Research Article

Existence of Positive Solutions to a Nonlocal Boundary Value Problem with p-Laplacian on Time Scales

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The nonlocal boundary value problem, with p-Laplacian of the form $(\Phi_p(u^\Delta))^\Delta(t) + h(t)f(t,u(t)) = 0$, $t \in [t_1, t_m)_{\mathbb{T}}$, $u^\Delta(t_1) - \sum_{j=1}^{n} \theta_j u^\Delta(\eta_j) - \sum_{i=1}^{m-2} \epsilon_i u(\xi_i) = 0$, $u^\Delta(t_m) = 0$, has been considered. Two existence criteria of at least one and three positive solutions are presented. The first one is based on the Four functionals fixed point theorem in the work of R. Avery et al. (2008), and the second one is based on the Five functionals fixed point theorem. Meanwhile an example is worked out to illustrate the main result.

1. Introduction

Due to the unification of the theory of differential and difference equations, there have been many investigations working on the existence of positive solutions to boundary value problems for dynamic equations on time scales. Also there is much attention paid to the study of multipoint boundary value problem with p-Laplacian; see [1–10].

For convenience, throughout this paper we denote $\Phi_p(s)$ as the p-Laplacian operator, that is, $\Phi_p(s) = |s|^{p-2}s, p > 1$. $(\Phi_p)^{-1} = \Phi_q$, where $1/p + 1/q = 1$.

In [11], the author discussed the positive solutions of a m-point boundary value problem for a second-order dynamic equation on a time scale

$$u^{\Delta\nabla}(t) + q(t)f(u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}},$$

$$u^\Delta(0) = \sum_{i=1}^{m-2} b_i u^\Delta(\xi_i), \qquad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

$$(1.1)$$

where $a_i, b_i \geq 0$ $(i = 1, 2, \ldots, m - 2)$, and $\xi_i \in (0, \rho(T))_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$. And he got the existence of at least two positive solutions of the above problem by means of a fixed point theorem in a cone.

Zhao and Ge [9] considered the following multi-point boundary value problem with one-dimensional p-Laplacian:

$$
\left(\Phi_p(x'(t))\right)' + f(t, x(t)) = 0, \quad 0 < t < 1,
$$

$$
x'(0) - \sum_{i=1}^{m-1} a_i x(\xi_i) = 0, \qquad x'(1) + \sum_{i=1}^{m-1} \beta_i x(\eta_i) = 0,
$$

(1.2)

where $a_i, \beta_i > 0$, $0 < \sum_{i=1}^{m-1} a_i \xi_i \leq 1$, $0 < \sum_{i=1}^{m-1} \beta_i (1 - \eta_i) \leq 1$, $i = 1, 2, \ldots, m - 1$, $0 = \xi_1 < \xi_2 < \cdots < \xi_{m-1} < \eta_1 < \eta_2 < \cdots < \eta_{m-1} = 1$. By using a fixed point theorem in a cone, they obtained the existence of at least one, two, or three positive solutions under some sufficient conditions.

Motivated by the above results, in this paper, we investigate the nonlocal boundary value problem with p-Laplacian

$$
\left(\Phi_p(u^\triangle)\right)^\triangle (t) + h(t) f(t, u(t)) = 0, \quad t \in [t_1, t_m)_{\mathbb{T}},
$$

$$
u^\triangle(t_1) - \sum_{j=1}^{n} \theta_j u^\triangle(\eta_j) - \sum_{i=1}^{m-2} \epsilon_i u(\xi_i) = 0, \quad u^\triangle(t_m) = 0,
$$

(1.3)

where $0 \leq t_1 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < t_m$ and $t_1 < \eta_1 < \eta_2 < \cdots < \eta_n < t_m < +\infty$.

For convenience, we list the following hypotheses:

(H1) $\epsilon_i > 0$, $i = 1, 2, \ldots, m - 2$, $\theta_j \geq 0$, $j = 1, 2, \ldots, n$, $\sum_{i=1}^{m-2} \epsilon_i \xi_i + \sum_{j=1}^{n} \theta_j < 1$;

(H2) $f(t, u) \in C([t_1, t_m]_{\mathbb{T}} \times [0, +\infty), [0, +\infty))$ and f is not identically zero on any compact subinterval of $[t_1, t_m]_{\mathbb{T}} \times [0, +\infty)$;

(H3) $h(t) \in C_{rd}([t_1, t_m]_{\mathbb{T}}, [0, +\infty))$ and h is not identically zero on any compact subinterval of $[t_1, t_m]_{\mathbb{T}}$, also it satisfies

$$
\Phi_q\left(\int_{t_1}^{t_m} h(\tau) \triangle \tau\right) < +\infty, \quad \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_{s}^{t_m} h(\tau) \triangle \tau\right) \triangle s < +\infty.
$$

(1.4)

By using the Four functionals fixed point theorem and Five functionals fixed point theorem, we obtain the existence criteria of at least one positive solution and three positive solutions for the BVP (1.3). As an application, an example is worked out finally. The remainder of this paper is organized as follows. Section 2 is devoted to some preliminary discussions. We give and prove our main results in Section 3.

