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Research Article

Some Results for Integral Inclusions of Volterra Type in Banach Spaces

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We first present several existence results and compactness of solutions set for the following Volterra type integral inclusions of the form: $y(t) \in \int_0^t a(t-s)[Ay(s)+F(s,y(s))]ds$, a.e. $t \in J$, where $J=[0,b]$, A is the infinitesimal generator of an integral resolvent family on a separable Banach space E, and F is a set-valued map. Then the Filippov's theorem and a Filippov-Ważewski result are proved.

1. Introduction

In the past few years, several papers have been devoted to the study of integral equations on real compact intervals under different conditions on the kernel (see, e.g., [1–4]) and references therein. However very few results are available for integral inclusions on compact intervals, see [5–7]. Topological structure of the solution set of integral inclusions of Volterra type is studied in [8].

In this paper we present some results on the existence of solutions, the compactness of set of solutions, Filippov's theorem, and relaxation for linear and semilinear integral inclusions of Volterra type of the form

$$y(t) \in \int_0^t a(t-s)\left[Ay(s)+F\left(s,y(s)\right)\right]ds, \quad \text{a.e. } t \in J := [0,b], \tag{1.1}$$

where $a \in L^1([0,b],\mathbb{R})$ and $A : D(A) \subset E \to E$ is the generator of an integral resolvent family defined on a complex Banach space E, and $F : [0,b] \times E \to \mathcal{P}(E)$ is a multivalued map.

In 1980, Da Prato and Iannelli introduced the concept of resolvent families, which can be regarded as an extension of C_0-semigroups in the study of a class of integrodifferential equations [9]. It is well known that the following abstract Volterra equation

$$y(t) = f(t) + \int_0^t a(t-s)Ay(s)ds, \quad t \geq 0, \tag{1.2}$$

where $f : \mathbb{R}_+ \to E$ is a continuous function, is well-posed if and only if it admits a resolvent family, that is, there is a strongly continuous family $S(t)$, $t > 0$, of bounded linear operators defined in E, which commutes with A and satisfies the resolvent equation

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds, \quad t \geq 0,\ x \in D(A). \tag{1.3}$$

The study of diverse properties of resolvent families such as the regularity, positivity, periodicity, approximation, uniform continuity, compactness, and others are studied by several authors under different conditions on the kernel and the operator A (see [10–24]). An important kernel is given by

$$a(t) = f_{1-\alpha}(t)e^{-kt}, \quad t > 0,\ k \geq 0,\ \alpha \in (0,1), \tag{1.4}$$

where

$$f_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases} \tag{1.5}$$

is the Riemann-Liouville kernel. In this case (1.1) and (1.2) can be represented in the form of fractional differential equations and inclusions or abstract fractional differential equations and inclusions. Also in the case where $A \equiv 0$, and a is a Rieman-Liouville kernel, (1.1) and (1.2) can be represented in the form of fractional differential equations and inclusions, see for instants [25–27].

Our goal in this paper is to complement and extend some recent results to the case of infinite-dimensional spaces; moreover the right-hand side nonlinearity may be either convex or nonconvex. Some auxiliary results from multivalued analysis, resolvent family theory, and so forth, are gathered together in Sections 2 and 3. In the first part of this work, we prove some existence results based on the nonlinear alternative of Leray-Schauder type (in the convex case), on Bressan-Colombo selection theorem and on the Covitz combined the nonlinear alternative of Leray-Schauder type for single-valued operators, and Covitz-Nadler fixed point theorem for contraction multivalued maps in a generalized metric space (in the nonconvex case). Some topological ingredients including some notions of measure of noncompactness are recalled and employed to prove the compactness of the solution set in Section 4.2. Section 5 is concerned with Filippov's theorem for the problem (1.1).

In Section 6, we discuss the relaxed problem, namely, the density of the solution set of problem (1.1) in that of the convexified problem.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $(E, \|\cdot\|)$ be a separable Banach space, $J = [0, b]$ an interval in $\mathbb{R}$ and $C(J, E)$ the Banach space of all continuous functions from J into E with the norm

$$\|y\|_{\infty} = \sup\{\|y(t)\| : 0 \leq t \leq b\}. \tag{2.1}$$

$B(E)$ refers to the Banach space of linear bounded operators from E into E with norm

$$\|N\|_{B(E)} = \sup\{\|N(y)\| : \|y\| = 1\}. \tag{2.2}$$

A function $y : J \longrightarrow E$ is called measurable provided for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J : y(t) \in U\}$ is Lebesgue measurable. A measurable function $y : J \longrightarrow E$ is Bochner integrable if $\|y\|$ is Lebesgue integrable. For properties of the Bochner integral, see, for example, Yosida [28]. In what follows, $L^1(J, E)$ denotes the Banach space of functions $y : J \longrightarrow E$, which are Bochner integrable with norm

$$\|y\|_1 = \int_0^b \|y(t)\| dt. \tag{2.3}$$

Denote by $\mathcal{P}(E) = \{Y \subset E : Y \neq \emptyset\}$, $\mathcal{P}_{\text{cl}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ closed}\}$, $\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ bounded}\}$, $\mathcal{P}_{\text{cv}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ convex}\}$, $\mathcal{P}_{\text{cp}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact}\}$.

2.1. Multivalued Analysis

Let (X, d) and (Y, ρ) be two metric spaces and $G : X \longrightarrow \mathcal{P}_{\text{cl}}(Y)$ be a multivalued map. A single-valued map $g : X \longrightarrow Y$ is said to be a selection of G and we write $g \subset G$ whenever $g(x) \in G(x)$ for every $x \in X$.

G is called *upper semicontinuous (u.s.c. for short)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $G^{-1}(V) = \{x \in X,\ G(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Y. Equivalently, G is *u.s.c.* if the set $G^{+1}(V) = \{x \in X,\ G(x) \subset V\}$ is open for any open set V in Y.

The following two results are easily deduced from the limit properties.

Lemma 2.1 (see, e.g., [29, Theorem 1.4.13]). *If $G : X \longrightarrow \mathcal{P}_{cp}(x)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x \to x_0} G(x) = G(x_0). \tag{2.4}$$

Lemma 2.2 (see, e.g., [29, Lemma 1.1.9]). *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{\text{co}}\left(\limsup_{n\to\infty} K_n\right) = \bigcap_{N>0} \overline{\text{co}}\left(\bigcup_{n\geq N} K_n\right), \tag{2.5}$$

where $\overline{\text{co}}\, C$ refers to the closure of the convex hull of C.

G is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $G(A)$ is relatively compact, that is, there exists a relatively compact set $K = K(A) \subset X$ such that $G(A) = \cup\{G(x),\ x \in A\} \subset K$. G is compact if $G(X)$ is relatively compact. It is called locally compact if, for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $G(U)$ is relatively compact. G is quasicompact if, for each subset $A \subset X$, $G(A)$ is relatively compact.

Definition 2.3. A multivalued map $F : J = [0, b] \to \mathcal{P}_{\text{cl}}(Y)$ is said measurable provided for every open $U \subset Y$, the set $F^{+1}(U)$ is Lebesgue measurable.

We have

Lemma 2.4 (see [30, 31]). *The mapping F is measurable if and only if for each $x \in Y$, the function $\zeta : J \to [0, +\infty)$ defined by*

$$\zeta(t) = \text{dist}(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}, \quad t \in J, \tag{2.6}$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.5 (see [31, Theorem 19.7]). *Let Y be a separable metric space and $F : [a, b] \to \mathcal{P}(Y)$ a measurable multivalued map with nonempty closed values. Then F has a measurable selection.*

Lemma 2.6 (see [32, Lemma 3.2]). *Let $F : [0, b] \to \mathcal{P}(Y)$ be a measurable multivalued map and $u : [a, b] \to Y$ a measurable function. Then for any measurable $v : [a, b] \to (0, +\infty)$, there exists a measurable selection f_v of F such that for a.e. $t \in [a, b]$,*

$$\|u(t) - f_v(t)\| \leq d(u(t), F(t)) + v(t). \tag{2.7}$$

Corollary 2.7. *Let $F : [0, b] \to \mathcal{P}_{cp}(Y)$ be a measurable multivalued map and $u : [0, b] \to E$ a measurable function. Then there exists a measurable selection f of F such that for a.e. $t \in [0, b]$,*

$$\|u(t) - f(t)\| \leq d(u(t), F(t)). \tag{2.8}$$

2.1.1. Closed Graphs

We denote the graph of G to be the set $\mathcal{G}r(G) = \{(x, y) \in X \times Y,\ y \in G(x)\}$.

Definition 2.8. G is closed if $\mathcal{G}r(G)$ is a closed subset of $X \times Y$, that is, for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ as $n \to \infty$ with $y_n \in F(x_n)$, then $y_* \in G(x_*)$.

We recall the following two results; the first one is classical.

Lemma 2.9 (see [33, Proposition 1.2]). *If $G : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is u.s.c., then $\mathcal{G}r(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.10. *If $G : X \to \mathcal{P}_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

Given a separable Banach space $(E, \|\cdot\|)$, for a multivalued map $F : J \times E \to \mathcal{P}(E)$, denote

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{\|v\| : v \in F(t,x)\}. \tag{2.9}$$

Definition 2.11. A multivalued map F is called a Carathéodory function if

(a) the function $t \mapsto F(t,x)$ is measurable for each $x \in E$;

(b) for a.e. $t \in J$, the map $x \mapsto F(t,x)$ is upper semicontinuous.

Furthermore, F is L^1-Carathéodory if it is locally integrably bounded, that is, for each positive r, there exists $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \le h_r(t), \quad \text{for a.e. } t \in J \text{ and all } \|x\| \le r. \tag{2.10}$$

For each $x \in C(J,E)$, the set

$$S_{F,x} = \left\{ f \in L^1(J,E) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in J \right\} \tag{2.11}$$

is known as the set of selection functions.

Remark 2.12. (a) For each $x \in C(J,E)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t,x(t))$ is convex for a.e. $t \in J$.

