(m+1, x) = 0$, which, by Definition 2.8, implies $\lim_{m\to\infty} x_n(m) = 0$.

Now we claim for all $j \in \{r+1, \ldots, n-1\}$ and $m > N_0$, extinctions of species $j+1, \ldots, n$ yield that of species j.

By Definition 2.8, there exist a positive constant $\alpha_3 < 1$ and a sufficiently integer m_2 such that for all $m \geq m_2$, thus we have

$$\begin{aligned} U_j(m+1, x) &\leq U_j(m, x) \cdot Q_j(m, x_1, \ldots, x_j) \cdot F(m, x_j + 1, \ldots, x_n) \\ &\leq U_j(m, x) \cdot \alpha_3 \cdot \frac{1+\alpha_3}{2\alpha_3} = \frac{1+\alpha_3}{2} \cdot U_j(m, x), \end{aligned} \tag{2.13}$$

this yields $\lim_{m\to\infty} x_j(m) = 0$. Hence we get the extinctions of species $r+1, \ldots, n$.

Now we prove the permanence of species $1, \ldots, r$. Choose a $\sigma_0 \in (0, 1)$ with

$$\delta_0 \leq \frac{1}{2} \min_{1\leq i\leq n} x_i^0 \cdot \sup_{1\leq i\leq r,\ \|\phi\|\leq M} \left\{ \frac{1}{\|f_i(m, x_m)\|} \right\}, \tag{2.14}$$

such that $0 < x_i < \delta_0$ implies that $P_i(x_i) \geq \varepsilon_0$ for some $\varepsilon_0 > 1$, $(i = 1, 2, \ldots, r)$. if $0 < x_i < \sigma_0$, then $Q_i(x_i) > \varepsilon_0 > 1$ for all $i = 1, \ldots, r$, where ε_0 is a constant.

Consider σ_1 such that for any given $i \in \{1, \ldots, r\}$, we have; whenever x is in the interior of R_+^n, $\|x\| \leq M$ and $E_i(m, x) \leq \sigma_1$, then $x_i < \sigma_0$. Such a σ_1 exists because of property (i) of Definition 2.7. Further by Definition 2.8, we can choose

$$0 < \sigma_2 < \sigma_1 \cdot \frac{\inf_{\|\phi\|\leq M} H_i(m, \phi)}{\sup_{\|\phi\|\leq M} H_i(s, \phi)} \quad \forall m, s \in Z^+,\ i = 1, \ldots, r. \tag{2.15}$$

Let

$$\Omega' = \{ y = (x_1, \ldots, x_r) \in R_+^n : E_i(m, y) \geq \sigma_2,\ x_i \leq M,\ i = 1, \ldots, r \}. \tag{2.16}$$

It is clear that the property (i) of Definition 2.7 yields $\Omega' \subset \operatorname{int} R_+^r$. In the following, we prove that Ω' is the desired permanent region for species $x_1, \ldots, x_r$.

Definition 2.8 and extinctions of species $x_{r+1}, \ldots, x_n$ imply that there exist a positive constant ε_0 and some integer $m_* > 0$ such that $F_j(m, x_{r+1}, \ldots, x_n) > (1+\varepsilon_0)/2 > 1$ for all $j = r+1, \ldots, n$ and $m > m_*$.

Now we claim that for each $i \in \{1, \dots, r\}$, $\|x_i\| \le M$ for $m \ge m_*$ and $E_i(x) \ge \sigma_1$ imply that $E_i(x) \ge \sigma_2$ for all $m \ge m_*$.

For any $i \in \{i = 1, 2, \dots, n\}, m_0 \ge 0$, $\|x(m, \phi)\| \le M$ for $m \ge m_0$, and $E_i(x(m_0, \phi)) \ge \delta_1$ follows that $E_i(x(m_1, \phi)) \ge \delta_2$ for all $m \ge m_0$.

Suppose this false, then there exist some $m_2' > m_1' \ge m_0$ and some $j \in \{1, \dots, r\}$ such that $E_j(x(m_1', \phi)) \ge \sigma_1$ while $E_j(x(m_2, \phi)) < \sigma_2$ and $\sigma_2 \le E_j(x(m, \phi)) < \sigma_1$ for all $m_1' < m < m_2'$ unless $m_2' = m_1' + 1$. We have the following two cases to consider.

Case 1 ($m_2' = m_1' + 1$). By the definition of σ_1, σ_2, $E_j(x(m_2', \phi)) < \sigma_2 \le \sigma_1$ implies $x_j(m_2') < \sigma_0$. Then by system (1.3),

$$\begin{aligned} x_j(m_1') &= \frac{x_j(m_2')}{f_j\left(m_1', x_{m_1'}\right)} \\ &\le \sigma_0 \cdot \sup_{1 \le i \le r,\ \|\phi\| \le M} \left\{ \frac{1}{\|f_i(m, x_m)\|} \right\} \\ &\le \frac{1}{2} \cdot x_j^0. \end{aligned} \tag{2.17}$$

Using Definition 2.2 we have $V_j(m_2', x) > V_j(m_1', x)$.