2. Preliminaries

The basic definitions and notations on time scales can be found in [12, 13]. In the following, we will provide some background materials on the theory of cones in Banach spaces. For more details, please refer to [14, 15].

Definition 2.1. Let E be a Banach space. A nonempty, closed set $P \subset E$ is said to be a cone provided that the following hypotheses are satisfied:

(1) if $x, y \in P$, $\alpha, \beta \geq 0$, then $\alpha x + \beta y \in P$;

(2) if $x \in P$, $x \neq \theta$, then $x \in P$.

Every cone $P \subset E$ induces a partial ordering "$\leq$" on E defined by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and $\alpha(tx+(1-t)y) \geq t\alpha(x)+(1-t)\alpha(y)$ for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say that the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and $\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Let α and Ψ be nonnegative continuous concave functionals on P, and let β and θ be nonnegative continuous convex functionals on P; then for positive numbers r, ι, υ, and R, define the sets

$$Q(\alpha, \beta, r, R) = \{x \in P : r \leq \alpha(x), \ \beta(x) \leq R\},$$
$$U(\Psi, \iota) = \{x \in Q(\alpha, \beta, r, R) : \iota \leq \Psi(x)\}, \tag{2.1}$$
$$V(\theta, \upsilon) = \{x \in Q(\alpha, \beta, r, R) : \theta(x) \leq \upsilon\}.$$

The following lemma can be found in [16].

Lemma 2.3 (four functionals fixed point theorem). *If P is a cone in a real Banach space E, α and Ψ are nonnegative continuous concave functionals on P, β, and θ are nonnegative continuous convex functionals on P, and there exist nonnegative positive numbers r, ι, υ, and R, such that*

$$A : Q(\alpha, \beta, r, R) \longrightarrow P \tag{2.2}$$

is a completely continuous operator, and $Q(\alpha, \beta, r, R)$ is a bounded set. If

(i) $\{x \in U(\Psi, \iota) : \beta(x) < R\} \cap \{x \in V(\theta, \upsilon) : r < \alpha(x)\} \neq \emptyset$,

(ii) $\alpha(Ax) \geq r$, *for all $x \in Q(\alpha, \beta, r, R)$, with $\alpha(x) = r$ and $\upsilon < \theta(Ax)$,*

(iii) $\alpha(Ax) \geq r$, *for all $x \in V(\theta, \upsilon)$, with $\alpha(x) = r$,*

(iv) $\beta(Ax) \leq R$, *for all $x \in Q(\alpha, \beta, r, R)$, with $\beta(x) = R$ and $\Psi(Ax) < \iota$,*

(v) $\beta(Ax) \leq R$, *for all $x \in U(\Psi, \iota)$, with $\beta(x) = R$,*

then A has a fixed point x in $Q(\alpha, \beta, r, R)$.

We are now in a position to present the Five functionals fixed point theorem (see [17]). Let γ, β, θ be nonnegative continuous convex functionals on P and α, φ nonnegative continuous concave functionals on P. For nonnegative numbers h, a, b, d, and c, define the following convex sets:

$$P(\gamma, c) = \{x \in P : \gamma(x) < c\},$$

$$P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}, \tag{2.3}$$

$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : a \leq \alpha(x), \ \theta(x) \leq b, \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, \varphi, h, d, c) = \{x \in P : h \leq \varphi(x), \ \beta(x) \leq d, \gamma(x) \leq c\}.$$

Lemma 2.4 (five functionals fixed point theorem). *Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers c and M, nonnegative continuous concave functionals α and φ on P, and nonnegative continuous convex functionals γ, β, and θ on P, with*

$$\alpha(x) \leq \beta(x), \quad \|x\| \leq M\gamma(x) \quad \forall x \in \overline{P(\gamma, c)}. \tag{2.4}$$

Suppose that $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(A(x)) > b$ for $x \in P(\gamma, \theta, \alpha, b, k, c)$,

(ii) $\{x \in Q(\gamma, \beta, \varphi, h, a, c) : \beta(x) < a\} \neq \emptyset$ and $\beta(A(x)) < a$ for $x \in Q(\gamma, \beta, \varphi, h, a, c)$,

(iii) $\alpha(A(x)) > b$ for $x \in P(\gamma, \alpha, b, c)$ with $\theta(A(x)) > k$,

(iv) $\beta(A(x)) < a$ for $x \in Q(\gamma, \beta, a, c)$ with $\varphi(A(x)) < h$,

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < a, \quad \alpha(x_2) > b, \quad \beta(x_3) > a \quad \text{with } \alpha(x_3) < b. \tag{2.5}$$