(b) From [34] (see also [35] when E is finite-dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \mapsto \inf\{\|v\| : v \in F(t,x(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t,x(t))\|_{\mathcal{P}}$ belongs to $L^1(J)$; this particularly holds true when F is L^1-Carathéodory. For the sake of completeness, we refer also to Theorem 1.3.5 in [36] which states that $S_{F,x}$ contains a measurable selection whenever x is measurable and F is a Carathéodory function.

Lemma 2.13 (see [35]). *Given a Banach space E, let $F : [a,b] \times E \to \mathcal{P}_{cp,cv}(E)$ be an L^1-Carathéodory multivalued map, and let Γ be a linear continuous mapping from $L^1([a,b],E)$ into*

$C([a,b],E)$. *Then the operator*

$$\Gamma \circ S_F : C([a,b],E) \longrightarrow \mathcal{P}_{cp,cv}(C([a,b],E)),$$
$$y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \tag{2.12}$$

has a closed graph in $C([a,b],E) \times C([a,b],E)$.

For further readings and details on multivalued analysis, we refer to the books by Andres and Górniewicz [37], Aubin and Cellina [38], Aubin and Frankowska [29], Deimling [33], Górniewicz [31], Hu and Papageorgiou [34], Kamenskii et al. [36], and Tolstonogov [39].

2.2. *Semicompactness in* $L^1([0,b],E)$

Definition 2.14. A sequence $\{v_n\}_{n\in\mathbb{N}} \subset L^1(J,E)$ is said to be semicompact if

(a) it is integrably bounded, that is, there exists $q \in L^1(J,\mathbb{R}^+)$ such that

$$\|v_n(t)\|_E \le q(t), \quad \text{for a.e. } t \in J \text{ and every } n \in \mathbb{N}, \tag{2.13}$$

(b) the image sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in E for a.e. $t \in J$.

We recall two fundamental results. The first one follows from the Dunford-Pettis theorem (see [36, Proposition 4.2.1]). This result is of particular importance if E is reflexive in which case (a) implies (b) in Definition 2.14.

Lemma 2.15. *Every semicompact sequence* $L^1(J,E)$ *is weakly compact in* $L^1(J,E)$.

The second one is due to Mazur, 1933.

Lemma 2.16 (Mazur's Lemma, [28]). *Let* E *be a normed space and* $\{x_k\}_{k\in\mathbb{N}} \subset E$ *be a sequence weakly converging to a limit* $x \in E$. *Then there exists a sequence of convex combinations* $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ *with* $\alpha_{mk} > 0$ *for* $k = 1,2,\ldots,m$ *and* $\sum_{k=1}^{m} \alpha_{mk} = 1$, *which converges strongly to* x.

3. Resolvent Family

The Laplace transformation of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+,E)$ is defined by

$$\mathcal{L}(f)(\lambda) =: \hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \text{Re}(\lambda) > \omega, \tag{3.1}$$

if the integral is absolutely convergent for $\text{Re}(\lambda) > \omega$. In order to defined the mild solution of the problems (1.1) we recall the following definition.

Definition 3.1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space E. We call A the generator of an integral resolvent if there exists $\omega > 0$ and a strongly continuous function $S : \mathbb{R}^+ \to B(E)$ such that

$$\left(\frac{1}{\widehat{a}(\lambda)} I - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt, \quad \operatorname{Re} \lambda > \omega, \ x \in E. \tag{3.2}$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The following result is a direct consequence of ([16, Proposition 3.1 and Lemma 2.2]).

Proposition 3.2. *Let $\{S(t)\}_{t \geq 0} \subset B(E)$ be an integral resolvent family with generator A. Then the following conditions are satisfied*:

(a) *$S(t)$ is strongly continuous for $t \geq 0$ and $S(0) = I$;*

(b) *$S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$;*

(c) *for every $x \in D(A)$ and $t \geq 0$,*

$$S(t)x = a(t)x + \int_0^t a(t-s) AS(s) x ds, \tag{3.3}$$

(d) *let $x \in D(A)$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$, and*

$$S(t)x = a(t)x + A\int_0^t a(t-s) S(s) x ds. \tag{3.4}$$

In particular, $S(0) = a(0)$.

Remark 3.3. The uniqueness of resolvent is well known (see Prüss [24]).

If an operator A with domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function which satisfies $\lim_{t \to 0^+} \|S(t)\|_{B(E)} / a(t) < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - a(t)x}{(a * a)(t)}, \tag{3.5}$$

(see [22, Theorem 2.1]). For example, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup and $a(t) = t$ actually corresponds to the generator of a sine family; see [40]. A characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for C_0-semigroups, can be directly deduced from [22, Theorem 3.4]. More information on the C_0-semigroups and sine families can be found in [41–43].

Definition 3.4. A resolvent family of bounded linear operators, $\{S(t)\}_{t>0}$, is called uniformly continuous if

$$\lim_{t \to s} \|S(t) - S(s)\|_{B(E)} = 0. \tag{3.6}$$

Definition 3.5. The solution operator $S(t)$ is called exponentially bounded if there are constants $M > 0$ and $\omega \geq 0$ such that

$$\|S(t)\|_{B(E)} \leq M e^{\omega t}, \quad t \geq 0. \tag{3.7}$$

4. Existence Results

4.1. Mild Solutions

In order to define mild solutions for problem (1.1), we proof the following auxiliary lemma.

Lemma 4.1. *Let $a \in L^1(J, \mathbb{R})$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ on E, which is in addition integrable and $\overline{D(A)} = E$. Let $f : J \to E$ be a continuous function (or $f \in L^1(J, E)$), then the unique bounded solution of the problem*

$$y(t) = \int_0^t a(t-s)Ay(s)ds + \int_0^t a(t-s)f(s)ds, \quad t \in J, \tag{4.1}$$

is given by

$$y(t) = \int_0^t S(t-s)f(s)ds, \quad t \in J. \tag{4.2}$$

Proof. Let y be a solution of the integral equation (4.2), then

$$y(t) = \int_0^t S(t-s)f(s)ds. \tag{4.3}$$

Using the fact that S is solution operator and Fubini's theorem we obtain

$$\begin{aligned}
\int_0^t a(t-s)Ay(s)ds &= \int_0^t a(t-s)A\int_0^s S(s-r)f(r)dr\,ds \\
&= \int_0^t \int_r^t a(t-s)AS(s-r)ds f(r)dr \\
&= \int_0^t \int_0^{t-r} a(t-s-r)AS(s)ds f(r)dr \\
&= \int_0^t [S(t-r) - a(t-r)]f(r)dr \\
&= \int_0^t S(t-s)f(s)ds - \int_0^t a(t-s)f(s)ds.
\end{aligned} \tag{4.4}$$

Thus

$$y(t) = \int_0^t a(t-s)Ay(s)ds + \int_0^t a(t-s)f(s)ds, \quad t \in J. \tag{4.5}$$

This lemma leads us to the definition of a mild solution of the problem (1.1). □

Definition 4.2. A function $y \in C(J, E)$ is said to be a mild solution of problem (1.1) if there exists $f \in L^1(J, E)$ such that $f(t) \in F(t, y(t))$ a.e. on J such that

$$y(t) = \int_0^t S(t-s)f(s)ds, \quad t \in J. \tag{4.6}$$

Consider the following assumptions.

$(\mathcal{B}_1)$ The operator solution $\{S(t)\}_{t\in J}$ is compact for $t > 0$.

$(\mathcal{B}_2)$ There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|) \quad \text{for a.e. } t \in J \text{ and each } x \in E \tag{4.7}$$

with

$$Me^{\omega b} \int_0^b p(s)ds < \int_0^\infty \frac{du}{\psi(u)}. \tag{4.8}$$

$(\mathcal{B}_3)$ For every $t > 0$, $S(t)$ is uniformly continuous.

In all the sequel we assume that $S(\cdot)$ is exponentially bounded. Our first main existence result is the following.

Theorem 4.3. *Assume $F : J \times E \to \mathcal{P}_{cp,cv}(E)$ is a Carathéodory map satisfying $(\mathcal{B}_1)$-$(\mathcal{B}_2)$(or $(\mathcal{B}_2)$-$(\mathcal{B}_3)$). Then problem (1.1) has at least one solution. If further E is a reflexive space, then the solution set is compact in $C(J, E)$.*

The following so-called nonlinear alternatives of Leray-Schauder type will be needed in the proof (see [31, 44]).

Lemma 4.4. *Let $(X, \|\cdot\|)$ be a normed space and $F : X \to \mathcal{P}_{cl,cv}(X)$ a compact, u.s.c. multivalued map. Then either one of the following conditions holds.*

(a) *F has at least one fixed point,*

(b) *the set $\mathcal{M} := \{x \in E,\ x \in \lambda F(x),\ \lambda \in (0,1)\}$ is unbounded.*

The single-valued version may be stated as follows.

Lemma 4.5. *Let X be a Banach space and $C \subset X$ a nonempty bounded, closed, convex subset. Assume U is an open subset of C with $0 \in U$ and let $G : \overline{U} \to C$ be a a continuous compact map. Then*

(a) *either there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$,*

(b) *or G has a fixed point in $\overline{U}$.*

Proof of Theorem 4.3. We have the following parts.

Part 1: Existence of Solutions

It is clear that all solutions of problem (1.1) are fixed points of the multivalued operator $N : C(J,E) \to \mathcal{P}(C(J,E))$ defined by

$$N(y) := \left\{ h \in C(J,E) \mid h(t) = \int_0^t S(t-s)f(s)ds, \text{ for } t \in J \right\} \tag{4.9}$$

where

$$f \in S_{F,y} = \left\{ f \in L^1(J,E) : f(t) \in F(t,y(t)), \text{ for a.e. } t \in J \right\}. \tag{4.10}$$

Notice that the set $S_{F,y}$ is nonempty (see Remark 2.12,(b)). Since, for each $y \in C(J,E)$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and therefore N has convex values.