Case 2 ($m_2' > m_1' + 1$). Since $E_j(x(m, \phi)) < \delta_1$ for all $m_1' + 1 \le m \le m_2'$, we have $x_j(m) < \sigma_0$. Using the analogous arguments to Case (i), we have $x_j(m_1') < (1/2) \cdot x_j^0$. Then we have

$$V_j(m_1', x) < V_j(m + 1, x) < \cdots < V_j(m_2', x). \tag{2.18}$$

Thus we obtain $V_j(m_2', x) > V_j(m_1', x)$. However, we also have

$$\begin{aligned} V_j(m_2', x) &= E_j(x(m_2', \phi)) \cdot H_j(m_2', x) \le \sigma_2 \cdot \sup_{\|\phi\| \le M} H_j(m_2', x) \\ &\le \sigma_1 \cdot \frac{\inf_{\|\phi\| \le M} H_i(m, x(m, \phi))}{\sup_{\|\phi\| \le M} H_i(k, x(k, \phi))} \cdot \sup_{\|\phi\| \le M} H_j(m, x) \\ &\le \sigma_1 H_j(m_1', x(m_1', \phi)) \\ &\le E_j(x(m_1', \phi)) \cdot H_j(m_1', x(m_1', \phi)) \\ &= V_j(m_1', x(m_1', \phi)), \end{aligned} \tag{2.19}$$

a contradiction, which proves $E_i(x) \ge \sigma_2$ for all $m \ge m_*$. □

Then using the similar arguments to Step 3–5 for Theorem 2.3, we prove Theorem 2.9.

3. Applications to Lotka-Volterra a System

Consider the following nonautonomous discrete competitive systems with time delays:

$$x_i(m+1) = x_i(m) \cdot \exp\left\{ b_i(m) - \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \right\}, \quad i = 1, \ldots, n, \tag{3.1}$$

where $x_i(m)$ represents the density of population i at the mth generation; $\tau_{ij}^{(k)}$ is the nonnegative integer delay to the competition between species i and species j.

Denote

$$\overline{f} = \sup_{m \in Z^+} f(m); \quad \underline{f} = \inf_{m \in Z^+} f(m) \tag{3.2}$$

for the bounded function $f(m)$ with $m \in Z^+$, and let $a_{ij}(m) = \sum_{k=1}^{p} a_{ij}^{(k)}(m)$, $i, j = 1, 2, \ldots, n$. We assume $0 \le a_{ij}^{(k)}(m), b_i(m) < +\infty$ and $\underline{a_{ii}} > 0$ for all $1 \le i, j \le n$, $m \in Z^+$.

Denote

$$A_i = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} & \cdots & \underline{a_{1n}} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{a_{i-11}} & \underline{a_{i-12}} & \cdots & \underline{a_{i-1n}} \\ \overline{a_{i1}} & \overline{a_{i2}} & \cdots & \overline{a_{in}} \\ \underline{a_{i+11}} & \underline{a_{i+12}} & \cdots & \underline{a_{i+1n}} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{a_{n1}} & \underline{a_{n2}} & \cdots & \underline{a_{nn}} \end{pmatrix}, \quad B_i = \begin{pmatrix} \overline{b_1} \\ \cdots \\ \overline{b_{i-1}} \\ \underline{b_i} \\ \overline{b_{i+1}} \\ \cdots \\ \overline{b_n} \end{pmatrix}. \tag{3.3}$$

Then we have the following.

Theorem 3.1. *Assume*

(H_4) A_i *is a nonsymmetric matrix; the vector equation* $A_i X^{(i)} = B_i$ *admits a positive solution* $X^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)})^T$.

(H_5) *Let* $(\beta_{jk}^{(i)})_{n\times n}$ *be the inverse matrix of* $A^{(i)}$ *with* $\beta_{kk}^{(i)} > 0$ *and* $\beta_{jk} \le 0$ *for all* $j, k = 1, \ldots, n$, $j \ne k$.

Then system (3.1) *is permanent.*

Proof. Using the similar arguments to those in [37], we can prove system (3.1) is dissipative. Let

$$V_l(m+1) = \prod_{i=1}^{n} (x_i(m+1))^{\beta_{li}^{(l)}} \cdot \exp\left\{ -\sum_{i,j=1}^{n} \beta_{ij}^{(l)} \sum_{k=1}^{p} \sum_{s=m+1-\tau_{ij}^{(k)}}^{m} a_{ij}^{(k)}\left(s + \tau_{ij}^{(k)}\right) x_j(s) \right\}, \quad l = 1, \ldots, n, \tag{3.4}$$

where $\beta_{ij}^k, (i,j = 1,\dots,n, k = 1,\dots,p)$ are defined in (H_5). By Theorem 2.3, we only need to prove the vector function $(V_1(m+1),\dots,V_n(m+1))$ is a vector Liapunov boundary function of system (3.1). With the similar arguments in [45], we can prove that $(\prod_{i=1}^n (x_i(m+1))^{\beta_{1i}^{(1)}},\dots,\prod_{i=1}^n (x_i(m+1))^{\beta_{ni}^{(n)}})$ is a boundary function. Hence we only need to prove the (ii) of Definition 2.2. □