Consider the Banach space $E = C[t_1, \sigma(t_m)]_{\mathbb{T}}$ equipped with the norm $\|u\| = \max_{t \in [t_1, \sigma(t_m)]_{\mathbb{T}}} |u(t)|$. Suppose $\varrho, \eta \in \mathbb{T}$ with $t_1 < \varrho < \eta < \sigma(t_m)$. For the sake of convenience, we take the notations

$$\varpi = \sum_{i=1}^{m-2} \epsilon_i, \qquad h_0 = \Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right), \qquad M_0 = \int_{t_1}^{\varrho} \Phi_q\left(\int_{\varrho}^{\eta} h(\tau)\triangle\tau\right)\triangle s,$$

$$M_{\varrho} = \int_{t_1}^{\varrho} \Phi_q\left(\int_{s}^{t_m} h(\tau)\triangle\tau\right)\triangle s, \tag{2.6}$$

$$M_{\eta} = \int_{t_1}^{\eta} \Phi_q\left(\int_{s}^{t_m} h(\tau)\triangle\tau\right)\triangle s, \qquad M_{\sigma(t_m)} = \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_{s}^{t_m} h(\tau)\triangle\tau\right)\triangle s.$$

Define a cone

$$P = \left\{ u \in E : u(t) \geq 0, \ u^{\triangle}(t_1) - \sum_{j=1}^{n} \theta_j u^{\triangle}(\eta_j) - \sum_{i=1}^{m-2} \epsilon_i u(\xi_i) = 0, \right.$$

$$\left. \text{for } t \in [t_1, \sigma(t_m)]_{\mathbb{T}} \text{ and } u^{\triangle\triangle}(t) \leq 0, \ u^{\triangle}(t) \geq 0 \text{ for } t \in [t_1, t_m)_{\mathbb{T}}, \ u^{\triangle}(t_m) = 0 \right\} \tag{2.7}$$

and an operator $A : P \to E$ by

$$Au = \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)}{\varpi} - \frac{\sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s}{\varpi}$$

$$- \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)}{\varpi} + \int_{t_1}^{t} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s. \tag{2.8}$$

Lemma 2.5. $A : P \to P$.

Proof. For $u \in P, t \in [t_1, \sigma(t_m)]_{\mathbb{T}}$,

$$Au(t) \geq \frac{1 - \sum_{j=1}^{n} \theta_j - \sum_{i=1}^{m-2} \epsilon_i \xi_i}{\varpi} \Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)$$

$$+ \int_{t_1}^{t} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s \tag{2.9}$$

$$\geq 0.$$

From the definition of A, it is clear that

$$(Au)^{\triangle}(t) = \Phi_q\left(\int_{t}^{t_m} h(s)f(s, u(s))\triangle s\right) \geq 0, \quad t \in [t_1, t_m]_{\mathbb{T}}, \tag{2.10}$$

is continuous, $(Au)^{\triangle}(t_1) - \sum_{j=1}^{n} \theta_j(Au)^{\triangle}(\eta_j) - \sum_{i=1}^{m-2} \epsilon_i(Au)(\xi_i) = 0$, and $(Au)(\sigma(t_m))$ is the maximum value of $Au(t)$ on $[t_1, \sigma(t_m)]_{\mathbb{T}}$.

Let $g(t) = \int_{t}^{t_m} h(s)f(s, u(s))\triangle s$, then $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $[t_1, t_m]_{\mathbb{T}^k}$, and $\Phi_q : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Moreover $\Phi_q(s)$ is monotonically increasing for $s \geq 0$ and $g^{\triangle}(t) = -h(t)f(t, u(t)) \leq 0$. Then by the chain rule [12, Theorem 1.87, page 31], we obtain

$$(Au)^{\triangle\triangle}(t) = \Phi_q'(g(c))g^{\triangle}(t) \leq 0, \tag{2.11}$$

where c is in the interval $[t, \sigma(t)]$. So, $A : P \to P$. This completes the proof. $\square$

3. Main Results and an Example

Theorem 3.1. *Assume that (H1), (H2), and (H3) hold, if there exist constants r, ι, v, R with $R > \max\{(h_0 + \varpi M_{\sigma(t_m)})/\varpi M_0, (\sigma(t_m) - t_1)/(\varrho - t_1)\}\iota$, $v \geq \max\{((h_0 + \varpi M_\eta)/\varpi M_\varrho)\iota, ((\eta - t_1)/(\varrho - t_1))r\}$, $r < \iota$ and suppose that $f(t, u)$ satisfies the following conditions:*

(A1) $f(t, u) \leq \Phi_p(R\varpi/(h_0 + \varpi M_{\sigma(t_m)}))$ *for all* $(t, u) \in [t_1, t_m]_{\mathbb{T}} \times [0, R]$,