Step 1 (N is completely continuous). (a) *N sends bounded sets into bounded sets in $C(J,E)$.*
Let $q > 0$, $B_q := \{y \in C(J,E) : \|y\|_\infty \le q\}$ be a bounded set in $C(J,E)$, and $y \in B_q$. Then for each $h \in N(y)$, there exists $f \in S_{F,y}$ such that

$$h(t) = \int_0^t S(t-s)f(s)ds, \quad \text{for } t \in J. \tag{4.11}$$

Thus for each $t \in J$,

$$\|h\|_\infty \le e^{\omega b}\psi(q)\int_0^b p(t)dt. \tag{4.12}$$

(b) *N maps bounded sets into equicontinuous sets of $C(J,E)$.*
Let $\tau_1, \tau_2 \in J$, $0 < \tau_1 < \tau_2$ and B_q be a bounded set of $C(J,E)$ as in (a). Let $y \in \mathcal{B}_q$; then for each $t \in J$

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\le \psi(q)\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} p(s)ds \\ &\quad + \psi(q)\int_0^{\tau_1} \|S(\tau_1 - s) - S(\tau_2 - s)\|_{B(E)} p(s)ds. \end{aligned} \tag{4.13}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$ since $S(t)$ is uniformly continuous.

(c) As a consequence of parts (a) and (b) together with the Arzéla-Ascoli theorem, it suffices to show that N maps B_q into a precompact set in E. Let $0 < t \leq b$ and let $0 < \varepsilon < t$. For $y \in B_q$, define

$$h_{\varepsilon}(t) = \int_0^{t-\varepsilon} S(t-s-\varepsilon) f(s) ds. \tag{4.14}$$

Then

$$\begin{aligned} |h(t) - h_{\varepsilon}(t)| \leq \psi(q) \int_0^{t-\varepsilon} &\|S(t-s) - S(t-s-\epsilon)\|_{B(E)} p(s) ds \\ &+ \psi(q) \int_{t-\varepsilon}^{t} \|S(t-s)\|_{B(E)} p(s) ds, \end{aligned} \tag{4.15}$$

which tends to 0 as $\varepsilon \to 0$. Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t) : h \in N(y)\}$. This set is then precompact in E.

Step 2 (N has a closed graph). Let $h_n \to h_*$, $h_n \in N(y_n)$ and $y_n \to y_*$. We will prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that for each $t \in J$

$$h_n(t) = \int_0^t S(t-s) f_n(s) ds. \tag{4.16}$$

First, we have

$$\|h_n - h_*\|_{\infty} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.17}$$

Now, consider the linear continuous operator $\Gamma : L^1(J, E) \to C(J, E)$ defined by

$$(\Gamma v)(t) = \int_0^t S(t-s) v(s) ds. \tag{4.18}$$

From the definition of Γ, we know that

$$h_n(t) \in \Gamma\left(S_{F,y_n}\right). \tag{4.19}$$

Since $y_n \to y_*$ and $\Gamma \circ S_F$ is a closed graph operator by Lemma 2.13, then there exists $f_* \in S_{F,y_*}$ such that

$$h_*(t) = \int_0^t S(t-s) f_*(s) ds. \tag{4.20}$$

Hence $h_* \in N(y_*)$, proving our claim. Lemma 2.10 implies that N is *u.s.c.*

Step 3 (a priori bounds on solutions). Let $y \in C(J,E)$ be such that $y \in N(y)$. Then there exists $f \in S_{F,y}$ such that

$$y(t) = \int_0^t S(t-s)f(s)ds, \quad \text{for } t \in J. \tag{4.21}$$

Then

$$\begin{aligned} \|y(t)\| &\leq \int_0^t \|S(t-s)\|_{B(E)} \|f(s)\| ds \\ &\leq \int_0^t \|S(t-s)\|_{B(E)} p(s)\psi(\|y(s)\|)ds \\ &\leq Me^{\omega b}\int_0^t p(s)\psi(\|y(s)\|)ds. \end{aligned} \tag{4.22}$$

Set

$$v(t) = Me^{\omega b}\int_0^t p(s)\psi(\|y(s)\|)ds, \tag{4.23}$$

then $v(0) = 0$ and for a.e. $t \in J$ we have

$$v'(t) = Me^{\omega b}p(t)\psi(\|y(t)\|) \leq Me^{\omega b}p(t)\psi(v(t)). \tag{4.24}$$

Thus

$$\int_0^t \frac{v(s)}{\psi(v(s))}ds \leq Me^{\omega b}\int_0^t p(s)ds. \tag{4.25}$$

Using a change of variable we get

$$\int_0^{v(t)} \frac{du}{\psi(u)} \leq Me^{\omega b}\int_0^b p(s)ds. \tag{4.26}$$

From $(\mathcal{B}_2)$ there exists $\widetilde{M} > 0$ such that

$$\|y(t)\| \leq v(t) \leq \widetilde{M} \quad \text{for each } t \in J. \tag{4.27}$$

Let

$$U := \left\{ y \in C(J,E) : \|y\|_\infty < \widetilde{M} + 1 \right\}, \tag{4.28}$$

and consider the operator $N : \overline{U} \to \mathcal{P}_{cv,cp}(C(J,E))$. From the choice of U, there is no $y \in \partial U$ such that $y \in \gamma N(y)$ for some $\gamma \in (0,1)$. As a consequence of the Leray-Schauder nonlinear alternative (Lemma 4.4), we deduce that N has a fixed point y in U which is a mild solution of problem (1.1).

Part 2: Compactness of the Solution Set

Let

$$S_F = \{y \in C(J,E) \mid y \text{ is a solution of problem (1.1)}\}. \tag{4.29}$$

From Part 1, $S_F \neq \emptyset$ and there exists $\widetilde{M}$ such that for every $y \in S_F$, $\|y\|_\infty \leq \widetilde{M}$. Since N is completely continuous, then $N(S_F)$ is relatively compact in $C(J,E)$. Let $y \in S_F$; then $y \in N(y)$ and $S_F \subset \overline{N(S_F)}$. It remains to prove that S_F is closed set in $C(J,E)$. Let $y_n \in S_F$ such that y_n converge to y. For every $n \in \mathbb{N}$, there exists $v_n(t) \in F(t, y_n(t))$, a.e. $t \in J$ such that

$$y_n(t) = \int_0^t S(t-s)v_n(s)ds. \tag{4.30}$$

$(\mathcal{B}_1)$ implies that $v_n(t) \in p(t)B(0,1)$, hence $(v_n)_{n\in\mathbb{N}}$ is integrably bounded. Note this still remains true when $(\mathcal{B}_2)$ holds for S_F is a bounded set. Since E is reflexive, $(v_n)_{n\in\mathbb{N}}$ is semicompact. By Lemma 2.15, there exists a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,E)$. Moreover, the mapping $\Gamma : L^1(J,E) \to C(J,E)$ defined by

$$\Gamma(g)(t) = \int_0^t S(t-s)g(s)ds \tag{4.31}$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies. Therefore for a.e. $t \in J$, the sequence $y_n(t)$ converges to $y(t)$, it follows that

$$y(t) = \int_0^t S(t-s)v(s)ds. \tag{4.32}$$

It remains to prove that $v \in F(t, y(t))$, for a.e. $t \in J$. Lemma 2.16 yields the existence of $\alpha_i^n \geq 0$, $i = n, \dots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinaisons

$g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\cdot)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.2, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{g_n(t)\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{\text{co}}\{v_k(t), k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{\text{co}}\left\{\bigcup_{k\geq n} F(t, y_k(t))\right\} \\ &= \overline{\text{co}}\left(\limsup_{k\to\infty} F(t, y_k(t))\right). \end{aligned} \tag{4.33}$$

Since F is *u.s.c.* with compact values, then by Lemma 2.1, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t)), \quad \text{for a.e. } t \in J. \tag{4.34}$$

This with (4.33) imply that $v(t) \in \overline{\text{co}}\, F(t, y(t))$. Since $F(\cdot,\cdot)$ has closed, convex values, we deduce that $v(t) \in F(t, y(t))$, for a.e. $t \in J$, as claimed. Hence $y \in S_F$ which yields that S_F is closed, hence compact in $C(J, E)$. □

4.2. The Convex Case: An MNC Approach

First, we gather together some material on the measure of noncompactness. For more details, we refer the reader to [36, 45] and the references therein.

Definition 4.6. Let E be a Banach space and $(\mathcal{A}, \geq)$ a partially ordered set. A map $\beta : \mathcal{P}(E) \to \mathcal{A}$ is called a measure of noncompactness on E, MNC for short, if

$$\beta(\overline{\text{co}}\, \Omega) = \beta(\Omega) \tag{4.35}$$

for every $\Omega \in \mathcal{P}(E)$.

Notice that if D is dense in Ω, then $\overline{\text{co}}\, \Omega = \overline{\text{co}}\, D$ and hence

$$\beta(\Omega) = \beta(D). \tag{4.36}$$

Definition 4.7. A measure of noncompactness β is called

(a) monotone if $\Omega_0,\ \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.

(b) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E,\ \Omega \in \mathcal{P}(E)$.

(c) invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$, and $\Omega \in \mathcal{P}(E)$.

(d) real if $\mathcal{A} = \overline{\mathbb{R}}_+ = [0, \infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω.

(e) semiadditive if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(E)$.

(f) regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As example of an MNC, one may consider the Hausdorf MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}. \tag{4.37}$$

Recall that a bounded set $A \subset E$ has a finite ε-net if there exits a finite subset $S \subset E$ such that $A \subset S + \varepsilon\overline{B}$ where $\overline{B}$ is a closed ball in E.

Other examples are given by the following measures of noncompactness defined on the space of continuous functions $C(J, E)$ with values in a Banach space E:

(i) the modulus of fiber noncompactness

$$\varphi(\Omega) = \sup_{t \in J} \chi_E(\Omega(t)), \tag{4.38}$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) the modulus of equicontinuity

$$\operatorname{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{\|\tau_1 - \tau_2\| \le \delta} \left\| y(\tau_1) - y(\tau_2) \right\|. \tag{4.39}$$

It should be mentioned that these MNC satisfy all above-mentioned properties except regularity.

