Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 714891, 11 pages
doi:10.1155/2010/714891

Research Article

Strictly Increasing Solutions of Nonautonomous Difference Equations Arising in Hydrodynamics

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Received 19 December 2009; Revised 14 February 2010; Accepted 10 March 2010

Academic Editor: Ağacik Zafer

The paper provides conditions sufficient for the existence of strictly increasing solutions of the second-order nonautonomous difference equation $x(n+1) = x(n) + (n/(n+1))^2(x(n) - x(n-1) + h^2 f(x(n)))$, $n \in \mathbb{N}$, where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$. In particular we prove that for each sufficiently small $h > 0$ there exists a solution $\{x(n)\}_{n=0}^{\infty}$ such that $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(0) = x(1) \in (L_0, 0)$, and $\lim_{n\to\infty} x(n) > L$. The problem is motivated by some models arising in hydrodynamics.

1. Formulation of Problem

We will investigate the following second-order non-autonomous difference equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \left(x(n) - x(n-1) + h^2 f(x(n))\right), \quad n \in \mathbb{N}, \tag{1.1}$$

where f is supposed to fulfil

$$L_0 < 0 < L, \quad f \in \mathrm{Lip_{loc}}[L_0, \infty), \quad f(L_0) = f(0) = f(L) = 0, \tag{1.2}$$

$$x f(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \qquad f(x) \geq 0 \quad \text{for } x \in (L, \infty), \tag{1.3}$$

$$\exists \overline{B} \in (L_0, 0) \text{ such that } \int_{\overline{B}}^{L} f(z)\mathrm{d}z = 0. \tag{1.4}$$

Let us note that $f \in \mathrm{Lip}_{\mathrm{loc}}[L_0, \infty)$ means that for each $[L_0, A] \subset [L_0, \infty)$ there exists $K_A > 0$ such that $|f(x) - f(y)| \le K_A|x - y|$ for all $x, y \in [L_0, A]$. A simple example of a function f satisfying (1.2)–(1.4) is $f(x) = c(x - L_0)x(x - L)$, where c is a positive constant.

A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (1.1) is called a solution of (1.1). For each values $B, B_1 \in [L_0, \infty)$ there exists a unique solution $\{x(n)\}_{n=0}^{\infty}$ of (1.1) satisfying the initial conditions

$$x(0) = B, \qquad x(1) = B_1. \tag{1.5}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called a solution of problem (1.1), (1.5).

In [1] we have shown that (1.1) is a discretization of differential equations which generalize some models arising in hydrodynamics or in the nonlinear field theory; see [2–6]. Increasing solutions of (1.1), (1.5) with $B = B_1 \in (L_0, 0)$ has a fundamental role in these models. Therefore, in [1], we have described the set of all solutions of problem (1.1), (1.6), where

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0). \tag{1.6}$$

In this paper, using [1], we will prove that for each sufficiently small $h > 0$ there exists at least one $B \in (L_0, 0)$ such that the corresponding solution of problem (1.1), (1.6) fulfils

$$x(0) = x(1), \quad \lim_{n \to \infty} x(n) > L, \quad \{x(n)\}_{n=1}^{\infty} \text{ is increasing.} \tag{1.7}$$

Note that an autonomous case of (1.1) was studied in [7]. We would like to point out that recently there has been a huge interest in studying the existence of monotonous and nontrivial solutions of nonlinear difference equations. For papers during last three years see, for example, [8–22]. A lot of other interesting references can be found therein.

2. Four Types of Solutions

Here we present some results of [1] which we need in next sections. In particular, we will use the following definitions and lemmas.

Definition 2.1. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) such that

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n \to \infty} x(n) = 0. \tag{2.1}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a damped solution*.

Definition 2.2. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) which fulfils

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n \to \infty} x(n) = L. \tag{2.2}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a homoclinic solution*.

Definition 2.3. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and

$$x(b) \leq L < x(b+1). \tag{2.3}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *an escape solution*.

Definition 2.4. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and

$$0 < x(b) < L, \quad x(b+1) \leq x(b). \tag{2.4}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a non-monotonous solution*.