By (3.4) and system (3.1), we have

$$\begin{aligned}
\left.\frac{V_l(m+1)}{V_l(m)}\right|_{(3)}
&= \exp\left\{\sum_{i=1}^n\left[\beta_{li}^{(l)} b_i(m) - \sum_{j=1}^n\sum_{k=1}^p \beta_{li}^{(l)} a_{ij}^{(k)}(m) x_j\left(m-\tau_{ij}^{(l)}\right)\right]\right.\\
&\quad\left. - \sum_{i,j=1}^n\sum_{k=1}^p \beta_{li}^{(l)}\left(\sum_{s=m+1-\tau_{ij}^{(k)}}^{m} a_{ij}^{(k)}\left(s+\tau_{ij}^{(k)}\right)x_j(s) - \sum_{s=m-\tau_{ij}^{(k)}}^{m-1} a_{ij}^{(k)}\left(s+\tau_{ij}^{(k)}\right)x_j(s)\right)\right\}\\
&= \exp\left\{\sum_{i=1}^n\sum_{i=1}^n \beta_{li}^{(l)} b_i(m) - \sum_{i,j=1}^n\sum_{k=1}^p a_{ij}^{(k)}(m) x_j\left(m-\tau_{ij}^{(l)}\right)\right.\\
&\quad\left. - \sum_{i,j=1}^n\sum_{k=1}^p \beta_{li}^{(l)} a_{ij}^{(k)}\left(m+\tau_{ij}^{(k)}\right)x_j(m) + \sum_{i,j=1}^n\sum_{k=1}^p \beta_{li}^{(l)} m a_{ij}^{(k)}(m) x_j\left(m-\tau_{ij}^{(k)}\right)\right\}\\
&= \exp\left\{\sum_{i=1}^n\sum_{i=1}^n \beta_{li}^{(l)} b_i(m) - \sum_{i,j=1}^n\sum_{k=1}^p \beta_{li}^{(l)} m a_{ij}^{(k)}\left(m+\tau_{ij}^{(k)}\right)x_j(m)\right\}\\
&\ge \exp\left\{\beta_{ll}^{(l)}\underline{b_i} + \sum_{i\neq l}^n\sum_{i=1}^n \beta_{li}^{(l)}\overline{b_i} - \sum_{j=1}^n\sum_{k=1}^p \beta_{ll}^{(l)}\underline{a_{lj}^{(k)}} x_j(m) - \sum_{i\neq l}^n\sum_{j=1}^n\sum_{k=1}^p \beta_{li}^{(l)}\overline{a_{ij}^{(k)}} x_j(m)\right\}\\
&= \exp\left\{x_l^{(l)} - \left(\beta_{ll}^{(l)}\underline{a_{lj}} + \sum_{j\neq l}^n \beta_{li}^{(l)}\overline{a_{ij}}\right)\cdot\sum_{j=1}^n x_j(m)\right\}.
\end{aligned} \tag{3.5}$$

By (H_4), $x_l^{(l)} > 0$ for all $l = 1,\dots,n$. Noticing that $(\underline{a_{1j}},\dots,\underline{a_{l-1j}},\overline{a_{lj}},\underline{a_{l+1j}},\dots,\underline{a_{nj}})^T$ is the jth column vector of A_l and $(\beta_{l1}^{(l)},\dots,\beta_{ln}^{(l)})$ the lth row vector of the matrix A_l^{-1}, respectively. Then we have

$$\beta_{ll}^{(l)}\underline{a_{lj}} + \sum_{j\neq l}^n \beta_{li}^{(l)}\overline{a_{ij}} = \begin{cases} 1, & j = l,\\ 0, & j\neq l,\end{cases} \tag{3.6}$$

which follows

$$\left.\frac{V_l(m+1)}{V_l(m)}\right|_{(3)} \geq \exp\left\{x_l^{(l)} - x_l(m)\right\}. \tag{3.7}$$

This proves the (ii) of Definition 2.2. Thus we prove $V(m, x_m)$ is a Liapunov boundary function, proving Theorem 3.1.

Let $2 \leq q < n$, $A_q = (a_{ij})_{q\times q}$, $B_q = (b_1, \ldots, b_q)^T$, $X_q = (x_1, \ldots, x_q)^T$. Now, we consider the balancing survival of the system (3.1). Denote

$$A_i^r = \begin{pmatrix} \underline{a_{21}} & \underline{a_{12}} & \cdots & \underline{a_{1r}} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{a_{i-11}} & \underline{a_{i-12}} & \cdots & \underline{a_{i-1r}} \\ \overline{a_{i1}} & \overline{a_{i2}} & \cdots & \overline{a_{ir}} \\ \underline{a_{i+11}} & \underline{a_{i+12}} & \cdots & \underline{a_{i+1r}} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{a_{r1}} & \underline{a_{r2}} & \cdots & \underline{a_{rr}} \end{pmatrix}, \qquad B_i^r = \begin{pmatrix} \overline{b_1} \\ \cdots \\ \overline{b_{i-1}} \\ \underline{b_i} \\ \overline{b_{i+1}} \\ \cdots \\ \overline{b_r} \end{pmatrix}, \quad i = 1, 2, \ldots, r. \tag{3.8}$$

Theorem 3.2. *Assume*

(H_4) *A_i^r is a nonsymmetric matrix and the vector equation $A_i^r X_i = B_i^r$ admits a positive solution vector $Y_i = (y_1^{(2)}, \ldots, y_i^{(r)})^T$ with $i = 1, \ldots, r$.*