Definition 4.8. Let M be a closed subset of a Banach space E and $\beta : \mathcal{P}(E) \to (\mathcal{A}, \ge)$ an MNC on E. A multivalued map $\mathcal{F} : \mathcal{M} \to \mathcal{P}_{cp}(E)$ is said to be β-condensing if for every $\Omega \subset \mathcal{M}$, the relation

$$\beta(\Omega) \le \beta(\mathcal{F}(\Omega)), \tag{4.40}$$

implies the relative compactness of Ω.

Some important results on fixed point theory with MNCs are recalled hereafter (see, e.g., [36] for the proofs and further details). The first one is a compactness criterion.

Lemma 4.9 (see [36, Theorem 5.1.1]). *Let $N : L^1([a,b], E) \to C([a,b], E)$ be an abstract operator satisfying the following conditions:*

($\mathcal{S}_1$) *N is ξ-Lipschitz: there exists $\xi > 0$ such that for every $f, g \in L^1([a,b], E)$*

$$\left\| Nf(t) - Ng(t) \right\| \le \xi \int_a^b \left\| f(s) - g(s) \right\| ds, \quad \forall t \in [a,b]. \tag{4.41}$$

$(\mathcal{S}_2)$ *N is weakly-strongly sequentially continuous on compact subsets: for any compact $K \subset E$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [a,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $N(f_n) \to N(f_0)$ as $n \to +\infty$.*

Then for every semicompact sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(J,E)$, the image sequence $N(\{f_n\}_{n=1}^{\infty})$ is relatively compact in $C([a,b],E)$.

Lemma 4.10 (see [36, Theorem 5.2.2]). *Let an operator $N : L^1([a,b],E) \to C([a,b],E)$ satisfy conditions $(\mathcal{S}_1)$-$(\mathcal{S}_2)$ together with the following:*

$(\mathcal{S}_3)$ *there exits $\eta \in L^1([a,b])$ such that for every integrable bounded sequence $\{f_n\}_{n=1}^{\infty}$, one has*

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \eta(t), \quad \text{for a.e. } t \in [a,b], \tag{4.42}$$

where χ is the Hausdorff MNC.

Then

$$\chi\left(\{N(f_n)(t)\}_{n=1}^{\infty}\right) \leq 2\xi \int_a^b \eta(s)ds, \quad \forall t \in [a,b], \tag{4.43}$$

where ξ is the constant in $(\mathcal{S}_1)$.

The next result is concerned with the nonlinear alternative for β-condensing *u.s.c.* multivalued maps.

Lemma 4.11 (see [36]). *Let $V \subset E$ be a bounded open neighborhood of zero and $N : \overline{V} \to \mathcal{P}_{cp,cv}(E)$ a β-condensing u.s.c. multivalued map, where β is a nonsingular measure of noncompactness defined on subsets of E, satisfying the boundary condition*

$$x \notin \lambda N(x) \tag{4.44}$$

for all $x \in \partial V$ and $0 < \lambda < 1$. Then $\operatorname{Fix} N \neq \emptyset$.

Lemma 4.12 (see [36]). *Let W be a closed subset of a Banach space E and $\mathcal{F} : W \to \mathcal{P}_{cp}(E)$ is a closed β-condensing multivalued map where β is a monotone MNC on E. If the fixed point set Fix $\mathcal{F}$ is bounded, then it is compact.*

4.2.1. Main Results

In all this part, we assume that there exists $M > 0$ such that

$$\|S(t)\|_{B(E)} \leq M \quad \text{for every } t \in J. \tag{4.45}$$

Let $F : J \times E \to \mathcal{P}_{\mathrm{cp},cv}(E)$ be a Carathéodory multivalued map which satisfies Lipschitz conditions with respect to the Hausdorf MNC.

($\mathcal{B}_4$) There exists $\overline{p} \in L^1(J, \mathbb{R}_+)$ such that for every bounded D in E,

$$\chi(F(t, D)) \le \overline{p}(t)\chi(D), \tag{4.46}$$

Lemma 4.13. *Under conditions* ($\mathcal{B}_2$) *and* ($\mathcal{B}_4$), *the operator N is closed and $N(y) \in \mathcal{P}_{cp,cv}(C(J, E))$, for every $y \in C(J, E)$ where N is as defined in the proof of Theorem 4.3.*

Proof. We have the following steps.

Step 1 (N is closed). Let $h_n \to h_*$, $h_n \in N(y_n)$, and $y_n \to y_*$. We will prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that for a.e. $t \in J$

$$h_n(t) = \int_0^t S(t-s) f_n(s) ds. \tag{4.47}$$

Since $\{f_n(t) : n \in \mathbb{N}\} \subseteq F(t, y_n(t))$, Assumption ($\mathcal{B}_1$) implies that $(f_n)_{n\in\mathbb{N}}$ is integrably bounded. In addition, the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact for a.e. $t \in J$ because Assumption ($\mathcal{B}_4$) both with the convergence of $\{y_n\}_{n\in\mathbb{N}}$ imply that

$$\chi(\{f_n(t) : n \in \mathbb{N}\}) \le \chi(F(t, y_n(t))) \le \overline{p}(t)\chi(\{y_n(t) : n \in \mathbb{N}\}) = 0. \tag{4.48}$$

Hence the sequence $\{f_n : n \in \mathbb{N}\}$ is semicompact, hence weakly compact in $L^1(J; E)$ to some limit f_* by Lemma 2.15. Arguing as in the proof of Theorem 4.3 Part 2, and passing to the limit in (4.47), we obtain that $f_* \in S_{F,y_*}$ and for each $t \in J$

$$h_*(t) = \int_0^t S(t-s) f_*(s) ds. \tag{4.49}$$

As a consequence, $h_* \in N(y_*)$, as claimed.

Step 2 (N has compact, convex values). The convexity of $N(y)$ follows immediately by the convexity of the values of F. To prove the compactness of the values of F, let $N(y) \in \mathcal{P}(E)$ for some $y \in C(J, E)$ and $h_n \in N(y)$. Then there exists $f_n \in S_{F,y}$ satisfying (4.47). Arguing again as in Step 1, we prove that $\{f_n\}$ is semicompact and converges weakly to some limit $f_* \in F(t, y(t))$, a.e. $t \in J$ hence passing to the limit in (4.47), h_n tends to some limit h_* in the closed set $N(y)$ with h_* satisfying (4.49). Therefore the set $N(y)$ is sequentially compact, hence compact.

□

Lemma 4.14. *Under the conditions* ($\mathcal{B}_2$) *and* ($\mathcal{B}_4$), *the operator N is u.s.c.*

Proof. Using Lemmas 2.10 and 4.13, we only prove that N is quasicompact. Let K be a compact set in $C(J, E)$ and $h_n \in N(y_n)$ such that $y_n \in K$. Then there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = \int_0^t S(t-s) f_n(s) ds. \tag{4.50}$$

Since K is compact, we may pass to a subsequence, if necessary, to get that $\{y_n\}$ converges to some limit y_* in $C(J,E)$. Arguing as in the proof of Theorem 4.3 Step 1, we can prove the existence of a subsequence $\{f_n\}$ which converges weakly to some limit f_* and hence h_n converges to h_*, where

$$h_*(t) = \int_0^t S(t-s)f_*(s)ds. \tag{4.51}$$

As a consequence, N is *u.s.c.* □

We are now in position to prove our second existence result in the convex case.

Theorem 4.15. *Assume that F satisfies Assumptions $(\mathcal{B}_2)$ and $(\mathcal{B}_4)$. If*

$$q := 2M\int_0^b \overline{p}(s)ds < 1, \tag{4.52}$$

then the set of solutions for problem (1.1) *is nonempty and compact.*

Proof. It is clear that all solutions of problem (1.1) are fixed points of the multivalued operator N defined in Theorem 4.3. By Lemmas 4.13 and 4.14, $N(\cdot) \in \mathcal{P}_{\mathrm{cv,cp}}(C(J,E))$ and it is u.s.c. Next, we prove that N is a β-condensing operator for a suitable MNC β. Given a bounded subset $D \subset C(J,E)$, let $\mathrm{mod}_C(D)$ the modulus of quasiequicontinuity of the set of functions D denote

$$\mathrm{mod}_C(D) = \lim_{\delta\to 0}\sup_{x\in D}\max_{\|\tau_2-\tau_1\|\le\delta}\|x(\tau_1)-x(\tau_2)\|. \tag{4.53}$$

It is well known (see, e.g., [36, Example 2.1.2]) that $\mathrm{mod}_C(D)$ defines an MNC in $C(J,E)$ which satisfies all of the properties in Definition 4.7 except regularity. Given the Hausdorff MNC χ, let γ be the real MNC defined on bounded subsets on $C(J,E)$ by

$$\gamma(D) = \sup_{t\in J}\chi(D(t)). \tag{4.54}$$

Finally, define the following MNC on bounded subsets of $C(J,E)$ by

$$\beta(D) = \max_{D\in\Delta(C(J,E))}(\gamma(D), \mathrm{mod}_C(D)), \tag{4.55}$$

where $\Delta(C(J,E))$ is the collection of all countable subsets of B. Then the MNC β is monotone, regular and nonsingular (see [36, Example 2.1.4]).