Lemma 2.5 (see [1] (on four types of solutions)). *Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6). *Then $\{x(n)\}_{n=0}^{\infty}$ is just one of the following four types:*

(I) $\{x(n)\}_{n=0}^{\infty}$ *is an escape solution;*

(II) $\{x(n)\}_{n=0}^{\infty}$ *is a homoclinic solution;*

(III) $\{x(n)\}_{n=0}^{\infty}$ *is a damped solution;*

(IV) $\{x(n)\}_{n=0}^{\infty}$ *is a non-monotonous solution.*

Lemma 2.6 (see [1] (estimates of solutions)). *Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6). *Then there exists a maximal $b \in \mathbb{N} \cup \{\infty\}$ satisfying*

$$\begin{aligned} x(n) &\in [B, L) \quad \textit{for } n = 1, \ldots, b, \ \textit{if } b \in \mathbb{N}, \\ x(n) &\in [B, L) \quad \textit{for } n \in \mathbb{N}, \ \textit{if } b = \infty. \end{aligned} \tag{2.5}$$

Further, if $b > 1$, then moreover

$$\{x(n)\}_{n=1}^{b} \ \textit{is increasing}, \tag{2.6}$$

$$\Delta x(n) < h\sqrt{(L - 2L_0)M_0} + h^2 M_0 \tag{2.7}$$

for $n = 1, \ldots, b-1$ if $b \in \mathbb{N}$, and for $n \in \mathbb{N}$ if $b = \infty$, where

$$M_0 = \max\{|f(x)| : x \in [L_0, L]\}. \tag{2.8}$$

In [1] we have proved that the set consisting of damped and non-monotonous solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. This is contained in the next lemma.

Lemma 2.7 (see [1] (on the existence of non-monotonous or damped solutions)). *Let $B \in (\overline{B}, 0)$, where $\overline{B}$ is defined by* (1.4). *There exists $h_B > 0$ such that if $h \in (0, h_B]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (1.1), (1.6) *is non-monotonous or damped.*

In Section 4 of this paper we prove that also the set of escape solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. Note that in our next paper [23] we prove this assertion for the set of homoclinic solutions.

3. Properties of Solutions

Now, we provide other properties of solutions important in the investigation of escape solutions.

Lemma 3.1. *Let $\{x(n)\}_{n=0}^{\infty}$ be an escape solution of problem* (1.1), (1.6). *Then $\{x(n)\}_{n=1}^{\infty}$ is increasing.*

Proof. Due to (1.1), $\{x(n)\}_{n=0}^{\infty}$ fulfils

$$\Delta x(n) = \left(\frac{n}{n+1}\right)^2 \left(\Delta x(n-1) + h^2 f(x(n))\right), \quad n \in \mathbb{N}. \tag{3.1}$$

According to Definition 2.3 there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and (2.3) holds. By (1.3) we get $f(x(b+1)) \geq 0$. Consequently, by (3.1) and (2.3), $\Delta x(b+1) \geq (b+1)^2/(b+2)^2 \Delta x(b) > 0$ and $f(x(b+2)) \geq 0$. Similarly $\Delta x(b+j) \geq (b+j)^2/(b+1+j)^2 \Delta x(b+j-1)$ and

$$\Delta x(b+j) \geq \left(\frac{b+1}{b+1+j}\right)^2 \Delta x(b), \quad j \in \mathbb{N}. \tag{3.2}$$

This yields that $\{x(n)\}_{n=1}^{\infty}$ is increasing. □

Lemma 3.2. *Assume that $f(x) = 0$ for $x > L$. Choose an arbitrary $\varrho > 0$. Let $B_1, B_2 \in (L_0, 0)$ and let $\{x(n)\}_{n=0}^{\infty}$ and $\{y(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6) *with $B = B_1$ and $B = B_2$, respectively. Let K_L be the Lipschitz constant for f on $[L_0, L]$. Then*

$$\left|x(n) - y(n)\right| \leq |B_1 - B_2| e^{\varrho^2 K_L}, \tag{3.3}$$

$$\left|\frac{\Delta x(n) - \Delta y(n)}{h}\right| \leq |B_1 - B_2| \varrho K_L \, e^{\varrho^2 K_L}, \tag{3.4}$$

where $n \in \mathbb{N}$, $n \leq \varrho/h$.

