

Research Article

Existence of Solutions for Nonlinear Fractional Integro-Differential Equations with Three-Point Nonlocal Fractional Boundary Conditions

Ahmed Alsaedi and Bashir Ahmad

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Bashir Ahmad, bashir_qau@yahoo.com

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We prove the existence and uniqueness of solutions for nonlinear integro-differential equations of fractional order $q \in (1, 2]$ with three-point nonlocal fractional boundary conditions by applying some standard fixed point theorems.

1. Introduction

Fractional calculus (differentiation and integration of arbitrary order) is proved to be an important tool in the modelling of dynamical systems associated with phenomena such as fractal and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, and fitting of experimental data [1–4].

Recently, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. For some recent work on fractional differential equations, see [5–11] and the references therein.

In this paper, we study the following nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions

$$\begin{aligned} D^q x(t) + f(t, x(t), (\phi x)(t), (\psi x)(t)) &= 0, \quad 0 < t < 1, \quad 1 < q \leq 2, \\ D^{(q-1)/2} x(0) &= 0, \quad a D^{(q-1)/2} x(1) + x(\eta) = 0, \quad 0 < \eta < 1, \end{aligned} \quad (1.1)$$

where D is the standard Riemann-Liouville fractional derivative, $f : [0, 1] \times X \times X \times X \rightarrow X$ is continuous, for $\gamma, \delta : [0, 1] \times [0, 1] \rightarrow [0, \infty)$,

$$(\phi x)(t) = \int_0^t \gamma(t, s)x(s)ds, \quad (\psi x)(t) = \int_0^t \delta(t, s)x(s)ds, \quad (1.2)$$

and $a \in \mathbb{R}$ satisfies the condition $a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2) \neq 0$. Here, $(X, \|\cdot\|)$ is a Banach space and $C = C([0, 1], X)$ denotes the Banach space of all continuous functions from $[0, 1] \rightarrow X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

We remark that fractional boundary conditions result in the existence of both electric and magnetic surface currents on the strip and are similar to the impedance boundary conditions with pure imaginary impedance, and in the physical optics approximation, the ratio of the surface currents is the same as for the impedance strip. For the comparison of the physical characteristics of the fractional and impedance strips such as radiation pattern, monostatic radar cross-section, and surface current densities, see [12]. The concept of nonlocal multipoint boundary conditions is quite important in various physical problems of applied nature when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located at intermediate points. Some recent results on nonlocal fractional boundary value problems can be found in [13–15].

2. Preliminaries

Let us recall some basic definitions [1–3] on fractional calculus.

Definition 2.1. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \quad (2.1)$$

provided the integral exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n-1 < q \leq n, q > 0, \quad (2.2)$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3 (see [16]). For $q > 0$, let $x, D^q x \in C(0, 1) \cap L(0, 1)$. Then

$$I^q D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n}, \quad (2.3)$$

where $c_i \in \mathbb{R}, i = 1, 2, \dots, n$ (n is the smallest integer such that $n \geq q$).

Lemma 2.4 (see [2]). *Let $x \in L(0, 1)$. Then*

- (i) $D^\nu I^\mu x(t) = I^{\mu-\nu} x(t), \mu > \nu > 0;$
- (ii) $D^\mu t^{\xi-1} = (\Gamma(\xi)/\Gamma(\xi-\mu))t^{\xi-\mu-1}, \mu > 0, \xi > 0.$

Lemma 2.5. *For a given $\sigma \in C[0, 1] \cap L(0, 1)$, the unique solution of the boundary value problem*

$$\begin{aligned} D^q x(t) + \sigma(t) &= 0, \quad 0 < t < 1, 1 < q \leq 2, \\ D^{(q-1)/2} x(0) &= 0, \quad aD^{(q-1)/2} x(1) + x(\eta) = 0, \quad 0 < \eta < 1, \end{aligned} \tag{2.4}$$

is given by

$$\begin{aligned} x(t) &= - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \\ &\quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} \sigma(s) ds \right\}. \end{aligned} \tag{2.5}$$

Proof. In view of Lemma 2.3, the fractional differential equation in (2.4) is equivalent to the integral equation

$$x(t) = -I^q \sigma(t) + b_1 t^{q-1} + b_2 t^{q-2} = - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + b_1 t^{q-1} + b_2 t^{q-2}, \tag{2.6}$$

where $b_1, b_2 \in \mathbb{R}$ are arbitrary constants. Applying the boundary conditions for (2.4), we find that $b_2 = 0$ and

$$b_1 = \frac{\Gamma((q+1)/2)}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \left(\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} \sigma(s) ds \right). \tag{2.7}$$

Substituting the values of b_1 and b_2 in (2.6), we obtain (2.5). This completes the proof. □

3. Main Results

To establish the main results, we need the following assumptions.