(H_5) *Let $(\gamma_{ij}^r)_{r\times r}$ be the inverse matrix of A_i^r with $\gamma_{ii}^r > 0$ and $\gamma_{ij}^r \leq 0$ for all $i, j = 1, \ldots, r$ and $j \neq i$.*

(H_6) *For all $r+1 \leq k \leq n$, there exists $i_k < k$ such that $\underline{b_k}\,\underline{a_{i_k j}} - \overline{b_{i_k}}\,\overline{a_{kj}} < 0$ holds for all $j = 1, 2, \ldots, k$.*

Then system (3.1) *is r-balancing survival, that is, species $1, \ldots, r$ are permanent while species $r+1, \ldots, n$ will go extinct.*

Corollary 3.3. *Assume*

(H_7) *For each $2 \leq r \leq n$, there exists a positive integer i_r with $i_r < r$ such that $\underline{b_r}\,\underline{a_{i_r j}} - \overline{b_{i_r}}\,\overline{a_{rj}} < 0$ holds for all $j = 1, 2, \ldots, r$.*

Then all species in system (3.1) *except species 1 are going extinct while species 1 is permanent.*

Remark 3.4. In [37], we considered permanence and balancing survive of system (1.1)—the autonomous case of system (3.1). Theorems 3.1 and 3.2 in this paper generalize the corresponding results in [37].

Remark 3.5. Kuang [40], Tang and Kuang [45], Liu and Chen [43] obtained the sufficient conditions for the permanence in the delayed n-species Lotka-Volterra differential equations. They also proved that time-delays are harmless for the permanence of the continuous Lotka-Volterra system. Our results in Theorem 3.1 and Corollary 3.3 are analogous to theirs.

Remark 3.6. Theorem 3.1 and Corollary 3.3 can be regarded as the two extreme cases of r-balancing survival of system (3.1) with $r = n$ and $r = 1$, respectively. Then Theorem 3.2 unifies Theorem 3.1 and Corollary 3.3.

Remark 3.7. Noting all conditions in Theorems 3.1 and 3.2 and Corollary 3.3 are independent of the delays $\tau_{ij}^{(k)}$, then once conditions for this propositions are satisfied, the inclusion, exclusion or the variations of the time-delays will not affect the conclusions any more.

Proof of Theorem 3.2. Let $W(m, x_m) = (W_1(m, x_m), \ldots, W_n(m, x_m))$ with $W_i(m, x_m) = E_i(x(m)) \cdot H_i(m, x_m)$, where

$$E_l(m) = \prod_{i=1}^{r} (x_i(m))^{\gamma_{li}^l}, \qquad H_l(m, x_m) = \exp\left\{ -\sum_{i,j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^l \sum_{s=m-\tau_{ij}^{(k)}}^{m} a_{ij}^{(k)}\left(s + \tau_{ij}^{(k)}\right) x_j(s) \right\} \quad (3.9)$$

as $l = 1, 2, \ldots, r$, and

$$\begin{aligned} E_l(x(m)) &= x_l(m+1)^{-\underline{b_{i_l}}} \ x_{i_l}(m+1)^{\overline{b_l}} \\ H_l(m, x_m) &= \exp\left\{ -\underline{b_{i_l}} \sum_{j=1}^{l} \sum_{k=1}^{p} \sum_{s=m+1-\tau_{lj}^{(k)}}^{m} a_{lj}^{(k)}\left(s + \tau_{lj}^{(k)}\right) x_j(s) \right. \\ &\qquad \left. + \overline{b_l} \sum_{j=1}^{l} \sum_{k=1}^{p} \sum_{s=m+1-\tau_{i_l j}^{(k)}}^{m} a_{i_l j}^{(k)}\left(s + \tau_{i_l j}^{(k)}\right) x_j(s) \right\}, \end{aligned} \quad (3.10)$$

when $\quad l = r + 1, \ldots, n$.

By Theorem 2.9, we only need to prove that $W(m, x_m)$ is a vector r-balancing survival function for system (3.1). With the similar arguments to Theorem 2 in [37], we can prove $E(x(m) = (E_1(x(m)), \ldots, E_n(x(m)))$ is an r-boundary function for system (3.1); by the dissipative property of system (3.1), we can prove that $H(m, x_m) = (H_1(m, x_m), \ldots, H_n(m, x_m))$ satisfies conditions for part (1) in Definition 2.8.

For $l = r + 1, \ldots, n$, we have

$$\begin{aligned} &\left.\frac{W_l(m+1, x_{m+1})}{W_l(m, x_m)}\right|_{(3)} \\ &\quad = \exp\left\{ \underline{b_{i_l}} b_k(m) - \overline{b_l} b_{i_l}(m) - \underline{b_{i_l}} \sum_{j=1}^{l} \sum_{k=1}^{p} a_{lj}^{(k)}(m) x_j\left(m - \tau_{lj}^{(k)}\right) \right. \\ &\qquad \left. + \overline{b_l} \sum_{j=1}^{l} \sum_{k=1}^{p} a_{i_l j}^{(k)}(m) x_j\left(m - \tau_{i_l j}^{(k)}\right) \right. \end{aligned}$$