(A2) $f(t, u) \geq \Phi_p(r/M_0)$ *for all* $(t, u) \in [\varrho, \eta]_{\mathbb{T}} \times [r, v]$,

then the BVP (1.3) *has a fixed point* $u \in P$ *such that*

$$\min_{t \in [\varrho, \eta]_{\mathbb{T}}} u(t) \geq r, \qquad \max_{t \in [t_1, \sigma(t_m)]_{\mathbb{T}}} u(t) \leq R. \tag{3.1}$$

Define maps

$$\alpha(u) = \Psi(u) = \min_{t \in [\varrho, \eta]_{\mathbb{T}}} u(t), \qquad \theta(u) = \max_{t \in [\varrho, \eta]_{\mathbb{T}}} u(t), \qquad \beta(u) = \max_{t \in [t_1, \sigma(t_m)]_{\mathbb{T}}} u(t), \tag{3.2}$$

and let $Q(\alpha, \beta, r, R)$, $U(\Psi, \iota)$ and $V(\theta, v)$ be defined by (2.1).

In order to complete the proof of Theorem 3.1, we first need to prove the following lemma.

Lemma 3.2. $Q(\alpha, \beta, r, R)$ *is bounded and* $A : Q(\alpha, \beta, r, R) \rightarrow P$ *is completely continuous.*

Proof. For all $u \in Q(\alpha, \beta, r, R)$, $\|u\| = \max_{t \in [t_1, \sigma(t_m)]_{\mathbb{T}}} |u(t)| = \beta(u) \leq R$, which means that $Q(\alpha, \beta, r, R)$ is a bounded set.

According to Lemma 2.5, it is clear that $A : Q(\alpha, \beta, r, R) \rightarrow P$.

In view of the continuity of f, there exists a constant $C > 0$ such that $f(t, u) < \Phi_p(C)$, for all $t \in [t_1, \sigma(t_m)]_{\mathbb{T}}$, $u \in Q(\alpha, \beta, r, R)$. Consider

$$\|Au\| = Au(\sigma(t_m)) \leq \left(\frac{h_0}{\varpi} + M_{\sigma(t_m)}\right)C, \tag{3.3}$$

which means that $AQ(\alpha, \beta, r, R)$ is uniformly bounded.

In addition, for all $t_1 \leq \bar{t} \leq t^* \leq \sigma(t_m)$, we have

$$\left|Au(\bar{t}) - Au(t^*)\right| = \left|\int_{\bar{t}}^{t^*} \Phi_q\left(\int_s^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s\right| \leq Ch_0\left|\bar{t} - t^*\right|. \tag{3.4}$$

Applying the Arzelà-Ascoli theorem on time scales [18], one can show that $AQ(\alpha, \beta, r, R)$ is relatively compact.

Now we prove that $A : Q(\alpha, \beta, r, R) \rightarrow P$ is continuous. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $Q(\alpha, \beta, r, R)$ which converges to $u_0 \in Q(\alpha, \beta, r, R)$ uniformly on $[t_1, \sigma(t_m)]_{\mathbb{T}}$. Because

$AQ(\alpha, \beta, r, R)$ is relatively compact, the sequence $\{Au_n\}$ admits a subsequence $\{Au_{n_m}\}$ converging to $v(t)$ uniformly on $[t_1, \sigma(t_m)]_{\mathbb{T}}$. In addition,

$$0 \leq Au_n(t) \leq C\left(\frac{h_0}{\omega} + M_{\sigma(t_m)}\right). \tag{3.5}$$

Observe that

$$Au_n(t) = \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u_n(\tau)) \triangle \tau\right)}{\varpi} - \frac{\sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u_n(\tau)) \triangle \tau\right) \triangle s}{\varpi}$$
$$- \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau) f(\tau, u_n(\tau)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{t} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u_n(\tau)) \triangle \tau\right) \triangle s. \tag{3.6}$$

Hence, by the Lebesgue's dominated convergence theorem on time scales [19], insert u_{n_m} into the above equality and then let $m \to \infty$, we obtain

$$v(t) = \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u_0(\tau)) \triangle \tau\right)}{\varpi} - \frac{\sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u_0(\tau)) \triangle \tau\right) \triangle s}{\varpi}$$
$$- \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau) f(\tau, u_0(\tau)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{t} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u_0(\tau)) \triangle \tau\right) \triangle s. \tag{3.7}$$

From the definition of A, we know that $v(t) = Au_0(t)$ on $[t_1, \sigma(t_m)]_{\mathbb{T}}$. This shows that each subsequence of $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au_0(t)$. Therefore the sequence $\{Au_n(t)\}_{n=1}^{\infty}$ uniformly converges to $Au_0(t)$. This means that A is continuous at $u_0 \in Q(\alpha, \beta, r, R)$. So, A is continuous on $Q(\alpha, \beta, r, R)$ since u_0 is arbitrary. Thus, $A : Q(\alpha, \beta, r, R) \to P$ is completely continuous. This completes the proof. $\square$