- (A₁) There exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\begin{aligned} &\|f(t, x(t), (\phi x)(t), (\psi x)(t)) - f(t, y(t), (\phi y)(t), (\psi y)(t))\| \\ &\leq L_1(t) \|x - y\| + L_2(t) \|\phi x - \phi y\| + L_3(t) \|\psi x - \psi y\|, \quad \forall t \in [0, 1], x, y \in X. \end{aligned} \tag{3.1}$$

Further,

$$\begin{aligned} \gamma_0 &= \sup_{t \in [0,1]} \left| \int_0^t \gamma(t,s) ds \right|, & \delta_0 &= \sup_{t \in [0,1]} \left| \int_0^t \delta(t,s) ds \right|, \\ I_L^q &= \sup_{t \in [0,1]} \{ |I^q L_1(t)|, |I^q L_2(t)|, |I^q L_3(t)| \}, \\ I^{(q+1)/2} L(1) &= \max \left\{ \left| I^{(q+1)/2} L_1(1) \right|, \left| I^{(q+1)/2} L_2(1) \right|, \left| I^{(q+1)/2} L_3(1) \right| \right\}, \\ I^q L(\eta) &= \max \{ |I^q L_1(\eta)|, |I^q L_2(\eta)|, |I^q L_3(\eta)| \}. \end{aligned} \quad (3.2)$$

(A₂) There exists a number κ such that $\Lambda \leq \kappa < 1$, where

$$\begin{aligned} \Lambda &= (1 + \gamma_0 + \delta_0) \left\{ I_L^q + \lambda_1 \left(I^q L(\eta) + |a| I^{(q+1)/2} L(1) \right) \right\}, \\ \lambda_1 &= \frac{\Gamma((q+1)/2)}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]}. \end{aligned} \quad (3.3)$$

(A₃) $\|f(t, x(t), (\phi x)(t), (\psi x)(t))\| \leq \mu(t)$, for all $(t, x, \phi x, \psi x) \in [0, 1] \times X \times X \times X$, $\mu \in L^1([0, 1], \mathbb{R}^+)$.

Theorem 3.1. Assume that $f : [0, 1] \times X \times X \times X \rightarrow X$ is a jointly continuous function and satisfies the assumption (A₁). Then the boundary value problem (1.1) has a unique solution provided $\Lambda < 1$, where Λ is given in the assumption (A₂).

Proof. Define $F : C \rightarrow C$ by

$$\begin{aligned} (Fx)(t) &= - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds \\ &\quad + \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \\ &\quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds \right. \\ &\quad \left. + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds \right\}, \quad t \in [0, 1]. \end{aligned} \quad (3.4)$$

Let us set $\sup_{t \in [0,1]} |f(t, 0, 0, 0)| = M$, and choose

$$r \geq \frac{M}{(1-\lambda)} \left\{ \frac{1 + \lambda_1 \eta^q}{\Gamma(q+1)} + \frac{\lambda_1 |a|}{\Gamma((q+3)/2)} \right\}, \quad (3.5)$$

where λ is such that $\Lambda \leq \lambda < 1$. Now we show that $FB_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned} & \| (Fx)(t) \| \\ & \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \| f(s, x(s), (\phi x)(s), (\psi x)(s)) \| ds \\ & \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\ & \quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \| f(s, x(s), (\phi x)(s), (\psi x)(s)) \| ds \right. \\ & \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} \| f(s, x(s), (\phi x)(s), (\psi x)(s)) \| ds \right\} \\ & \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (\| f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, 0, 0, 0) \| + \| f(s, 0, 0, 0) \|) ds \\ & \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\ & \quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (\| f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, 0, 0, 0) \| + \| f(s, 0, 0, 0) \|) ds \right. \\ & \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} (\| f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, 0, 0, 0) \| + \| f(s, 0, 0, 0) \|) ds \right\} \\ & \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M) ds \\ & \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\ & \quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M) ds \right. \\ & \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} (L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M) ds \right\} \\ & \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M) ds \\ & \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M) ds \right. \\
& \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} (L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M) ds \right\} \\
& \leq (I^q L_1(t) + \gamma_0 I^q L_2(t) + \delta_0 I^q L_3(t))r + \frac{Mt^q}{\Gamma(q+1)} + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\
& \quad \times \left\{ \left((I^q L_1(\eta) + \gamma_0 I^q L_2(\eta) + \delta_0 I^q L_3(\eta))r + \frac{M\eta^q}{\Gamma(q+1)} \right) \right. \\
& \quad \left. + |a| \left((I^{(q+1)/2} L_1(1) + \gamma_0 I^{(q+1)/2} L_2(1) + \delta_0 I^{(q+1)/2} L_3(1))r + \frac{M}{\Gamma((q+3)/2)} \right) \right\} \\
& \leq (1 + \gamma_0 + \delta_0) \left\{ I_L^q + \lambda_1 (I^q L(\eta) + |a| I^{(q+1)/2} L(1)) \right\} r + M \left\{ \frac{1 + \lambda_1 \eta^q}{\Gamma(q+1)} + \frac{\lambda_1 |a|}{\Gamma((q+3)/2)} \right\} \\
& \leq (\Lambda + 1 - \lambda)r \leq r.
\end{aligned} \tag{3.6}$$