$$
\begin{aligned}
&-\underline{b_{i_l}}\sum_{j=l+1}^{n}\sum_{k=1}^{p}a_{lj}^{(k)}(m)x_j\left(m-\tau_{lj}^{(k)}\right)+\overline{b_l}\sum_{j=l+1}^{n}\sum_{k=1}^{p}a_{i_lj}^{(k)}(m)x_j\left(m-\tau_{i_lj}^{(k)}\right)\\
&-\underline{b_{i_l}}\sum_{j=1}^{l}\sum_{k=1}^{p}\left(\sum_{s=m+1-\tau_{lj}^{(k)}}^{m}a_{lj}^{(k)}\left(s+\tau_{lj}^{(k)}\right)x_j(s)-\sum_{s=m-\tau_{lj}^{(k)}}^{m-1}a_{lj}^{(k)}\left(s+\tau_{lj}^{(k)}\right)x_j(s)\right)\\
&+\overline{b_l}\sum_{j=1}^{l}\sum_{k=1}^{p}\left(\sum_{s=m+1-\tau_{i_lj}^{(k)}}^{m}a_{i_lj}^{(k)}\left(s+\tau_{i_lj}^{(k)}\right)x_j(s)-\sum_{s=m-\tau_{i_lj}^{(k)}}^{m-1}a_{i_lj}^{(k)}\left(s+\tau_{i_lj}^{(k)}\right)x_j(s)\right)\Bigg\}\\
\le \exp&\Bigg\{-\frac{\alpha_l}{4}-\underline{b_{i_l}}\sum_{j=1}^{l}\sum_{k=1}^{p}a_{lj}^{(k)}(m)x_j\left(m-\tau_{lj}^{(k)}\right)+\overline{b_l}\sum_{j=1}^{l}\sum_{k=1}^{p}a_{i_lj}^{(k)}(m)x_j\left(m-\tau_{i_lj}^{(k)}\right)\\
&-\underline{b_{i_l}}\sum_{j=1}^{l}\sum_{k=1}^{p}\left(a_{lj}^{(k)}\left(m+\tau_{lj}^{(k)}\right)x_j(m)-a_{lj}^{(k)}(m)x_j\left(m-\tau_{lj}^{(k)}\right)\right)\\
&+\overline{b_l}\sum_{j=1}^{l}\sum_{k=1}^{p}\left(a_{i_lj}^{(k)}\left(m+\tau_{i_lj}^{(k)}\right)x_j(m)-a_{i_lj}^{(k)}(m)x_j\left(m-\tau_{i_lj}^{(k)}\right)\right)\Bigg\}.
\end{aligned}
\tag{3.11}
$$

Thus we have

$$
\begin{aligned}
&\frac{W_l(m+1,x_{m+1})}{W_l(m,x_m)}\bigg|_{(3)}\\
&\quad\le \exp\left\{-\frac{\alpha_l}{4}-\underline{b_{i_l}}\sum_{j=1}^{l}\sum_{k=1}^{p}a_{lj}^{(k)}\left(m+\tau_{lj}^{(k)}\right)x_j(m)+\overline{b_l}\sum_{j=1}^{l}\sum_{k=1}^{p}a_{i_lj}^{(k)}\left(m+\tau_{i_lj}^{(k)}\right)x_j(m)\right\}\\
&\quad\le \exp\left\{-\frac{\alpha_l}{4}-\sum_{j=1}^{l}\left(\underline{b_{i_l}}\,\underline{a_{lj}}x_j(m)-\overline{b_l}\,\overline{a_{i_lj}}\right)x_j(m)\right\}\\
&\quad\le \exp\left\{-\frac{\alpha_l}{4}-\min_{1\le j\le l}\left\{\underline{b_{i_l}}\,\underline{a_{lj}}x_j(m)-\overline{b_l}\,\overline{a_{i_lj}}\right\}\sum_{j=1}^{l}x_j(m)\right\}\\
&\quad\le \exp\left\{-\frac{\alpha_l}{4}\right\}<1.
\end{aligned}
\tag{3.12}
$$