To show that N is β-condensing, let $B \subset$ be a bounded set in $C(J,E)$ such that

$$\beta(B) \le \beta(N(B)). \tag{4.56}$$

We will show that B is relatively compact. Let $\{y_n : n \in \mathbb{N}\} \subset B$ and let $N = \Gamma \circ S_F$ where $S_F : C(J,E) \to L^1(J,E)$ is defined by

$$S_F(y) = S_{F,y} = \left\{v \in L^1(J,E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\right\}, \tag{4.57}$$

$\Gamma : L^1(J,E) \to C(J,E)$ is defined by

$$\Gamma(f)(t) = \int_0^t S(t-s)f(s)ds, \quad t \in J. \tag{4.58}$$

Then

$$\|\Gamma f_1(t) - \Gamma f_2(t)\| \le \int_0^t \|S(t-s)\| \cdot \|f_1(s) - f_2(s)\| ds \le M \int_0^t \|f_1(s) - f_2(s)\| ds. \tag{4.59}$$

Moreover, each element h_n in $N(y_n)$ can be represented as

$$h_n = \Gamma(f_n), \quad \text{with } f_n \in S_F(y_n). \tag{4.60}$$

Moreover (4.56) yields

$$\beta(\{h_n : n \in \mathbb{N}\}) \ge \beta(\{y_n : n \in \mathbb{N}\}). \tag{4.61}$$

From Assumption $(\mathcal{B}_4)$, it holds that for a.e. $t \in J$,

$$\begin{aligned}
\chi(\{f_n(t) : n \in \mathbb{N}\}) &\le \chi(F(t, \{y_n(t)\}_{n=1}^{\infty})) \\
&\le \overline{p}(t)\chi(\{y_n(t)\}_{n=1}^{\infty}) \\
&\le \overline{p}(t)\sup_{0\le s\le t}\chi(\{y_n(s)\}_{n=1}^{\infty}) \\
&\le \overline{p}(t)\gamma(\{y_n\}_{n=1}^{\infty}).
\end{aligned} \tag{4.62}$$

Lemmas 4.9 and 4.10 imply that

$$\chi(\{\Gamma(f_n)(t)\}_{n=1}^{\infty}) \le \gamma(\{y_n\}_{n=1}^{\infty})2M\int_0^t \overline{p}(s)ds. \tag{4.63}$$

Therefore

$$\gamma(\{y_n\}_{n=1}^{\infty}) \le \gamma(\{h_n\}_{n=1}^{\infty}) = \sup_{t\in J}\chi(\{h_n(t)\}_{n=1}^{\infty}) \le q\gamma(\{y_n\}_{n=1}^{\infty}). \tag{4.64}$$

Since $0 < q < 1$, we infer that

$$\gamma(\{y_n\}_{n=1}^{\infty}) = 0. \tag{4.65}$$

$\gamma(y_n) = 0$ implies that $\chi(\{y_n(t)\}) = 0$, for a.e. $t \in J$. In turn, (4.62) implies that

$$\chi(\{f_n(t)\}) = 0, \quad \text{for a.e. } t \in J. \tag{4.66}$$

Hence (4.60) implies that $\chi(\{h_n\}_{n=1}^{\infty}) = 0$. To show that $\text{mod}_C(B) = 0$, i.e, the set $\{h_n\}$ is equicontinuous, we proceed as in the proof of Theorem 4.3 Step 1 Part (b). It follows that $\beta(\{h_n\}_{n=1}^{\infty}) = 0$ which implies, by (4.61), that $\beta(\{y_n\}_{n=1}^{\infty}) = 0$. We have proved that B is relatively compact. Hence $N : \overline{U} \to \mathcal{P}_{\text{cp,cv}}(C(J,E))$ is u.s.c. and β-condensing, where U is as in the proof of Theorem 4.3. From the choice of U, there is no $z \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type for condensing maps (Lemma 4.11), we deduce that N has a fixed point y in U, which is a solution to problem (1.1). Finally, since $\text{Fix}\, N$ is bounded, by Lemma 4.12, $\text{Fix}\, N$ is further compact. □

4.3. The Nonconvex Case

In this section, we present a second existence result for problem (1.1) when the multivalued nonlinearity is not necessarily convex. In the proof, we will make use of the nonlinear alternative of Leray-Schauder type [44] combined with a selection theorem due to Bressan and Colombo [46] for lower semicontinuous multivalued maps with decomposable values. The main ingredients are presented hereafter. We first start with some definitions (see, e.g., [47]). Consider a topological space E and a family A of subsets of E.

Definition 4.16. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in E.

Definition 4.17. A subset $A \subset L^1(J,E)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, we have:

$$u\chi_I + v\chi_{J\setminus I} \in A, \tag{4.67}$$

where χ_A stands for the characteristic function of the set A.

Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty closed values. Assign to F the multivalued operator $\mathcal{F} : C(J,E) \to \mathcal{P}(L^1(J,E))$ defined by $\mathcal{F}(y) = S_{F,y}$. The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 4.18. Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. We say that F is of lower semicontinuous type (l.s.c. type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 4.19 (see [46, 47]). *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multivalued operator $N : X \to \mathcal{P}_{cl}(L^1(J,E))$ with closed decomposable values has a continuous selection, that is, there exists a continuous single-valued function $f : X \to L^1(J,E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis.

($\mathcal{H}_1$) $F : J \times E \to \mathcal{P}(E)$ is a nonempty compact valued multivalued map such that

(a) the mapping $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;

(b) the mapping $y \mapsto F(t, y)$ is lower semicontinuous for a.e. $t \in J$.

The following lemma is crucial in the proof of our existence theorem.

Lemma 4.20 (see, e.g., [48]). *Let $F : J \times E \to \mathcal{P}_{cp}(E)$ be an integrably bounded multivalued map satisfying ($\mathcal{H}_1$). Then F is of lower semicontinuous type.*

Theorem 4.21. *Suppose that the hypotheses ($\mathcal{B}_1$) or ($\mathcal{B}_3$)-($\mathcal{B}_2$) and ($\mathcal{H}_1$) are satisfied. Then problem* (1.1) *has at least one solution.*

Proof. ($\mathcal{H}_1$) imply, by Lemma 4.20, that F is of lower semicontinuous type. From Lemma 4.19, there is a continuous selection $f : C(J,E) \to L^1(J,E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C(J,E)$. Consider the problem

$$y(t) = \int_0^t a(t-s)\left[Ay(s) + f(y)(s)\right]ds, \quad t \in J, \tag{4.68}$$

and the operator $G : C(J,E) \to C(J,E)$ defined by

$$G(y)(t) = \int_0^t S(t-s)f(y)(s)ds, \quad \text{for } t \in J. \tag{4.69}$$

As in Theorem 4.3, we can prove that the single-valued operator G is compact and there exists $M_* > 0$ such that for all possible solutions y, we have $\|y\|_\infty < M_*$. Now, we only check that G is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$, as $n \to +\infty$. Then

$$\|G(y_n(t)) - G(y(t))\| \leq M\int_0^b \|f(y_n(s)) - f(y(s))\|ds. \tag{4.70}$$

Since the function f is continuous, we have

$$\|G(y_n) - G(y)\|_\infty \leq M\|f(y_n) - f(y)\|_{L^1} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.71}$$

Let

$$U = \{y \in C(J,E) \mid \|y\|_\infty < M_*\}. \tag{4.72}$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N y$ for in $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of the Leray-Schauder type (Lemma 4.5), we deduce that G has a fixed point $y \in U$ which is a solution of problem (4.68), hence a solution to the problem (1.1). □

4.4. A Further Result

In this part, we present a second existence result to problem (1.1) with a nonconvex valued right-hand side. First, consider the Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\} \tag{4.73}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}, \tag{4.74}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,\mathrm{cl}}(E), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [49]). In particular, H_d satisfies the triangle inequality.

Definition 4.22. A multivalued operator $N : E \to \mathcal{P}_{\mathrm{cl}}(E)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x,\ y \in E, \tag{4.75}$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is γ-Lipschitz, then for every $\gamma' > \gamma$,

$$N(x) \subset N(y) + \gamma' d(x,y) B(0,1), \quad \forall x, y \in E. \tag{4.76}$$

Our proofs are based on the following classical fixed point theorem for contraction multivalued operators proved by Covitz and Nadler in 1970 [50] (see also Deimling [33, Theorem 11.1]).

Lemma 4.23. *Let (X,d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* Fix $N \neq \emptyset$.

Let us introduce the following hypotheses:

($\mathcal{A}_1$) $F : J \times E \to \mathcal{P}_{cp}(E)$; $t \mapsto F(t,x)$ is measurable for each $x \in E$;

($\mathcal{A}_2$) there exists a function $l \in L^1(J, \mathbb{R}_+)$ such that

$$H_d(F(t,x), F(t,y)) \le l(t)\|x - y\|, \quad \text{for a.e. } t \in J \text{ and all } x, y \in E, \tag{4.77}$$

with

$$F(t,0) \subset l(t)\overline{B}(0,1), \quad \text{for a.e. } t \in J. \tag{4.78}$$

Theorem 4.24. *Let Assumptions* $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ *be satisfied. Then problem* (1.1) *has at least one solution.*

Proof. In order to transform the problem (1.1) into a fixed point problem, let the multivalued operator $N : C(J,E) \to \mathcal{P}(C(J,E))$ be as defined in Theorem 4.3. We will show that N satisfies the assumptions of Lemma 4.23.

(a) $N(y) \in \mathcal{P}_{cl}(C(J,E))$ for each $y \in C(J,E)$. Indeed, let $\{h_n : n \in \mathbb{N}\} \subset N(y)$ be a sequence converge to h. Then there exists a sequence $g_n \in S_{F,y}$ such that

$$y_n(t) = \int_0^t S(t-s)g_n(s)ds, \quad t \in J. \tag{4.79}$$

Since $F(\cdot,\cdot)$ has compact values, let $w(\cdot) \in F(\cdot,0)$ be such that $\|g(t) - w(t)\| = d(g(t), F(t,0))$. From $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we infer that for a.e. $t \in J$

$$\begin{aligned} \|g_n(t)\| &\le \|g_n(t) - w(t)\| + \|w(t)\| \\ &\le l(t)\|y\|_\infty + l(t) := \widehat{M}(t), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.80}$$

Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|g_n - g\|_{L^1} \longrightarrow 0 \quad \text{and thus } h_n(t) \longrightarrow h(t) \tag{4.81}$$

with

$$h(t) = \int_0^t S(t-s)g(s)ds, \quad t \in J, \tag{4.82}$$

proving that $h \in N(y)$.