Proof. By (3.1) we have

$$(j+1)^2 \Delta x(j) - j^2 \Delta x(j-1) = h^2 j^2 f(x(j)), \quad j \in \mathbb{N}. \tag{3.5}$$

Summing it for $j = 1, \dots, k$, we get by (1.6)

$$\Delta x(k) = h^2 \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad k \in \mathbb{N}. \tag{3.6}$$

Summing it again for $k = 1, \dots, n-1$, we get

$$x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad n \in \mathbb{N}, \tag{3.7}$$

and similarly

$$y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(y(j)), \quad n \in \mathbb{N}. \tag{3.8}$$

From this and by using summation by parts we easily obtain

$$\begin{aligned} |x(n) - y(n)| &\leq |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 |f(x(j)) - f(y(j))| \\ &\leq |B_1 - B_2| + (n-1)h^2 K_L \sum_{j=1}^{n-1} |x(j) - y(j)|, \quad n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

By the discrete analogue of the Gronwall-Bellman inequality (see, e.g., [24, Lemma 4.34]), we get

$$|x(n) - y(n)| \leq |B_1 - B_2| e^{(n-1)^2 h^2 K_L} \quad \text{for } n \in \mathbb{N}, \tag{3.10}$$

which yields (3.3).

By (3.6) and (3.3) we have for $n \in \mathbb{N}$, $n \leq \varrho/h$,

$$\begin{aligned} \left| \frac{\Delta x(n) - \Delta y(n)}{h} \right| &\leq h \frac{1}{(n+1)^2} \sum_{j=1}^{n} j^2 |f(x(j)) - f(y(j))| \\ &\leq h K_L \sum_{j=1}^{n} |x(j) - y(j)| \leq |B_1 - B_2| \varrho K_L e^{\varrho^2 K_L}. \end{aligned} \tag{3.11}$$

□

4. Existence of Escape Solutions

Lemma 4.1. *Assume that $C \in (L_0, \overline{B})$ and $\{B_k\}_{k=1}^{\infty} \subset (L_0, C)$. Let $\{x_k(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) with $B = B_k$, $k \in \mathbb{N}$. For $k \in \mathbb{N}$ choose a maximal $b_k \in \mathbb{N} \cup \{\infty\}$ such that $x_k(n) \in [B_k, L)$ for $n = 1, \dots, b_k$ if b_k is finite, and for $n \in \mathbb{N}$ if $b_k = \infty$, and $\{x_k(n)\}_{n=1}^{b_k}$ is increasing if $b_k > 1$. Then there exists $h^* > 0$ such that for any $h \in (0, h^*]$ there exists a unique $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, such that*

$$x_k(\gamma_k) \geq C, \qquad x_k(\gamma_k - 1) < C. \tag{4.1}$$

Moreover, if the sequence $\{\gamma_k\}_{k=1}^{\infty}$ *is unbounded, then there exists* $\ell \in \mathbb{N}$ *such that the solution* $\{x_\ell(n)\}_{n=0}^{\infty}$ *of problem* (1.1), (1.6) *with* $B = B_\ell \in (L_0, \overline{B})$ *is an escape solution.*

Proof. Choose $h_0 > 0$ such that

$$h_0\sqrt{(L-2L_0)M_0} + h_0^2 M_0 < |C|. \tag{4.2}$$

For $k \in \mathbb{N}$ denote by $\{x_k(n)\}_{n=0}^{\infty}$ a solution of problem (1.1), (1.6) with $B = B_k$. The existence of b_k is guaranteed by Lemma 2.6. By Lemma 2.5, $\{x_k(n)\}_{n=0}^{\infty}$ is just one of the types (I)–(IV), and if $h \in (0, h_0]$, then the monotonicity of $\{x_k(n)\}_{n=0}^{b_k}$ yields a unique $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, satisfying (4.1).

For $h \in (0, h_0)$, consider the sequence $\{\gamma_k\}_{k=1}^{\infty}$ and assume that it is unbounded. Then we have

$$\lim_{k\to\infty} \gamma_k = \infty \tag{4.3}$$

(otherwise we take a subsequence.) Assume on the contrary that for any $k \in \mathbb{N}$, $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution. Choose $k \in \mathbb{N}$. If $\{x_k(n)\}_{n=0}^{\infty}$ is damped, then by Definition 2.1, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = 0, \qquad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.4}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is homoclinic, then by Definition 2.2, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = L, \qquad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.5}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is non-monotonous, then by Definition 2.4, we have $b_k < \infty$ and

$$x_k(b_k) \in (0, L), \quad \Delta x_k(b_k) \le 0. \tag{4.6}$$

To summarize if $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution, then by (4.4), (4.5), and (4.6), we have

$$x_k(b_k) \in [0, L], \quad \Delta x_k(b_k) \le 0. \tag{4.7}$$

Since $\Delta x_k(0) = 0$, there exists $\overline{\gamma}_k \in \mathbb{N}$ satisfying

$$\gamma_k \le \overline{\gamma}_k < b_k, \quad \Delta x_k(\overline{\gamma}_k) = \max\{\Delta x_k(j) : \gamma_k \le j \le b_k - 1\}. \tag{4.8}$$