While for $l = 1, \ldots, r$, we have

$$
\begin{aligned}
&\left.\frac{W_l(m+1, x_{m+1})}{W_l(m, x_m)}\right|_{(3)} \\
&\quad = \exp\Bigg\{ \sum_{i=1}^{r} \gamma_{li}^{l} b_i(m) - \sum_{j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \\
&\qquad\quad - \sum_{j=r+1}^{n} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \\
&\qquad\quad - \sum_{i,j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^{l} \left(\sum_{s=m+1-\tau_{ij}^{(k)}}^{m} a_{ij}^{(k)}\left(s + \tau_{ij}^{(k)}\right) x_j(s) - \sum_{s=m-\tau_{ij}^{(k)}}^{m-1} a_{ij}^{(k)}\left(s + \tau_{ij}^{(k)}\right) x_j(s) \right) \Bigg\} \\
&\quad \geq \exp\Bigg\{ \sum_{i=1}^{r} \gamma_{li}^{l} b_i(m) - \sum_{j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \\
&\qquad\quad - \sum_{i,j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^{l} \left(a_{ij}^{(k)}\left(m + \tau_{ij}^{(k)}\right) x_j(m) - a_{ij}^{(k)}(m) x_j\left(s - \tau_{ij}^{(k)}\right) \right) \Bigg\} \\
&\qquad \cdot \exp\Bigg\{ - \sum_{j=r+1}^{n} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \Bigg\} \\
&\quad = \exp\Bigg\{ \sum_{i=1}^{r} \gamma_{li}^{l} b_i(m) - \sum_{i,j=1}^{r} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}\left(m + \tau_{ij}^{(k)}\right) x_j(m) \Bigg\} \\
&\qquad \cdot \exp\Bigg\{ - \sum_{j=r+1}^{n} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \Bigg\} \\
&\quad \geq \exp\Bigg\{ \gamma_{ll}^{l} \underline{b_l} + \sum_{i \neq l}^{r} \gamma_{li}^{l} \overline{b_i} - \sum_{j=1}^{r} \gamma_{ll}^{l} \overline{a_{lj}} x_j(m) - \sum_{i \neq l}^{r} \sum_{j=1}^{r} \gamma_{li}^{l} \underline{a_{ij}} x_j(m) \Bigg\} \\
&\quad = \exp\Bigg\{ \gamma_{ll}^{l} \underline{b_l} + \sum_{i \neq l}^{r} \gamma_{li}^{l} \overline{b_i} - \left(\sum_{j=1}^{r} \gamma_{ll}^{l} \overline{a_{lj}} + \sum_{i \neq l}^{r} \sum_{j=1}^{r} \gamma_{li}^{l} \underline{a_{ij}} \right) x_j(m) \Bigg\} \\
&\qquad \cdot \exp\Bigg\{ - \sum_{j=r+1}^{n} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right) \Bigg\}.
\end{aligned}
\tag{3.13}
$$

By (H_4), $A_l^r \cdot Y_l = B_l^r$, then $\gamma_{ll}^l \underline{b_l} + \sum_{i\neq l}^r \gamma_{li}^l \overline{b_i} = y_l^{(l)} > 0$. Using (H_5), we have

$$\gamma_{ll}^{(l)} \underline{a_{lj}} + \sum_{j\neq l}^{r} \gamma_{li}^{(l)} \overline{a_{ij}} = \begin{cases} 1, & j = l, \\ 0, & j \neq l, \end{cases} \tag{3.14}$$

which follows

$$\left.\frac{W_l(m+1, x_{m+1})}{W_l(m, x_m)}\right|_{(3)} \geq \exp\left\{y_l^{(l)} - x_l(m)\right\} \cdot \exp\left\{-\sum_{j=r+1}^{n} \sum_{k=1}^{p} \gamma_{li}^{l} a_{ij}^{(k)}(m) x_j\left(m - \tau_{ij}^{(k)}\right)\right\}. \tag{3.15}$$

Then $W(m, x_m)$ also satisfies conditions for part (2) in Definition 2.8, this proves Theorem 3.2. □

4. Conclusions

Many authors have studied the effects of time delays on dynamics of population difference systems. Levin and May [31] showed excessive time lags could lead to stable oscillations behaviors. Crone [32] showed that the inclusion of time delays can dramatically change the dynamics and lead to chaos and cyclical. Further, Crone and Taylor [15] proved that inclusion of delays into the density dependence can destabilize the dynamics that may be stabilized by the nondelayed density dependence. Ginzburg and Taneybill [17] obtained that delays can produce patterns of population fluctuation. Keeling et al. [22, 45] showed that time delays might be one of the causes to stabilize the natural enemy victim interactions and allow the long term coexistence of the two species.

Harmless delays have been well-known for some continuous population since Wang and Ma [46] proved that delays are "harmless" for the permanence of a continuous Lotka-Volterra predator-prey system, similar conclusions can also be found in some competitive Lotka-Volterra systems (see [43, 49, 50]). Recently, Liu and Chen [43] proved the existence of "profitless delays", that is, the delays do not affect on species' extinction. For the discrete system, to study the effects of time delays on permanence, Tang and Xiao [35], Saito et al. [33] and Liu et al. [37] study the effects of time delays on the two-species competitive systems and they prove that time delays are "harmless" for the uniform persistence or permanence. Saito et al. [34] also discover the same conclusions for the two-species predator-prey systems.

Different from the above results, we consider the long-time behaviors of the discrete nonautonomous Kolmogorov-type population system with delays. We obtained the sufficient conditions for its permanence and balancing survival behaviors. These results have the advantage that we do not assume any sign condition on $\partial f_i/\partial x_i$. So, we can study simultaneously several population models: competing species, predator-prey, mutualism, and so forth. In this paper, we have only applied the main results to Lotka-Volterra competing species.

When applying the results of Kolmogorov system into the nonautonomous the competitive system of Lotka-Volterra type, we construct the sufficient conditions for the permanence and balancing survival behaviors of these systems, with all the conditions independent of the time-delays. Hence if the nondelayed system is permanent, its corresponding delayed system will be permanent, too. If several species of the nondelayed systems are balancing survival, so will be in the corresponding delayed system. On the other hand, under the corresponding conditions, if the delayed system is permanent or some of its species go extinct (balancing survival), so will be in the relative nondelayed system.