Proof of Theorem 3.1. Let

$$u_0 = \frac{\iota}{M_Q}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) \triangle \tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau) \triangle \tau\right) \triangle s}{\varpi}\right)$$
$$- \frac{\iota}{M_Q}\left(\frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau) \triangle \tau\right)}{\varpi}\right) + \frac{\iota}{M_Q} \int_{t_1}^{t} \Phi_q\left(\int_s^{t_m} h(\tau) \triangle \tau\right) \triangle s. \tag{3.8}$$

Clearly, $u_0 \in P$. By direct calculation,

$$
\begin{aligned}
\Psi(u_0) &= \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi} \right) \\
&\quad - \frac{\iota}{M_\varrho} \left(\frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} \right) + \frac{\iota}{M_\varrho} \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
&\geq \frac{\iota}{M_\varrho} \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s = \iota, \\
\beta(u_0) &= \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi} \right) \\
&\quad - \frac{\iota}{M_\varrho} \left(\frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} \right) + \frac{\iota}{M_\varrho} \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
&< \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\Delta\tau\right)}{\varpi} + \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \right) \leq R, \\
\theta(u_0) &= \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi} \right) \\
&\quad - \frac{\iota}{M_\varrho} \left(\frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} \right) + \frac{\iota}{M_\varrho} \int_{t_1}^{\eta} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
&\leq \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} + \int_{t_1}^{\eta} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \right) \leq \upsilon, \\
\alpha(u_0) &= \frac{\iota}{M_\varrho} \left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi} \right) \\
&\quad - \frac{\iota}{M_\varrho} \left(\frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} \right) + \frac{\iota}{M_\varrho} \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
&> \frac{\iota}{M_\varrho} \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s > r.
\end{aligned}
$$

(3.9)

So, $u_0 \in \{u \in U(\Psi, \iota) : \beta(u) < R\} \cap \{u \in V(\theta, \upsilon) : r < \alpha(u)\}$, which means that (i) in Lemma 2.3 is satisfied.

For all $u \in Q(\alpha, \beta, r, R)$, with $\alpha(u) = r$ and $\upsilon < \theta(Au)$, we have $\alpha(Au) = Au(\varrho) \geq ((\varrho - t_1)/(\eta - t_1))Au(\eta) = ((\varrho - t_1)/(\eta - t_1))\theta(Au) > ((\varrho - t_1)/(\eta - t_1))\upsilon > r$, and for all $u \in Q(\alpha, \beta, r, R)$, with $\beta(u) = R$ and $\Psi(Au) < \iota$, we obtain that $\beta(Au) = Au(\sigma(t_m)) \leq ((\sigma(t_m) - t_1)/(\varrho - t_1))Au(\varrho) = ((\sigma(t_m) - t_1)/(\varrho - t_1))\Psi(Au) < ((\sigma(t_m) - t_1)/(\varrho - t_1))\iota < R$. Hence, (ii) and (iv) in Lemma 2.3 are fulfilled.

For any $u \in V(\theta, \upsilon)$, with $\alpha(u) = r$,

$$
\begin{aligned}
\alpha(Au) &= \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s}{\varpi} \\
&\quad - \frac{\sum_{j=1}^{n-2}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)}{\varpi} + \int_{t_1}^{\varrho}\Phi_q\left(\int_s^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s \qquad (3.10) \\
&\geq \int_{t_1}^{\varrho}\Phi_q\left(\int_{\varrho}^{\eta} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s \geq \int_{t_1}^{\varrho}\Phi_q\left(\int_{\varrho}^{\eta} h(\tau)\Phi_p\left(\frac{r}{M_0}\right)\triangle\tau\right)\triangle s = r,
\end{aligned}
$$

and for all $u \in U(\Psi, \iota)$, with $\beta(u) = R$,

$$
\begin{aligned}
\beta(Au) &= \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s}{\varpi} \\
&\quad - \frac{\sum_{j=1}^{n-2}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)}{\varpi} + \int_{t_1}^{\eta}\Phi_q\left(\int_s^{t_m} h(\tau)f(\tau, u(\tau))\triangle\tau\right)\triangle s \\
&\leq \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\Phi_p\left(R\varpi/(h_0 + \varpi M_{\sigma(t_m)})\right)\triangle\tau\right)}{\varpi} \qquad (3.11) \\
&\quad + \int_{t_1}^{\sigma(t_m)}\Phi_q\left(\int_s^{t_m} h(\tau)\Phi_p\left(\frac{R\varpi}{h_0 + \varpi M_{\sigma(t_m)}}\right)\triangle\tau\right)\triangle s \\
&= R.
\end{aligned}
$$

Thus (iii) and (v) in Lemma 2.3 hold true. So, by Lemma 2.3, the BVP (1.3) has a fixed point u in $Q(\alpha, \beta, r, R)$. This completes the proof. $\qquad\square$