Consider (3.5) with $x = x_k$. By dividing it by j^2, multiplying such obtained equality by $x_k(j+1) - x_k(j-1)$ and summing in j from 1 to n we get

$$(\Delta x_k(n))^2 - h^2\sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)) = -\sum_{j=1}^{n}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}. \tag{4.9}$$

Denote

$$E_k(n+1) = (\Delta x_k(n))^2 - h^2\sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)). \tag{4.10}$$

Then we get

$$E_k(n+1) = -\sum_{j=1}^{n}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}. \tag{4.11}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.11) and subtract. By (4.7) and (4.8) we get

$$E_k(\gamma_k) - E_k(b_k) = \sum_{j=\gamma_k}^{b_k-1}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)) \leq 2\frac{2\gamma_k+1}{\gamma_k^2}\Delta x_k(\overline{\gamma}_k)(L - L_0). \tag{4.12}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.10) and subtract. We get

$$E_k(\gamma_k) - E_k(b_k) = (\Delta x_k(\gamma_k - 1))^2 - (\Delta x_k(b_k - 1))^2 + 2h^2\sum_{j=\gamma_k}^{b_k-1} f(x_k(j))\frac{x_k(j+1) - x_k(j-1)}{2}. \tag{4.13}$$

Choose $\varepsilon > 0$ and $h_1 > 0$ such that

$$\varepsilon < \frac{1}{2}\int_C^L f(z)\mathrm{d}z, \quad h_1 M_0 < \sqrt{\varepsilon}. \tag{4.14}$$

Let $b_k < \infty$. Then (4.6) holds. Since $\Delta x_k(b_k - 1) > 0$, $f(x_k(b_k)) < 0$ and $\Delta x_k(b_k) \leq 0$, (3.1) yields

$$\left(\frac{b_k+1}{b_k}\right)^2 |\Delta x_k(b_k)| + \Delta x_k(b_k - 1) = h^2 |f(x_k(b_k))|, \tag{4.15}$$

and hence

$$0 < \Delta x_k(b_k - 1) \le -h^2 f(x_k(b_k)) < h^2 M_0 < h\sqrt{\varepsilon} \quad \text{for } h \in (0, h_1]. \tag{4.16}$$

Clearly, if $b_k = \infty$, then by (4.4) and (4.5), inequality (4.16) holds, as well. By (1.2), f is integrable on $[L_0, L]$. So, having in mind (4.1), we can find $\delta > 0$ such that if

$$\frac{x_k(j+1) - x_k(j-1)}{2} < \delta, \quad j = \gamma_k, \dots, b_k - 1, \tag{4.17}$$

then

$$\left| \sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2} - \int_C^{b_k} f(z)\mathrm{d}z \right| < \varepsilon. \tag{4.18}$$

Therefore, due to (1.3) and (4.7),

$$\sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2} > \int_C^{b_k} f(z)\ \mathrm{d}z - \varepsilon \ge \int_C^{L} f(z)\mathrm{d}z - \varepsilon. \tag{4.19}$$

Let $h_2 > 0$ be such that

$$h_2\left(\sqrt{(L - 2L_0)M_0} + h_2 M_0\right) < \delta. \tag{4.20}$$

If $h \in (0, h_2]$, then (2.7) implies (4.17) and hence (4.19) holds.

Now, let us put $h^* = \min\{h_0, h_1, h_2\}$ and choose $h \in (0, h^*]$. Then, (4.2), (4.14), (4.20), and (4.13)–(4.19) yield

$$\begin{aligned} E_k(\gamma_k) - E_k(b_k) &> -h^2\varepsilon + 2h^2\left(\int_C^L f(z)\mathrm{d}z - \varepsilon\right) \\ &= 2h^2\left(\int_C^L f(z)\mathrm{d}z - \frac{3}{2}\varepsilon\right) > h^2\varepsilon > 0. \end{aligned} \tag{4.21}$$

Finally, (4.12) and (4.21) imply

$$\begin{gathered} 0 < h^2\varepsilon < E_k(\gamma_k) - E_k(b_k) \le 2\frac{2\gamma_k + 1}{\gamma_k^2}\Delta x_k(\overline{\gamma}_k)(L - L_0), \\ \frac{h^2\varepsilon}{2(L - L_0)} \cdot \frac{\gamma_k^2}{2\gamma_k + 1} < \Delta x_k(\overline{\gamma}_k). \end{gathered} \tag{4.22}$$

