Research Article
On a Max-Type Difference Equation

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We prove that every positive solution of the max-type difference equation $x_n = \max\{A/x_{n-p}^{\alpha}, B/x_{n-k}^{\beta}\}$, $n = 0, 1, 2, \ldots$ converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$ where p, k are positive integers, $0 < \alpha, \beta < 1$, and $0 < A, B$.

1. Introduction

Recently, the study of max-type difference equations attracted a considerable attention. Although max-type difference equations are relatively simple in form, it is unfortunately extremely difficult to understand thoroughly the behavior of their solutions; see, for example, [1–20] and the relevant references cited therein. The max operator arises naturally in certain models in automatic control theory (see [13, 14]). Furthermore, difference equation appear naturally as a discrete analogue and as a numerical solution of differential and delay differential equations having applications and various scientific branches, such as in ecology, economy, physics, technics, sociology, and biology.

In [20], Yang et al. proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{1}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

converges to $\overline{x} = 1$ or eventually periodic with period 4, where $0 < \alpha < 1$ and $0 < A$.

In [9], We proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^\alpha}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to $\overline{x} = 1$ or eventually periodic with period 2, where $0 < \alpha < 1$ and $0 < A$.

In [17], Sun proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}^\alpha}, \frac{B}{x_{n-2}^\beta}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$ where $0 < \alpha, \beta < 1$, and $0 < A, B$.

The following difference equation is more general than (1.3):

$$x_n = \max\left\{\frac{A}{x_{n-p}^\alpha}, \frac{B}{x_{n-k}^\beta}\right\}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where p, k are positive integers, $0 < \alpha, \beta < 1$, $0 < A, B$, and initial conditions are positive real numbers.

In this paper, we investigate the asymptotic behavior of the positive solutions of (1.4). We prove that every positive solution of (1.4) converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$. Clearly, we can assume that $p \leq k$ without loss of generality.

2. Main Results

2.1. The Case $B^{1/(1+\beta)} \leq A^{1/(1+\alpha)}$

In this section, we consider the asymptotic behavior of the positive solutions of (1.4) in the case $B^{1/(1+\beta)} \leq A^{1/(1+\alpha)}$.

It is easy to see that by the change

$$x_n = A^{1/(1+\alpha)} C^{y_n} \quad \text{for } 0 < C < 1, \ n \geq -k. \tag{2.1}$$

Equation (1.4) is transformed into the difference equation

$$C^{y_n} = \max\left\{C^{-\alpha y_{n-p}}, DC^{-\beta y_{n-k}}\right\}, \tag{2.2}$$

where $D = B/A^{(1+\beta)/(1+\alpha)}$ and the initial conditions are real numbers. Since $B^{1/(1+\beta)} \leq A^{1/(1+\alpha)}$, we have $D \leq 1$.

We need the following two lemmas in order to prove the main result of this section.

Lemma 2.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of* (2.2). *If $D = 1$, then*

$$|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.3}$$

Proof. Clearly, (2.2) implies the following difference equation:

$$y_n = \min\{-\alpha y_{n-p}, -\beta y_{n-k}\} \quad \forall n \geq -k. \tag{2.4}$$

From (2.4), we get the following statements.

(i) If $y_{n-p} \geq 0$ and $y_{n-k} \geq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$.

(ii) If $y_{n-p} \leq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$.

(iii) If $y_{n-p} \geq 0$ and $y_{n-k} \leq 0$, then $|y_n| = \alpha|y_{n-p}|$.

(iv) If $y_{n-p} \leq 0$ and $y_{n-k} \geq 0$, then $|y_n| = \beta|y_{n-k}|$.

From the above statements, we have $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$ for all $n \geq -k$. Therefore, the proof is complete. □

Lemma 2.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of* (2.2). *If $D < 1$, then*

$$|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\} \quad \forall n \geq -k. \tag{2.5}$$

Proof. Assume that $C = D$. Then (2.2) implies the following difference equation:

$$y_n = \min\{-\alpha y_{n-p}, 1 - \beta y_{n-k}\} \quad \forall n \geq -k. \tag{2.6}$$

From (2.6), we get the following statements.