Theorem 3.3. *Assume that (H1), (H2), and (H3) hold. If there exist constants h, a, b, c, k, with $b < a\varpi M_0/(h_0 + \varpi M_\eta)$, $c > ((h_0 + \varpi M_{\sigma(t_m)})/(h_0 + \varpi M_\eta))a$, $k < ((h_0 + \varpi M_\eta)/(h_0 + \varpi M_{\sigma(t_m)}))c$, $k \geq \max\{(\eta - t_1)/(\varrho - t_1), (h_0 + \varpi M_\eta)/\varpi M_0\}b$, $a \geq \max\{(\eta - t_1)/(\varrho - t_1), (h_0 + \varpi M_\eta)/\varpi M_\varrho\}h$, further suppose that $f(t, u)$ satisfies the following conditions:*

(B1) $f(t, u) < \Phi_p(a\varpi/(h_0 + \varpi M_\eta))$ *for all* $(t, u) \in [t_1, t_m]_{\mathbb{T}} \times [0, c]$,

(B2) $f(t, u) \geq \Phi_p(b/M_0)$ *for all* $(t, u) \in [\varrho, \eta]_{\mathbb{T}} \times [b, k]$,

then the BVP (1.3) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{t\in[\varrho,\eta]_{\mathbb{T}}} u_1(t) < a < \max_{t\in[\varrho,\eta]_{\mathbb{T}}} u_3(t), \qquad \min_{t\in[\varrho,\eta]_{\mathbb{T}}} u_3(t) < b < \min_{t\in[\varrho,\eta]_{\mathbb{T}}} u_2(t). \tag{3.12}$$

Proof. Define these maps

$$\alpha(u) = \varphi(u) = \min_{t\in[\varrho,\eta]} u(t), \qquad \beta(u) = \theta(u) = \max_{t\in[\varrho,\eta]} u(t), \qquad \gamma(u) = \max_{t\in[t_1,\sigma(t_m)]} u(t), \tag{3.13}$$

and let $P(\gamma,c), P(\gamma,\alpha,b,c), Q(\gamma,\beta,a,c), P(\gamma,\theta,\alpha,b,k,c)$ and $Q(\gamma,\beta,\varphi,h,a,c)$ be defined by (2.3). It is clear that

$$\alpha(u) \leq \beta(u), \quad \|u\| \leq \gamma(u), \quad \forall u \in \overline{P(\gamma,c)}. \tag{3.14}$$

Using similar methods as those in Lemma 3.2, we obtain that $A : \overline{P(\gamma,c)} \to P$ is completely continuous. Thus, we only need to show that $A : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$. Let $u \in \overline{P(\gamma,c)}$, then

$$
\begin{aligned}
\gamma(Au) &= \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)}{\varpi} - \frac{\sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)\triangle s}{\varpi} \\
&\quad - \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)}{\varpi} + \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)\triangle s \\
&\leq \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)}{\varpi} + \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_{s}^{t_m} h(\tau)f(\tau,u(\tau))\triangle\tau\right)\triangle s \\
&\leq \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\Phi_p(a\varpi/(h_0+\varpi M_\eta))\triangle\tau\right)}{\varpi} \\
&\quad + \int_{t_1}^{\sigma(t_m)} \Phi_q\left(\int_{s}^{t_m} h(\tau)\Phi_p\left(\frac{a\varpi}{h_0+\varpi M_\eta}\right)\triangle\tau\right)\triangle s \\
&\leq c,
\end{aligned}
\tag{3.15}
$$

which implies that $A(\overline{P(\gamma,c)}) \subset \overline{P(\gamma,c)}$.

Let $N = h_0/\varpi + M_\eta$ and

$$
\begin{aligned}
u_1 = {} & \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} e_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right) \\
& - \frac{b}{M_\varrho}\left(\frac{\sum_{j=1}^n \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{b}{M_\varrho}\int_{t_1}^t \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s,
\end{aligned}
$$

$$
\begin{aligned}
u_2 = {} & \frac{a}{N}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} e_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right) \\
& - \frac{a}{N}\left(\frac{\sum_{j=1}^n \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{a}{N}\int_{t_1}^t \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s,
\end{aligned}
$$