Letting $k \to \infty$, we obtain, by (4.3), that $\lim_{k\to\infty} \Delta x_k(\overline{\gamma}_k) = \infty$, contrary to (4.17). Therefore an escape solution $\{x_\ell(n)\}_{n=0}^{\infty}$ of problem (1.1), (1.6) with $B = B_\ell \in (L_0, \overline{B})$ must exist. □

Now, we are in a position to prove the next main result.

Theorem 4.2 (On the existence of escape solutions). *There exists $h^* > 0$ such that for any $h \in (0, h^*]$ the initial value problem* (1.1), (1.6) *has an escape solution for some $B \in (L_0, \overline{B})$.*

Proof. We have the following steps.

Step 1. Let us define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \le L, \\ 0 & \text{for } x > L, \end{cases} \tag{4.23}$$

and consider an auxiliary equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \left(x(n) - x(n-1) + h^2 \tilde{f}(x(n))\right), \quad n \in \mathbb{N}. \tag{4.24}$$

Let $h^* > 0$ be the constant of Lemma 4.1 for problem (4.24), (1.6). Choose $h \in (0, h^*]$, $C \in (L_0, \overline{B})$ and let K_L be the Lipschitz constant for $\tilde{f}$ on $[L_0, \infty)$. Consider a sequence $\{B_k\}_{k=1}^{\infty} \subset (L_0, C)$ such that $\lim_{k\to\infty} B_k = L_0$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$|B_{k_m} - L_0| < e^{-m^2 K_L}(C - L_0). \tag{4.25}$$

Let $\tilde{x}_0(0) = \tilde{x}_0(n) = L_0$ for $n \in \mathbb{N}$. Then the sequence $\{\tilde{x}_0(n)\}_{n=0}^{\infty}$ is the unique solution of problem (4.24), (1.6) with $B = L_0$. Let $\{\tilde{x}_k(n)\}_{n=0}^{\infty}$ be a solution of problem (4.24), (1.6) with $B = B_k$, $k \in \mathbb{N}$, and let $\{\gamma_k\}_{k=1}^{\infty}$ be the sequence corresponding to $\{\tilde{x}_k(n)\}_{n=0}^{\infty}$ by Lemma 4.1. We prove that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. According to Lemma 3.2, for each $m \in \mathbb{N}$,

$$|\tilde{x}_{k_m}(n) - \tilde{x}_0(n)| \le |B_{k_m} - L_0| e^{m^2 K_L}, \quad n \le \frac{m}{h}. \tag{4.26}$$

Consequently, (4.25) and (4.26) give

$$|\tilde{x}_{k_m}(n) - \tilde{x}_0(n)| \le C - L_0, \quad n \le \frac{m}{h}, \tag{4.27}$$

and hence

$$\tilde{x}_{k_m}(n) \le C, \quad n \le \frac{m}{h}. \tag{4.28}$$

Therefore

$$\gamma_{k_m}(n) \ge \frac{m}{h}, \quad m \in \mathbb{N}, \tag{4.29}$$

which yields that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. By Lemma 4.1, the auxiliary initial value problem (4.24), (1.6) has an escape solution for some $B = B_\ell \in (L_0, \widetilde{B})$. Denote this solution by $\{\widetilde{x}_\ell(n)\}_{n=0}^{\infty}$.

Step 2. By Definition 2.3, there exists $b \in \mathbb{N}$ such that

$$\{\widetilde{x}(n)\}_{n=1}^{b+1} \text{ is increasing}, \qquad \widetilde{x}_\ell(b) \le L < \widetilde{x}_\ell(b+1). \tag{4.30}$$

Now, consider the solution $\{x_\ell(n)\}_{n=0}^{\infty}$ of our original problem (1.1), (1.6) with $B = B_\ell$. Due to (4.23), $x_\ell(n) = \widetilde{x}_\ell(n)$ for $n = 0, 1, \ldots, b+1$. Using (4.30) and Definition 2.3, we get that $\{x_\ell(n)\}_{n=0}^{\infty}$ is an escape solution of problem (1.1), (1.6). □

Acknowledgments

The paper was supported by the Council of Czech Government MSM 6198959214. The authors thank the referees for valuable comments.

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