(i) If $y_{n-p} \geq 0$ and $y_{n-k} \geq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$.

(ii) If $y_{n-p} \leq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \alpha|y_{n-p}|$.

(iii) If $y_{n-p} \geq 0$ and $y_{n-k} \leq 0$, then $|y_n| = \alpha|y_{n-p}|$.

(iv) If $y_{n-p} \leq 0$ and $y_{n-k} \geq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$.

From the above statements, we have $|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$ for all $n \geq -k$. Therefore, the proof is complete. □

Theorem 2.3. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of* (1.4) *where $B^{1/(1+\beta)} \leq A^{1/(1+\alpha)}$. Then $\{x_n\}_{n=-k}^{\infty}$ converges to $\overline{x} = A^{1/(1+\alpha)}$.*

Proof. Assume that $D = 1$. $\{y_n\}_{n=-k}^{\infty}$ is a solution of (2.2). If it is proved that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$, then $\{x_n\}_{n=-k}^{\infty}$ converges to $\overline{x} = A^{1/(1+\alpha)}$.

From Lemma 2.1, we have that

$$|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.7}$$

Let $\gamma = \max\{\alpha, \beta\}$. Immediately, we have that the following inequality

$$|y_n| \leq \gamma \max\{|y_{n-p}|, |y_{n-k}|\} \quad \forall n \geq -k. \tag{2.8}$$

From (2.8) and by induction, we get

$$|y_n| \leq \gamma^{[[n/k]]+1} \max_{1 \leq j \leq k}\{y_{-j}\} \quad \forall n \geq -k. \tag{2.9}$$

From (2.9), it is clear that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$.

Now, we assume that $D < 1$. From Lemma 2.2, we have that

$$|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\} \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.10}$$

Then, the rest of proof is similar to the case $D = 1$ and will be omitted. Therefore, the proof is complete. $\qquad\square$

2.2. The Case $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$

In this section, we consider the asymptotic behavior of the positive solutions of (1.4) in the case $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$.

It is easy to see that by the change

$$x_n = B^{1/(1+\beta)}C^{y_n} \quad \text{for } C = \frac{A}{B^{(1+\alpha)/(1+\beta)}}, \ n \geq -k. \tag{2.11}$$

Equation (1.4) is transformed into the difference equation:

$$y_n = \min\{1 - \alpha y_{n-p}, -\beta y_{n-k}\}, \tag{2.12}$$

where initial conditions are real numbers.

We need the following lemma in order to prove the main result of this section.

Lemma 2.4. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of (2.12). Then*

$$|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.13}$$

Proof. From (2.12), we get the following statements.

    (i) If $y_{n-p} \geq 0$ and $y_{n-k} \geq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$.

    (ii) If $y_{n-p} \leq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \beta|y_{n-k}|$.

    (iii) If $y_{n-p} \geq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$.

    (iv) If $y_{n-p} \leq 0$ and $y_{n-k} \geq 0$, then $|y_n| = \beta|y_{n-k}|$.

From the above statements, we have $|y_n| \le \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$ for all $n \ge -k$. Therefore, the proof is complete. □

Theorem 2.5. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.4) *where* $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$. *Then* $\{x_n\}_{n=-k}^{\infty}$ *converges to* $\overline{x} = B^{1/(1+\beta)}$.

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of (2.12). To prove the desired result, it suffices to prove that $\{y_n\}_{n=-k}^{\infty}$ converges to zero.

From Lemma 2.4, we have that

$$|y_n| \le \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\} \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \ge -k. \tag{2.14}$$

From (2.14) and by induction, we get

$$|y_n| \le \gamma^{[|n/k|]+1} \max_{1 \le j \le k}\{y_{-j}\} \quad \forall n \ge -k. \tag{2.15}$$

From (2.15), it is clear that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$. Therefore, the proof is complete. □

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