(b) There exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \le \gamma\|y - \overline{y}\|_\infty, \quad \forall y, \overline{y} \in C(J,E). \tag{4.83}$$

Let $y, \overline{y} \in C(J,E)$ and $h \in N(y)$. Then there exists $g(t) \in F(t,y(t))$ (g is a measurable selection) such that for each $t \in J$

$$h(t) = \int_0^t S(t-s)g(s)ds. \tag{4.84}$$

$(\mathcal{A}_2)$ tells us that

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \le l(t)\|y(t) - \overline{y}(t)\|, \quad \text{a.e. } t \in J. \tag{4.85}$$

Hence there is $w \in F(t, \overline{y}(t))$ such that

$$\|g(t) - w\| \leq l(t)\|y(t) - \overline{y}(t)\|, \quad t \in J. \tag{4.86}$$

Then consider the mapping $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g(t) - w\| \leq l(t)\|y(t) - \overline{y}(t)\|\}, \qquad t \in J, \tag{4.87}$$

that is $U(t) = \overline{B}(g(t), l(t)\|y(t) - \overline{y}(t)\|)$. Since $g, l, y, \overline{y}$ are measurable, Theorem III.4.1 in [30] tells us that the closed ball U is measurable. Finally the set $V(t) = U(t) \cap F(t, \overline{y}(t))$ is nonempty since it contains w. Therefore the intersection multivalued operator V is measurable with nonempty, closed values (see [29–31]). By Lemma 2.5, there exists a function $\overline{g}(t)$, which is a measurable selection for V. Thus $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$\|g(t) - \overline{g}(t)\| \leq l(t)\|y(t) - \overline{y}(t)\|, \quad \text{for a.e. } t \in J. \tag{4.88}$$

Let us define for a.e. $t \in J$

$$\overline{h}(t) = \int_0^t S(t-s)\overline{g}(s)ds. \tag{4.89}$$

Then

$$\begin{aligned}
\left\|h(t) - \overline{h}(t)\right\| &\leq \int_0^b l(s)\|y(s) - \overline{y}(s)\|ds \\
&\leq \int_0^b l(s)e^{\tau L(s)}e^{-\tau L(s)}\|y(s) - \overline{y}(s)\|ds \\
&\leq \frac{1}{\tau}e^{\tau L(t)}\|y - \overline{y}\|_*.
\end{aligned} \tag{4.90}$$

Thus

$$\left\|h - \overline{h}\right\|_* \leq \frac{1}{\tau}\|y - \overline{y}\|_*. \tag{4.91}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at

$$H_d(N(y), N(\overline{y})) \leq \frac{1}{\tau}\|y - \overline{y}\|_*, \tag{4.92}$$

where $\tau > 1$ and

$$\|y\|_* = \sup\left\{e^{-\tau L(t)}\|y(t)\| : t \in J\right\}, \qquad L(t) = Me^{\omega b}\int_0^t l(s)ds \tag{4.93}$$

is the Bielecki-type norm on $C(J,E)$. So, N is a contraction and thus, by Lemma 4.23, N has a fixed point y, which is a mild solution to (1.1).

Arguing as in Theorem 4.3, we can also prove the following result the proof of which is omitted. □

Theorem 4.25. *Let $(E,\|\cdot\|)$ be a reflexive Banach space. Suppose that all conditions of Theorem 4.24 are satisfied and $F : J \times E \to \mathcal{P}_{cp,cv}(E)$. Then the solution set of problem* (1.1) *is nonempty and compact.*

5. Filippov's Theorem

5.1. Filippov's Theorem on a Bounded Interval

Let $x \in C(J,E)$ be a mild solution of the integral equation:

$$x(t) = \int_0^t a(t-s)\left[Ax(s) + g(s)\right]ds, \quad \text{a.e. } t \in J. \tag{5.1}$$

We will consider the following two assumptions.

($\mathcal{C}_1$) The function $F : J \times E \to \mathcal{P}_{\mathrm{cl}}(E)$ is such that

(a) for all $y \in E$, themap $t \mapsto F(t,y)$ is measurable,

(b) the map $\gamma : t \mapsto \ d(g(t), F(t,x(t)))$ is integrable.

($\mathcal{C}_2$) There exist a function $p \in L^1(J,\ \mathbb{R}^+)$ and a positive constant $\beta > 0$ such that

$$H_d(F(t,z_1), F(t,z_2)) \le p(t)\|z_1 - z_2\|, \quad \forall z_1, z_2 \in E. \tag{5.2}$$

Theorem 5.1. *Assume that the conditions* ($\mathcal{C}_1$)-($\mathcal{C}_2$). *Then, for every $\epsilon > 0$ problem* (1.1) *has at least one solution y_ϵ satisfying, for a.e. $t \in J$, the estimates*

$$\left\|y_\epsilon(t) - x(t)\right\| \le M\int_0^t \left[\gamma(u) + \epsilon\right] \exp\left(2Me^{P(t)-P(s)}\right)ds, \quad t \in J, \tag{5.3}$$

where $P(t) = \int_0^t p(s)ds$.

Proof. We construct a sequence of functions $(y_n)_{n\in\mathbb{N}}$ which will be shown to converge to some solution of problem (1.1) on the interval J, namely, to

$$y(t) \in \int_0^t a(t-s)\left[Ay(s) + F(s,y(s))\right]ds, \quad t \in J. \tag{5.4}$$

Let $f_0 = g$ on J and $y_0(t) = x(t),\ t \in [0, b)$, that is,

$$y_0(t) = \int_0^t S(t-s) f_0(s) ds, \quad t \in J, \tag{5.5}$$

Then define the multivalued map $U_1 : J \to \mathcal{P}(E)$ by $U_1(t) = F(t, y_0(t)) \cap \mathcal{B}(g(t), \gamma(t) + \epsilon)$. Since g and γ are measurable, Theorem III.4.1 in [30] tells us that the ball $\mathcal{B}(g(t), \gamma(t) + \epsilon)$ is measurable. Moreover $F(t, y_0(t))$ is measurable (see [29]) and U_1 is nonempty. Indeed, since $v = \epsilon > 0$ is a measurable function, from Lemma 2.6, there exists a function u which is a measurable selection of $F(t, y_0(t))$ and such that

$$|u(t) - g(t)| \le d(g(t), F(t, y_0(t))) + \epsilon = \gamma(t) + \epsilon. \tag{5.6}$$

Then $u \in U_1(t)$, proving our claim. We deduce that the intersection multivalued operator $U_1(t)$ is measurable (see [29–31]). By Lemma 2.7 (Kuratowski-Ryll-Nardzewski selection theorem), there exists a function $t \to f_1(t)$ which is a measurable selection for U_1. Consider

$$y_1(t) = \int_0^t S(t-s) f_1(s) ds, \quad t \in J. \tag{5.7}$$

For each $t \in J$, we have

$$\|y_1(t) - y_0(t)\| \le M_0^t \|f_0(s) - f_1(s)\| ds. \tag{5.8}$$

Hence

$$\|y_1(t) - y_0(t)\| \le M \int_0^t \gamma(s) ds + Mt\epsilon. \tag{5.9}$$

Using the fact that $F(t, y_1(t))$ is measurable, the ball $\mathcal{B}(f_1(t), p(t)\|y_1(t) - y_0(t)\|)$ is also measurable by Theorm III.4.1 in [30]. From $(\mathcal{C}_2)$ we have

$$H_d(F(t, y_1(t))), F(t, y_0(t)) \le p(t)\|y_1(t) - y_0(t)\|. \tag{5.10}$$

Hence there exist $w \in F(t, y_1(t))$ such that

$$\|f_1(t) - w\| \le p(t)\|y_1(t) - y_0(t)\|. \tag{5.11}$$

We consider the following multivalued map $U_2(t) = F(t, y_1(t)) \cap \mathcal{B}(f_1(t), p(t)\|y_1(t) - y_0(t)\|)$ is nonempty. Therefore the intersection multivalued operator U_2 is measurable with nonempty, closed values (see [29–31]). By Lemma 2.5, there exists a function $f_2(t)$, which is a measurable selection for V. Thus $f_2(t) \in F(t, y_1(t))$ and

$$\|f_1(t) - f_2(t)\| \le p(t)\|y_1(t) - y_0(t)\|. \tag{5.12}$$

Define

$$y_2(t) = \int_0^t S(t-s) f_2(s) ds, \quad t \in J. \tag{5.13}$$

Using (5.8) and (5.12), a simple integration by parts yields the following estimates, valid for every $t \in J$:

$$\begin{aligned} \|y_2(t) - y_1(t)\| &\leq \int_0^t \|S(t-s)\|_{B(E)} \|f_2(s) - f_1(s)\| ds \\ &\leq \int_0^t Mp(s)\left(M \int_0^s \gamma(u) du + M\epsilon s\right) ds \\ &\leq M^2 \int_0^t [\gamma(s) + \epsilon] e^{P(t)-P(s)} ds, \quad t \in J. \end{aligned} \tag{5.14}$$

Let $U_3(t) = F(t, y_2(t)) \cap \mathcal{B}(f_2(t), p(t)\|y_2(t) - y_1(t)\|)$. Arguing as for U_2, we can prove that U_3 is a measurable multivalued map with nonempty values; so there exists a measurable selection $f_3(t) \in U_3(t)$. This allows us to define

$$y_3(t) = \int_0^t S(t-s) f_3(s) ds, \quad t \in J. \tag{5.15}$$

For $t \in J$, we have

$$\begin{aligned} \|y_3(t) - y_2(t)\| &\leq M \int_0^t \|f_2(s) - f_3(s)\| ds \\ &\leq M \int_0^t p(s) \|y_2(s) - y_1(s)\| ds. \end{aligned} \tag{5.16}$$

Then

$$\|y_2(s) - y_3(s)\| \leq M\left(\int_0^t M^2 \int_0^s (\gamma(r) + \epsilon) e^{P(s)-P(r)} ds\, dr\right). \tag{5.17}$$