Now, for $x, y \in C$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
& \|(Fx)(t) - (Fy)(t)\| \\
& \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, y(s), (\phi y)(s), (\psi y)(s))\| ds \\
& \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\
& \quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, y(s), (\phi y)(s), (\psi y)(s))\| ds \right. \\
& \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} \|f(s, x(s), (\phi x)(s), (\psi x)(s)) - f(s, y(s), (\phi y)(s), (\psi y)(s))\| ds \right\} \\
& \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x-y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|) ds \\
& \quad + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\
& \quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} (L_1(s)\|x-y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|) ds \right. \\
& \quad \left. + |a| \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} (L_1(s)\|x-y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|) ds \right\}
\end{aligned}$$

$$\begin{aligned}
 &\leq (I^q L_1(t) + \gamma_0 I^q L_2(t) + \delta_0 I^q L_3(t)) \|x - y\| + \left| \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \right| \\
 &\quad \times \left\{ I^q L_1(\eta) + \gamma_0 I^q L_2(\eta) + \delta_0 I^q L_3(\eta) \right. \\
 &\quad \left. + |a| \left(I^{(q+1)/2} L_1(1) + \gamma_0 I^{(q+1)/2} L_2(1) + \delta_0 I^{(q+1)/2} L_3(1) \right) \right\} \|x - y\| \\
 &\leq (1 + \gamma_0 + \delta_0) \left\{ I_L^q + \lambda_1 \left(I^q L(\eta) + |a| I^{(q+1)/2} L(1) \right) \right\} \|x - y\| \\
 &= \Lambda \|x - y\|,
 \end{aligned} \tag{3.7}$$

where we have used the assumption (A_2) . As $\Lambda < 1$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Now, we state Krasnoselskii's fixed point theorem [17] which is needed to prove the following result to prove the existence of at least one solution of (1.1). \square

Theorem 3.2. *Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3. *Let $f : [0, 1] \times X \times X \times X \rightarrow X$ be jointly continuous, and the assumptions (A_1) and (A_3) hold with*

$$\Lambda_1 = \lambda_1 (1 + \gamma_0 + \delta_0) \left(I^q L(\eta) + |a| I^{(q+1)/2} L(1) \right) < 1. \tag{3.8}$$

Then there exists at least one solution of the boundary value problem (1.1) on $[0, 1]$.

Proof. Let us fix

$$r \geq \|\mu\|_{L^1} \left\{ \frac{1 + \lambda_1 \eta^q}{\Gamma(q+1)} + \frac{\lambda_1 |a|}{\Gamma((q+3)/2)} \right\}, \tag{3.9}$$

and consider $B_r = \{x \in C : \|x\| \leq r\}$. We define the operators Θ_1 and Θ_2 on B_r as

$$\begin{aligned}
 (\Theta_1 x)(t) &= - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds, \\
 (\Theta_2 x)(t) &= \frac{\Gamma((q+1)/2)t^{q-1}}{[a\Gamma(q) + \eta^{(q-1)}\Gamma((q+1)/2)]} \\
 &\quad \times \left\{ \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds \right. \\
 &\quad \left. + a \int_0^1 \frac{(1-s)^{(q-1)/2}}{\Gamma((q+1)/2)} f(s, x(s), (\phi x)(s), (\psi x)(s)) ds \right\}.
 \end{aligned} \tag{3.10}$$

For $x, y \in B_r$, we find that

$$\|\Theta_1 x + \Theta_2 y\| \leq \|\mu\|_{L^1} \left\{ \frac{1 + \lambda_1 \eta^q}{\Gamma(q+1)} + \frac{\lambda_1 |a|}{\Gamma((q+3)/2)} \right\} \leq r. \quad (3.11)$$

Thus, $\Theta_1 x + \Theta_2 y \in B_r$. It follows from the assumption (A_1) that Θ_2 is a contraction mapping for $\Lambda_1 < 1$.