Thus, under the proper conditions, neither can time delays break the permanence of some species into extinction, nor can they save the extinction of some species. Therefore, time-lags in the discrete competitive Lotka-Volterra system with time-varying environments are both harmless for the permanence and profitless to the extinction of species in system (3.1), these results confirm and improve our previous conclusions for the discrete autonomous Lotka-Volterra systems [37].

Further, we show that the permanence and extinction of the discrete system (3.1) are equivalent to their corresponding continuous systems (see [40, 43, 45]), where time delays are also both harmless for the permanence and profitless to the extinction of species of the system.

Time delays have been shown to dramatically change the dynamics of the discrete populations systems (see [15, 17, 22, 32]) and they may even lead to some complicated dynamical behaviors such as Crone [32]. Based on our results, it would be interesting to consider the effects of time delay on the stability of discrete systems, we leave this as our future work.

Acknowledgments

The authors are grateful to the anonymous referees for their careful reading and valuable comments, which led to an improvement of their original manuscript. This work is supported by the National Natural Science Foundation of China (no.10601042), Science Research Foundation in Harbin Institute of Technology (HITC200714) and the program of excellent Team in Harbin Institute of Technology. The authors would like to thank Dr. S. Tang and Prof. M. J. Keeling for sending their reprints/preprints to them.

References

[1] R. M. May and G. F. Oster, “Bifurcations and dynamic complexity in simple ecological models,” *The American Naturalist*, vol. 110, no. 974, pp. 573–599, 1976.

[2] R. M. May, “The croonian lecture, 1985. When two and two do not make four: nonlinear phenomena in ecology,” *Proceedings of the Royal Society of London*, vol. 228, no. 1252, pp. 241–266, 1986.

[3] M. P. Hassell, *The Dynamics of Arthropod Predator-Prey Systems*, vol. 13 of *Monographs in Population Biology*, Princeton University Press, Princeton, NJ, USA, 1978.

[4] M. Basson and M. J. Fogarty, “Harvesting in discrete-time predator-prey systems,” *Mathematical Biosciences*, vol. 141, no. 1, pp. 41–74, 1997.

[5] J. R. Beddington, C. A. Free, and J. H. Lawton, “Dynamic complexity in predator prey models framed in difference equations,” *Nature*, vol. 255, no. 5503, pp. 58–60, 1975.

[6] R. P. Agarwal, *Difference Equations and Inequalities: Theorey, Methods and Applications*, vol. 155 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1992.

[7] D. M. Chan and J. E. Franke, “Extinction, weak extinction and persistence in a discrete, competitive Lotka-Volterra model,” *International Journal of Applied Mathematics and Computer Science*, vol. 10, no. 1, pp. 7–36, 2000.

[8] J. Hofbauer, V. Hutson, and W. Jansen, “Coexistence for systems governed by difference equations of Lotka-Volterra type,” *Journal of Mathematical Biology*, vol. 25, no. 5, pp. 553–570, 1987.

[9] V. Hutson and W. Moran, “Persistence of species obeying difference equations,” *Journal of Mathematical Biology*, vol. 15, no. 2, pp. 203–213, 1982.