(3.16)

we can verify that $u_1, u_2 \in P$. By calculation,

$$
\begin{aligned}
\alpha(u_1) = {} & \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} e_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right) \\
& - \frac{b}{M_\varrho}\left(\frac{\sum_{j=1}^n \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{b}{M_\varrho}\int_{t_1}^\varrho \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
\geq {} & \frac{b}{M_\varrho}\left(\frac{1 - \sum_{j=1}^n \theta_j - \sum_{i=1}^{m-2} e_i\xi_i}{\varpi}\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) + \int_{t_1}^\varrho \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s\right) \\
> {} & \frac{b}{M_\varrho}\int_{t_1}^\varrho \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s = b,
\end{aligned}
$$

$$
\begin{aligned}
\theta(u_1) = {} & \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2} e_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right) \\
& - \frac{b}{M_\varrho}\left(\frac{\sum_{j=1}^n \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{b}{M_\varrho}\int_{t_1}^\eta \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s \\
\leq {} & \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau)\triangle\tau\right)}{\varpi} + \int_{t_1}^\eta \Phi_q\left(\int_s^{t_m} h(\tau)\triangle\tau\right)\triangle s\right) \leq k,
\end{aligned}
$$

$$\gamma(u_1) = \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right)$$

$$- \frac{b}{M_\varrho}\left(\frac{\sum_{j=1}^{n}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m}h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{b}{M_\varrho}\int_{t_1}^{\sigma(t_m)}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s$$

$$\leq \frac{b}{M_\varrho}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right)}{\varpi} + \int_{t_1}^{\sigma(t_m)}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s\right) \leq c,$$

$$\beta(u_2) = \frac{a}{N}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right)$$

$$- \frac{a}{N}\left(\frac{\sum_{j=1}^{n}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m}h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{a}{N}\int_{t_1}^{\eta}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s$$

$$< \frac{a}{N}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right)}{\varpi} + \int_{t_1}^{\eta}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s\right) = a,$$

$$\varphi(u_2) = \frac{a}{N}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right)$$

$$- \frac{a}{N}\left(\frac{\sum_{j=1}^{n}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m}h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{a}{N}\int_{t_1}^{\varrho}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s$$

$$\geq \frac{a}{N}\left(\frac{1 - \sum_{j=1}^{n}\theta_j - \sum_{i=1}^{m-2}\epsilon_i\xi_i}{\varpi}\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right) + \int_{t_1}^{\varrho}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s\right)$$

$$> \frac{a}{N}\int_{t_1}^{\varrho}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s > h,$$

$$\gamma(u_2) = \frac{a}{N}\left(\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right) - \sum_{i=1}^{m-2}\epsilon_i\int_{t_1}^{\xi_i}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s}{\varpi}\right)$$

$$- \frac{a}{N}\left(\frac{\sum_{j=1}^{n}\theta_j\Phi_q\left(\int_{\eta_j}^{t_m}h(\tau)\triangle\tau\right)}{\varpi}\right) + \frac{a}{N}\int_{t_1}^{\sigma(t_m)}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s$$

$$\leq \frac{a}{N}\left[\frac{\Phi_q\left(\int_{t_1}^{t_m}h(\tau)\triangle\tau\right)}{\varpi} + \int_{t_1}^{\sigma(t_m)}\Phi_q\left(\int_s^{t_m}h(\tau)\triangle\tau\right)\triangle s\right] < c.$$

$$(3.17)$$

So, $u_1 \in P(\gamma, \theta, \alpha, b, k, c)$, $\alpha(u_1) > b$, $u_2 \in Q(\gamma, \beta, \varphi, h, a, c)$, $\beta(u_2) < a$, which means that $\{u \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(u) > b\}$ and $\{u \in Q(\gamma, \beta, \varphi, h, a, c) : \beta(u) < a\}$ are not empty.

For $u \in P(\gamma, \theta, \alpha, b, k, c)$,

$$
\begin{aligned}
\alpha(Au) &= \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s}{\varpi} \\
&\quad - \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s \\
&\geq \frac{1 - \sum_{j=1}^{n} \theta_j - \sum_{i=1}^{m-2} \epsilon_i \xi_i}{\varpi} \Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \\
&\quad + \int_{t_1}^{\varrho} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s \\
&> \int_{t_1}^{\varrho} \Phi_q\left(\int_{\varrho}^{\eta} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s \\
&\geq \int_{t_1}^{\varrho} \Phi_q\left(\int_{\varrho}^{\eta} h(\tau) \Phi_p\left(\frac{b}{M_0}\right) \triangle \tau\right) \triangle s = b,
\end{aligned}
\tag{3.18}
$$

and for $u \in Q(\gamma, \beta, \varphi, h, a, c)$,

$$
\begin{aligned}
\beta(Au) &= \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) - \sum_{i=1}^{m-2} \epsilon_i \int_{t_1}^{\xi_i} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s}{\varpi} \\
&\quad - \frac{\sum_{j=1}^{n} \theta_j \Phi_q\left(\int_{\eta_j}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{\eta} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s \\
&\leq \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{\eta} \Phi_q\left(\int_s^{t_m} h(\tau) f(\tau, u(\tau)) \triangle \tau\right) \triangle s \\
&< \frac{\Phi_q\left(\int_{t_1}^{t_m} h(\tau) \Phi_p(a\varpi/(h_0 + \varpi M_\eta)) \triangle \tau\right)}{\varpi} + \int_{t_1}^{\eta} \Phi_q\left(\int_s^{t_m} h(\tau) \Phi_p\left(\frac{a\varpi}{h_0 + \varpi M_\eta}\right) \triangle \tau\right) \triangle s = a.
\end{aligned}
\tag{3.19}
$$

Thus (i) and (ii) in Lemma 2.4 hold.