In order to prove that Θ_1 is compact and continuous, we follow the approach used in [6, 7]. Continuity of f implies that the operator $(\Theta_1 x)(t)$ is continuous. Also, $(\Theta_1 x)(t)$ is uniformly bounded on B_r as

$$\|\Theta_1 x\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q+1)}. \quad (3.12)$$

Now, we show that $(\Theta_1 x)(t)$ is equicontinuous. Since f is bounded on the compact set $[0, 1] \times B_r \times B_r \times B_r$, therefore, we define $\sup_{(t,x,\phi x,\psi x) \in [0,1] \times B_r \times B_r \times B_r} \|f(t, x, \phi x, \psi x)\| = f_{\max}$. Consequently, for $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} & \|(\Theta_1 x)(t_1) - (\Theta_1 x)(t_2)\| \\ &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) f(s, x(s), \phi x(s), \psi x(s)) ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s), \phi x(s), \psi x(s)) ds \right\| \\ & \leq \frac{f_{\max}}{\Gamma(q+1)} \left| 2(t_2 - t_1)^q + t_1^q - t_2^q \right|, \end{aligned} \quad (3.13)$$

which is independent of x . So, Θ_1 is relatively compact on B_r . Hence, By Arzela-Ascoli's Theorem, Θ_1 is compact on B_r . Thus all the assumptions of Theorem 3.2 are satisfied and the conclusion of Theorem 3.2 implies that the boundary value problem (1.1) has at least one solution on $[0, 1]$. \square

Example. Consider the following boundary value problem:

$$\begin{aligned} {}^c D^{3/2} x(t) + \frac{t}{8} \frac{|x|}{1+|x|} + \frac{1}{5} \int_0^t \frac{e^{-(s-t)}}{5} x(s) ds + \frac{1}{5} \int_0^t \frac{e^{-(s-t)/2}}{5} x(s) ds &= 0, \quad t \in [0, 1], \\ D^{1/4} x(0) = 0, \quad a D^{1/4} x(1) + x\left(\frac{1}{3}\right) &= 0. \end{aligned} \quad (3.14)$$

Here, $q = 3/2$, $\gamma(t, s) = e^{-(s-t)}/5$, $\delta = e^{-(s-t)/2}/5$, $a = 1$, $\eta = 1/3$. With $\gamma_0 = (e - 1)/5$, $\delta_0 = 2(\sqrt{e} - 1)/5$, we find that

$$\Lambda = \frac{8(e + 2(\sqrt{e} + 1))(9\sqrt{3\pi} + \Gamma(1/4))}{225\sqrt{\pi}(2\sqrt{3\pi} + \Gamma(1/4))} < 1. \quad (3.15)$$

Thus, by Theorem 3.1, the boundary value problem (3.14) has a unique solution on $[0, 1]$.

4. Conclusions

This paper studies the existence and uniqueness of solutions for nonlinear integro-differential equations of fractional order $q \in (1, 2]$ with three-point nonlocal fractional boundary conditions involving the fractional derivative $D^{(q-1)/2}x(\cdot)$. Our results are based on a generalized variant of Lipschitz condition given in (A_1) , that is, there exist positive functions $L_1(t)$, $L_2(t)$, and $L_3(t)$ such that

$$\begin{aligned} & \|f(t, x(t), (\phi x)(t), (\psi x)(t)) - f(t, y(t), (\phi y)(t), (\psi y)(t))\| \\ & \leq L_1(t)\|x - y\| + L_2(t)\|\phi x - \phi y\| + L_3(t)\|\psi x - \psi y\|, \quad \forall t \in [0, 1], \quad x, y \in X. \end{aligned} \quad (4.1)$$

In case $L_1(t)$, $L_2(t)$, and $L_3(t)$ are constant functions, that is, $L_1(t) = L_1$, $L_2(t) = L_2$, and $L_3(t) = L_3$ (L_1, L_2 , and L_3 are positive real numbers), then Lipschitz-generalized variant reduces to the classical Lipschitz condition and Λ in the assumption (A_2) takes the form

$$\Lambda = (L_1 + \gamma_0 L_2 + \delta_0 L_3) \left\{ \frac{1 + \lambda_1 \eta^q}{\Gamma(q + 1)} + \frac{\lambda_1 |a|}{\Gamma((q + 3)/2)} \right\}. \quad (4.2)$$

In the limit $q \rightarrow 2$, our results correspond to a second-order integro-differential equation with fractional boundary conditions:

$$\begin{aligned} D^2 x(t) + f(t, x(t), (\phi x)(t), (\psi x)(t)) &= 0, \quad 0 < t < 1, \\ D^{1/2} x(0) = 0, \quad a D^{1/2} x(1) + x(\eta) &= 0, \quad 0 < \eta < 1. \end{aligned} \quad (4.3)$$

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