Performing an integration by parts, we obtain, since P is a nondecreasing function, the following estimates:

$$\begin{aligned}
\|y_3(t)-y_2(t)\| &\le \frac{M^3}{2}\int_0^t 2p(s)\left(\int_0^s [\gamma(u)+\epsilon]e^{P(s)-P(u)}du\right)ds \\
&\le \frac{M^3}{2}\left(\int_0^t 2p(s)ds\int_0^s [\gamma(u)+\epsilon]e^{2(P(s)-P(u))}du\right) \\
&\le \frac{M^3}{2}\left(\int_0^t \left(e^{2P(s)}\right)'ds\int_0^s [\gamma(u)+\epsilon]e^{-2P(u))}du\right) \\
&\le \frac{M^3}{2}\left(e^{2P(t)}\int_0^t [\gamma(s)+\epsilon]e^{-2P(s)}ds-\int_0^t [\gamma(s)+\epsilon]ds\right) \\
&\le \frac{M^3}{2}\left(\int_0^t [\gamma(s)+\epsilon]e^{2(P(t)-P(s))}ds\right).
\end{aligned} \tag{5.18}$$

Let $U_4(t)=F(t,y_3(t))\cap\mathcal{B}(f_3(t),p(t)\|y_3(t)-y_2(t)\|)$. Then, arguing again as for U_1,U_2,U_3, we show that U_4 is a measurable multivalued map with nonempty values and that there exists a measurable selection $f_4(t)$ in $U_4(t)$. Define

$$y_4(t)=\int_0^t S(t-s)f_4(s)ds, \quad t\in J. \tag{5.19}$$

For $t\in J$, we have

$$\begin{aligned}
\|y_4(t)-y_3(t)\| &\le M\int_0^t \|f_4(s)-f_3(s)\|\ ds \\
&\le M\int_0^t p(s)\|y_3(s)-y_2(s)\|+M\epsilon\, ds \\
&\le \frac{M^4}{2}\int_0^t p(s)\left(\int_0^s [\gamma(s)+\epsilon]e^{2(P(s)-P(u))}du\right)ds \\
&\le \frac{M^4e^{4\omega b}}{6}\int_0^t 3p(s)e^{3P(s)}ds\int_0^s [\gamma(s)+\epsilon]e^{-3P(u)}du \\
&\le \frac{M^4}{6}\left(\int_0^t [\gamma(s)+\epsilon]e^{3(P(t)-P(s))}ds\right).
\end{aligned} \tag{5.20}$$

Repeating the process for $n=0,1,2,3,\ldots$, we arrive at the following bound:

$$\|y_n(t)-y_{n-1}(t)\| \le \frac{M^n}{(n-1)!}\int_0^t [\gamma(s)+\epsilon]e^{(n-1)(P(t)-P(s))}ds, \quad t\in J. \tag{5.21}$$

By induction, suppose that (5.21) holds for some n and check (5.21) for $n+1$. Let $U_{n+1}(t) = F(t, y_n(t)) \cap \mathcal{B}(f_n, p(t)\|y_n(t) - y_{n-1}(t)\| + \epsilon)$. Since U_{n+1} is a nonempty measurable set, there exists a measurable selection $f_{n+1}(t) \in U_{n+1}(t)$, which allows us to define for $n \in \mathbb{N}$

$$y_{n+1}(t) = \int_0^t S(t-s) f_{n+1}(s) ds, \quad t \in J. \tag{5.22}$$

Therefore, for a.e. $t \in J$, we have

$$\begin{aligned}
\|y_{n+1}(t) - y_n(t)\| &\le M \int_0^t \|f_{n+1}(s) - f_n(s)\| ds \\
&\le \frac{M^{n+1}}{(n-1)!} \int_0^t p(s) ds \left(\int_0^s [\gamma(u) + \epsilon] e^{(n-1)(P(s)-P(u))} du \right) \\
&\le \frac{M^{n+1}}{n!} \int_0^t n p(s) e^{nP(s)} ds \int_0^s [\gamma(u) + \epsilon] e^{-nP(u)} du.
\end{aligned} \tag{5.23}$$

Again, an integration by parts leads to

$$\|y_{n+1}(t) - y_n(t)\| \le \frac{M^{(n+1)}}{n!} \int_0^t [\gamma(s) + \epsilon] e^{n(P(t)-P(s))} ds. \tag{5.24}$$

Consequently, (5.21) holds true for all $n \in \mathbb{N}$. We infer that $\{y_n\}$ is a Cauchy sequence in $C(J, E)$, converging uniformly to a limit function $y \in C(J, E)$. Moreover, from the definition of $\{U_n\}$, we have

$$\|f_{n+1}(t) - f_n(t)\| \le p(t) \|y_n(t) - y_{n-1}(t)\|, \quad \text{for a.e } t \in J. \tag{5.25}$$

Hence, for a.e. $t \in J$, $\{f_n(t)\}$ is also a Cauchy sequence in E and then converges almost everywhere to some measurable function $f(\cdot)$ in E. In addition, since $f_0 = g$, we have for a.e. $t \in J$

$$\begin{aligned}
\|f_n(t)\| &\le \sum_{k=1}^{n} \|f_k(t) - f_{k-1}(t)\| + \|f_0(t)\| \\
&\le \sum_{k=1}^{n} p(t) \|y_{k-1}(t) - y_{k-2}(t)\| + \gamma(t) + \|g(t)\| + \epsilon \\
&\le p(t) \sum_{k=1}^{\infty} \|y_k(t) - y_{k-1}(t)\| + \gamma(t) + \|g(t)\| + \epsilon.
\end{aligned} \tag{5.26}$$

Hence

$$\|f_n(t)\| \le M H(t) p(t) + \gamma(t) + \|g(t)\| + \epsilon, \tag{5.27}$$

where

$$H(t) := M\int_0^t [\gamma(s)+\epsilon]\exp\Big(Me^{P(t)-P(s)}\Big)ds. \tag{5.28}$$

From the Lebesgue dominated convergence theorem, we deduce that $\{f_n\}$ converges to f in $L^1(J,E)$. Passing to the limit in (5.22), we find that the function

$$y(t) = \int_0^t S(t-s)f(s)ds, \quad t \in J \tag{5.29}$$

is solution to problem (1.1) on J. Moreover, for a.e. $t \in J$, we have

$$\begin{aligned}
\|x(t)-y(t)\| &= \left\|\int_0^t S(t-s)g(s)ds - \int_0^t S(t-s)f(s)ds\right\| \\
&\le M\int_0^t \|f(s)-f_0(s)\|ds \\
&\le \int_0^t \|f(s)-f_n(s)\|ds + M\int_0^t \|f_n(s)-f_0(s)\|ds \\
&\le M\int_0^t \|f(s)-f_n(s)\|ds + M\int_0^t p(s)H(s)ds \\
&\le M\int_0^t \|f(s)-f_n(s)\|ds + M\int_0^t p(s)\left[M\int_0^s [\gamma(u)+\epsilon]\right] \exp\Big(Me^{P(s)-P(u)}\Big)ds\,du \\
&\le M\int_0^t \|f(s)-f_n(s)\|ds \\
&\quad + M\int_0^t \Big(e^{P(s)}\Big)'\left[M\int_0^s e^{-P(u)}[\gamma(u)+\epsilon]\exp\Big(Me^{P(s)-P(u)}\Big)ds\,du\right].
\end{aligned} \tag{5.30}$$

Passing to the limit as $n \to \infty$, we get

$$\|x(t)-y(t)\| \le \eta(t), \quad \text{a.e. } t \in J \tag{5.31}$$

with

$$\eta(t) := M\int_0^t [\gamma(u)+\epsilon]\exp\Big(2Me^{P(t)-P(s)}\Big)ds. \tag{5.32}$$

□

6. The Relaxed Problem

More precisely, we compare, in this section, trajectories of the following problem,

$$y(t) \in \int_0^t a(t-s)\left[Ay(s) + F(s, y(s))\right] ds, \quad \text{a.e. } t \in J, \tag{6.1}$$

and those of the convexified Volttera integral inclusion problem

$$y(t) \in \int_0^t a(t-s)\left[Ay(s) + \overline{\text{co}}\ F(s, y(s))\right] ds, \quad \text{a.e. } t \in J, \tag{6.2}$$

where $\overline{\text{co}}\ C$ refers to the closure of the convex hull of the set C. We will need the following auxiliary results in order to prove our main relaxation theorem.

Lemma 6.1 (see [29]). *Let $U : J \to \mathcal{P}_{cl}(E)$ be a measurable, integrably bounded set-valued map and let $t \mapsto d(0, U(t))$ be an integrable map. Then the integral $\int_0^b U(t)dt$ is convex, the map $t \mapsto \text{co}\, U(t)$ is measurable and, for every $\varepsilon > 0$, and every measurable selection u of $\overline{\text{co}}\ U(t)$, there exists a measurable selection $\overline{u}$ of U such that*

$$\begin{gathered}\sup_{t \in J} \left\| \int_0^t u(s)ds - \int_0^t \overline{u}(s)ds \right\| \leq \varepsilon, \\ \overline{\int_0^b \overline{\text{co}}\ U(t)dt} = \overline{\int_0^b U(t)dt} = \int_0^b \overline{\text{co}}\ U(t)dt.\end{gathered} \tag{6.3}$$

With E being a reflexive Banach space, the following hypotheses will be assumed in this section:

($\overline{\mathcal{H}_1}$) The function $F : J \times E \to \mathcal{P}_{cl}(E)$ satisfies

(a) for all $y \in E$, the map $t \mapsto F(t, y)$ is measurable,

(b) the map $t \mapsto F(t, 0)$ is integrable bounded (i.e., there exists $k \in L^1(J, \mathbb{R}_+)$ such that $F(t, 0) \subset k(t)B(0, 1)$).

Then our main contribution is the following.