On the other hand, for $u \in P(\gamma, \alpha, b, c)$ with $\theta(Au) > k$, we have $\alpha(Au) = Au(\varrho) \geq ((\varrho - t_1)/(\eta - t_1))(Au(\eta)) = ((\varrho - t_1)/(\eta - t_1))\theta(Au) > ((\varrho - t_1)/(\eta - t_1))k > b$. And for $u \in P(\gamma, \beta, a, c)$ with $\varphi(Au) < h$, we can obtain $\beta(Au) = Au(\eta) \leq ((\eta - t_1)/(\varrho - t_1))Au(\varrho) = ((\eta - t_1)/(\varrho - t_1))\varphi(Au) < ((\eta - t_1)/(\varrho - t_1))h < a$. Thus, (iii) and (iv) in Lemma 2.4 hold.

So, by Lemma 2.4, we obtain that the BVP (1.3) has at least three positive solutions $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$\max_{t \in [\varrho, \eta]_{\mathbb{T}}} u_1(t) < a < \max_{t \in [\varrho, \eta]_{\mathbb{T}}} u_3(t), \qquad \min_{t \in [\varrho, \eta]_{\mathbb{T}}} u_3(t) < b < \min_{t \in [\varrho, \eta]_{\mathbb{T}}} u_2(t). \qquad (3.20)$$

This completes the proof.   □

Remark 3.4. Let $R = c$, $r = b$, $\upsilon = k$, we can find that the conditions of Theorem 3.1 are contained in Theorem 3.3.

Example 3.5. Let $\mathbb{T} = \{0.1, 0.18\} \cup [0.2, 1] \cup \{1.2\} \cup [1.5, 2]$, $p = 2$, consider the following eight-point BVP:

$$\left(\Phi_p(u^\Delta)\right)^\Delta(t) + h(t)f(t, u(t)) = 0, \quad t \in (0.1, 2)_{\mathbb{T}},$$

$$u^\Delta(0.1) - \sum_{j=1}^{3} \theta_j u^\Delta(\eta_j) - \sum_{i=1}^{3} \epsilon_i u(\xi_i) = 0, \quad u^\Delta(2) = 0, \qquad (3.21)$$

where $h(t) = t + \sigma(t)$, $\theta_1 = 1/12$, $\theta_2 = 1/7$, $\theta_3 = 1/42$, $\epsilon_1 = 1/6$, $\epsilon_2 = 1/24$, $\epsilon_3 = 1/8$, $\xi_1 = 0.33$, $\xi_2 = 0.45$, $\xi_3 = 1.65$, $\eta_1 = 0.88$, $\eta_2 = 1.86$, $\eta_3 = 1.95$, for all $t \in \mathbb{T}$, and

$$f(t, u) = \begin{cases} 0.00015, & [0.4, 1.8]_{\mathbb{T}} \times [0.0001, 0.055], \\ g(u), & \text{other,} \end{cases} \qquad (3.22)$$

where $g(u)$ is continuous, $0 \le g(u) \le 0.0026$, and $g(0.0001) = g(0.055) = 0.00015$.

Set $\varrho = 0.4$, $\eta = 1.8$, by calculation,

$$\sum_{j=1}^{3} \theta_j = \frac{1}{4}, \qquad \sum_{i=1}^{3} \epsilon_i = \frac{1}{3}, \qquad h_0 = 3.99, \qquad M_0 = 0.924,$$

$$M_\varrho = \frac{3.539656}{3}, \qquad M_\eta = \frac{15.050656}{3}, \qquad M_{\sigma(t_m)} = \frac{15.282656}{3}, \qquad (3.23)$$

and let $b = 0.0001$, $k = 0.055$, $c = 0.102\,585\,312$, $a = 0.045$, $h = 0.000125$. Clearly, we can verify that the conditions in Theorem 3.3 are fulfilled. Thus, by Theorem 3.3, the BVP (3.21) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{t \in [\varrho, \eta]_{\mathbb{T}}} u_1(t) < 0.045 < \max_{t \in [\varrho, \eta]_{\mathbb{T}}} u_3(t), \qquad \min_{t \in [\varrho, \eta]_{\mathbb{T}}} u_3(t) < 0.0001 < \min_{t \in [\varrho, \eta]_{\mathbb{T}}} u_2(t). \qquad (3.24)$$

Remark 3.6. If we let $R = 0.102\,585\,312$, $r = 0.0001$, $\iota = r + 10^{-5}$, $\upsilon = 0.055$, we can also verify that the conditions in Theorem 3.1 are satisfied.

Acknowledgments

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