[10] Z. Lu and W. Wang, "Permanence and global attractivity for Lotka-Volterra difference systems," *Journal of Mathematical Biology*, vol. 39, no. 3, pp. 269–282, 1999.
[11] R. Kon, "Permanence of discrete-time Kolmogorov systems for two species and saturated fixed points," *Journal of Mathematical Biology*, vol. 48, no. 1, pp. 57–81, 2004.
[12] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.
[13] D. Chitty, "Population proceses in the vole and their relevance to general theory," *Canadian Journal of Zoology*, vol. 38, pp. 99–113, 1960.
[14] D. Chitty, *Do Lemmings Commit Suicide? Beautiful Hypotheses and Ugly Facts*, Oxford University Press, New York, NY, USA, 1996.
[15] E. E. Crone and D. R. Taylor, "complex dynamics in experimental populations of an annual plant, Cardamine Pensylvanica," *Ecology*, vol. 77, no. 1, pp. 289–299, 1996.
[16] E. E. Crone, "Parental environmental effects and cyclical dynamics in plant populations," *The American Naturalist*, vol. 150, no. 6, pp. 708–729, 1997.
[17] L. R. Ginzburg and D. E. Taneyhill, "Population cycles of forest Lepidoptera: a maternal effect hypothesis," *Journal of Animal Ecology*, vol. 63, no. 1, pp. 79–92, 1994.
[18] L. R. Ginzburg, "Assuming reproduction to be a function of consumption raises doubts about some popular predator-prey models," *Journal of Animal Ecology*, vol. 67, no. 2, pp. 325–327, 1998.
[19] M. Holyoak, "Identifying delayed density dependence in time-series data," *Oikos*, vol. 70, no. 2, pp. 296–304, 1994.
[20] B. Hornfeldt, "Delayed density dependence as a determinant of vole cycles," *Ecology*, vol. 75, no. 3, pp. 791–806, 1994.
[21] P. Inchausti and L. R. Ginzburg, "Small mammals cycles in northern Europe: patterns and evidence for a maternal effect hypothesis," *Journal of Animal Ecology*, vol. 67, no. 2, pp. 180–194, 1998.
[22] M. J. Keeling, H. B. Wilson, and S. W. Pacala, "Reinterpreting space, time lags, and functional responses in ecological models," *Science*, vol. 290, no. 5497, pp. 1758–1761, 2000.
[23] T. A. Mousseau and H. Dingle, "Maternal effects in insect life histories," *Annual Review of Entomology*, vol. 36, no. 1, pp. 511–534, 1991.
[24] T. Mousseau and C. W. Fox, *Maternal Effects as Adaptations*, Oxford University Press, New York, NY, USA, 1998.
[25] D. A. Roach and R. D. Wulff, "Maternal effects in plants," *Annual Review of Ecology and Systematics*, vol. 18, pp. 209–235, 1987.
[26] M. C. Rossiter, "Maternal effects hypothesis of herbivore outbreak," *BioScience*, vol. 44, no. 11, pp. 752–763, 1994.
[27] P. Turchin and A. D. Taylor, "Complex dynamics in ecological time series," *Ecology*, vol. 73, no. 1, pp. 289–305, 1992.
[28] P. Turchin, "Rarity of density dependence or population regulation with lags?" *Nature*, vol. 344, no. 6267, pp. 660–663, 1990.
[29] A. Wikan, "From chaos to chaos. An analysis of a discrete age-structured prey-predator model," *Journal of Mathematical Biology*, vol. 43, no. 6, pp. 471–500, 2001.
[30] A. Wikan and E. Mjølhus, "Overcompensatory recruitment and generation delay in discrete age-structured population models," *Journal of Mathematical Biology*, vol. 35, no. 2, pp. 195–239, 1996.
[31] S. A. Levin and R. M. May, "A note on difference-delay equations," *Theoretical Population Biology*, vol. 9, no. 2, pp. 178–187, 1976.
[32] E. E. Crone, "Delayed density dependence and the stability of interacting populations and subpopulations," *Theoretical Population Biology*, vol. 51, no. 1, pp. 67–76, 1997.
[33] Y. Saito, W. Ma, and T. Hara, "A necessary and sufficient condition for permanence of a Lotka-Volterra discrete system with delays," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 1, pp. 162–174, 2001.
[34] Y. Saito, T. Hara, and W. Ma, "Harmless delays for permanence and impersistence of a Lotka-Volterra discrete predator-prey system," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 50, no. 5, pp. 703–715, 2002.
[35] S. Tang and Y. Xiao, "Permanence in Kolmogorov-type systems of delay difference equations," *Journal of Difference Equations and Applications*, vol. 7, no. 2, pp. 167–181, 2001.
[36] W. Wendi, G. Mulone, F. Salemi, and V. Salone, "Global stability of discrete population models with time delays and fluctuating environment," *Journal of Mathematical Analysis and Applications*, vol. 264, no. 1, pp. 147–167, 2001.

[37] S. Liu, L. Chen, and R. P. Agarwal, "Harmless and profitless delays in discrete competitive Lotka-Volterra systems," *Applicable Analysis*, vol. 83, no. 4, pp. 411–431, 2004.

[38] H. I. Freedman and S. G. Ruan, "Uniform persistence in functional-differential equations," *Journal of Differential Equations*, vol. 115, no. 1, pp. 173–192, 1995.

[39] H. I. Freedman, M. Agarwal, and S. Devi, "Analysis of stability and persistence in a ratio-dependent predator-prey resource model," *International Journal of Biomathematics*, vol. 2, no. 1, pp. 107–118, 2009.

[40] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, vol. 191 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1993.

[41] S. Liu and L. Chen, "Necessary-sufficient conditions for permanence and extinction in Lotka-Volterra system with distributed delays," *Applied Mathematics Letters*, vol. 16, no. 6, pp. 911–917, 2003.

[42] S. Liu and L. Chen, "Necessary-sufficient conditions for permanence and extinction in Lotka-Volterra system with discrete delays," *Applicable Analysis*, vol. 81, no. 3, pp. 575–587, 2002.

[43] S. Liu and L. Chen, "Permanence, extinction and balancing survival in nonautonomous Lotka-Volterra system with delays," *Applied Mathematics and Computation*, vol. 129, no. 2-3, pp. 481–499, 2002.

[44] Y. Nakata, "Permanence for the Lotka-Volterra cooperative system with several delays," *International Journal of Biomathematics*, vol. 2, no. 3, pp. 267–285, 2009.

[45] B. Tang and Y. Kuang, "Permanence in Kolmogorov-type systems of nonautonomous functional-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 197, no. 2, pp. 427–447, 1996.

[46] W. Wendi and M. Zhien, "Harmless delays for uniform persistence," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 1, pp. 256–268, 1991.

[47] F. Montes de Oca and M. L. Zeeman, "Balancing survival and extinction in nonautonomous competitive Lotka-Volterra systems," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 360–370, 1995.

[48] S. Ahmad and F. Montes de Oca, "Extinction in nonautonomous T-periodic competitive Lotka-Volterra system," *Applied Mathematics and Computation*, vol. 90, no. 2-3, pp. 155–166, 1998.

[49] Z. Y. Lu and Y. Takeuchi, "Permanence and global attractivity for competitive Lotka-Volterra systems with delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 22, no. 7, pp. 847–856, 1994.

[50] H. Fujimoto, "Dynamical behaviors for population growth equations with delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 31, no. 5-6, pp. 549–558, 1998.