Theorem 6.2. *Assume that $(\mathcal{C}_2)$ and $(\overline{\mathcal{H}_1})$ hold. Then problem* (6.2) *has at least one solution. In addition, for all $\varepsilon > 0$ and every solution x of problem* (6.2)*, has a solution y defined on J satisfying*

$$\|x - y\|_\infty \leq \varepsilon. \tag{6.4}$$

In particular $S^{\text{co}} = \overline{S}$, where

$$S^{\text{co}} = \{y : y \text{ is a solution to } (6.2)\}. \tag{6.5}$$

Remark 6.3. Notice that the multivalued map $t \mapsto \overline{\text{co}}\ F(t, \cdot)$ also satisfies $(\mathcal{H}_2)$.

Proof. Part 1 ($S^{co} \neq \emptyset$)

For this, we first transform problem (6.2) into a fixed point problem and then make use of Lemma 4.23. It is clear that all solutions of problem (6.2) are fixed points of the multivalued operator $N : C(J, E) \to \mathcal{P}(CJ, E)$ defined by

$$N(y) := \left\{ h \in C(J, E) \mid h(t) = \int_0^t S(t-s) f(s) ds,\ t \in [0, b] \right\}, \tag{6.6}$$

where

$$g \in S_{\overline{co}F,y} = \left\{ g \in L^1(J, E) : g(t) \in \overline{co}\ F(t, y(t)) \text{ for a.e. } t \in J \right\}. \tag{6.7}$$

To show that N satisfies the assumptions of Lemma 4.23, the proof will be given in two steps.

Step 1. $N(y) \in \mathcal{P}_{cl}(CJ, E)$ for each $y \in (CJ, E)$. Indeed, let $\{y_n\} \in N(y)$ be such that $y_n \to \tilde{y}$ in $C(J, E)$, as $n \to \infty$. Then $\tilde{y} \in (CJ, E)$ and there exists a sequence $g_n \in S_{\overline{co}\ F,y}$ such that

$$y_n(t) = \int_0^t S(t-s) g_n(s) ds, \quad t \in J. \tag{6.8}$$

Then $\{g_n\}$ is integrably bounded. Since $\overline{co}F(\cdot,\cdot)$ has closed values and integrable bounded, then from Corollary 4.14 for every $n \in \mathbb{N}$ we have $w_n(\cdot) \in \overline{co}F(\cdot, 0)$ such that

$$\|g_n(t) - w_n(t)\| \leq d\big(g_n(t), \overline{co}F(t, 0)\big). \tag{6.9}$$

Since $\{y_n\}_{n\in\mathbb{N}}$ is a convergent sequence the there exists $M_* > 0$ such that

$$\|y_n\|_\infty \leq M_*, \quad \forall n \in \mathbb{N}. \tag{6.10}$$

Then

$$\begin{aligned} \|g_n(t)\| &\leq \|g_n(t) - w_n(t)\| + \|w_n(t)\| \\ &\leq p(t)\|y_n\|_\infty + k(t) \\ &\leq M_* p(t) + k(t) := M(t), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{6.11}$$

that is

$$g_n(t) \in M(t)\mathcal{B}(0, 1), \quad \text{a.e. } t \in J. \tag{6.12}$$

Since $\mathcal{B}(0, 1)$ is weakly compact in the reflexive Banach space E, there exists a subsequence, still denoted $\{g_n\}$, which converges weakly to g by the Dunford-Pettis theorem. By Mazur's Lemma (Lemma 2.16), there exists a second subsequence which converges strongly to g in E, hence almost everywhere. Then the Lebesgue dominated convergence theorem implies that, as $n \longrightarrow \infty$,

$$\begin{gathered}
\|g_n - g\|_{L^1} \longrightarrow 0 \quad \text{thus } y_n(t) \longrightarrow \tilde{y}(t), \\
\tilde{y}(t) = \int_0^t S(t-s)g(s)ds, \qquad t \in J,
\end{gathered} \tag{6.13}$$

proving that $\tilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that $H_d(N(y), N(\overline{y})) \leq \gamma\|y - \overline{y}\|_{BC}$ for each $y, \overline{y} \in (CJ, E)$ where the norm $\|y - \overline{y}\|_{BC}$ will be chosen conveniently. Indeed, let $y, \overline{y} \in (CJ, E)$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in \overline{\text{co}}\ F(t, y(t))$ such that for each $t \in J$

$$h_1(t) = \int_0^t S(t-s)g_1(s)ds. \tag{6.14}$$

Since, for each $t \in J$,

$$H_d(\overline{\text{co}}\ F(t, y_t), \overline{\text{co}}\ F(t, \overline{y}_t)) \leq p(t)\|y(t) - \overline{y}(t)\|, \tag{6.15}$$

then there exists some $w(t) \in \overline{\text{co}}\ F(t, \overline{y}(t))$ such that

$$\|g_1(t) - w(t)\| \leq p(t)\|y(t) - \overline{y}(t)\|, \quad t \in J. \tag{6.16}$$

Consider the multimap $U_1 : J \longrightarrow \mathcal{P}_{\text{cl}}(E)$ defined by

$$U_1(t) = \{w \in E : \|g_1(t) - w\| \leq p(t)\|y(t) - \overline{y}(t)\|\}. \tag{6.17}$$

As in the proof of Theorem 5.1, we can show that the multivalued operator $V_1(t) = U_1(t) \cap \overline{\text{co}}\ F(t, \overline{y}(t))$ is measurable and takes nonempty values. Then there exists a function $g_2(t)$, which is a measurable selection for V_1. Thus, $g_2(t) \in \overline{\text{co}}\ F(t, \overline{y}(t))$ and

$$\|g_1(t) - g_2(t)\| \leq p(t)\|y(t) - \overline{y}(t)\|, \quad \text{for a.e. } t \in J. \tag{6.18}$$

For each $t \in J$, let

$$h_2(t) = \int_0^t S(t-s)g_2(s)ds. \tag{6.19}$$

Therefore, for each $t \in J$, we have

$$\begin{aligned} \|h_1(t) - h_2(t)\| &\leq M \int_0^t \|g_1(s) - g_2(s)\| ds \\ &\leq \int_0^t M p(s) \|y(s) - \overline{y}(s)\| ds \\ &\leq \int_0^t p(s) e^{\tau \int_0^s p(u) du} \left(\sup_{0 \leq z \leq b} e^{-\tau \int_0^z p(u) du} \|y(z) - \overline{y}(z)\| \right) ds \\ &\leq \frac{1}{\tau} \int_0^t \left(e^{\tau \int_0^s p(u) du} \right)' \|y - \overline{y}\|_* ds. \end{aligned} \tag{6.20}$$

Hence

$$\|h_1 - h_2\|_* \leq \frac{1}{\tau} \|y - \overline{y}\|_*, \tag{6.21}$$

where

$$\|y\|_* = \sup\left\{ e^{-\tau \int_0^t p(s) ds} \|y(t)\| : t \in J \right\}, \quad \tau > 1. \tag{6.22}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we find that

$$H_d(N(y), N(\overline{y})) \leq \frac{1}{\tau} \|y - \overline{y}\|_*. \tag{6.23}$$

Then N is a contraction and hence, by Lemma 4.23, N has a fixed point y, which is solution to problem (6.2).

Part 2

Let x be a solution of problem (6.2). Then, there exists $g \in S_{\overline{\text{co}}\, F,x}$ such that

$$x(t) = \int_0^t S(t-s) g(s) ds, \quad t \in J, \tag{6.24}$$

that is, x is a solution of the problem

$$x(t) = \int_0^t a(t-s) \left[Ax(s) + \overline{g}(s) \right] ds, \quad \text{a.e. } t \in J. \tag{6.25}$$

Let $\epsilon > 0$ and $\delta > 0$ be given by the relation $\delta = b\epsilon/(2\|p\|_{L^1})$. From Lemma 6.1, there exists a measurable selection f_* of $t \mapsto F(t, x(t))$ such that

$$\sup_{t\in J}\left\| \int_0^t S(t-s)g(s)ds - \int_0^t S(t-s)f_*(s)ds \right\| \leq M\delta. \tag{6.26}$$

Let

$$z(t) = \int_0^t S(t-s)f_*(s)ds, \quad t \in J. \tag{6.27}$$

Hence

$$\|x(t) - z(t)\| \leq M\delta. \tag{6.28}$$

With Assumption $(\overline{\mathcal{H}_1})$, we infer from Corollary 4.14 that there exists $u(t) \in \overline{\text{co}}F(t, z(t))$ such that

$$\|g(t) - u(t)\| \leq d\big(g(t), \overline{\text{co}}F(t, z(t))\big). \tag{6.29}$$

Then

$$\begin{aligned} \gamma(t) := d\big(g(t), F(t, x(t))\big) &\leq d\big(g(t), u\big) + H_d(\overline{\text{co}}F(t, z(t)), F(t, x(t))) \\ &\leq H_d(\overline{\text{co}}\ F(t, x(t)), \overline{\text{co}}\ F(t, z(t))) \\ &\quad + H_d(\overline{\text{co}}F(t, z(t)), \overline{\text{co}}F(t, x(t))) \\ &\leq 2p(t)\|x(t) - z(t)\| \leq 2\delta p(t). \end{aligned} \tag{6.30}$$

Since, under $(\overline{\mathcal{H}_1(a)})$ and $(\mathcal{C}_2)$, γ is measurable (see [29]), by the above inequality, we deduce that $\gamma \in L^1(J, E)$. From Theorem 5.1, problem (6.1) has a solution y which satisfies

$$\|y(t) - x(t)\| \leq \eta(t), \quad t \in J, \tag{6.31}$$

where

$$\begin{aligned} \eta(t) &= M\int_0^t [\gamma(s) + \epsilon] \exp\Big(2Me^{P(t)-P(s)}\Big)ds \\ &\leq M\big[2\delta\|p\|_{L^1} + \epsilon b\big] \exp\Big(2Me^{P(b)}\Big). \end{aligned} \tag{6.32}$$

Using the definition of δ, we obtain the upper bound

$$\|y - x\|_\infty \leq 2b\epsilon \exp\Big(2Me^{P(b)}\Big). \tag{6.33}$$

Since ε is arbitrary, $\|y - x\|_\infty \leq \varepsilon$, showing the density relation $S^{\text{co}} = \overline{S}$. □

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