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Research Article

Limit Cycles of a Class of Hilbert's Sixteenth Problem Presented by Fractional Differential Equations

G. H. Erjaee,[1,2] H. R. Z. Zangeneh,[3] and N. Nyamoradi[3]

[1] *Mathematics Department, Qatar University, Doha 2713, Qatar*
[2] *Mathematics Department, Shiraz University, Shiraz 13797-71467, Iran*
[3] *Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran*

Correspondence should be addressed to G. H. Erjaee, erjaee@qu.edu.qa

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The second part of Hilbert's sixteenth problem concerned with the existence and number of the limit cycles for planer polynomial differential equations of degree n. In this article after a brief review on previous studies of a particular class of Hilbert's sixteenth problem, we will discuss the existence and the stability of limit cycles of this class in the form of fractional differential equations.

1. Introduction

The second part of the well-known Hilbert's 16th problem is still unsolved since Hilbert proposed it in 1900. This problem is concerned with the maximum number of limit cycles and their relative distributions of the real planar polynomial systems of degree n in the form of

$$\begin{aligned} \frac{dx}{dt} &= P(x,y), \\ \frac{dy}{dt} &= Q(x,y), \end{aligned} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomial of degree n with real coefficients. The general form of this problem, even for $n = 2$, is yet an open problem that has attracted more researches but it is remarkably inflexible. With the development of computer's and graphical software, many recent new improvement results have been obtained. Some survey articles can be found in [1–5] and references therein. One of the classical methods to produce and study limit cycles in such system (1.1) is by perturbing a system which has a centre (e.g., see [6, 7]). In such methods the limit cycles are produced in the perturbed system from the periodic orbits of the periodic annulus of the unperturbed system. As we can see in [8] by perturbing the linear centre $dx/dt = -y$, $dy/dt = x$, using arbitrary polynomials P and Q of degree n, $[(n-1)/2]$ limit cycles bifurcated with the bifurcation parameter ε of order one. Almost the same argument can be seen in [9] by perturbing the system $dx/dt = -y(1+x)$, $dy/dt = x(1+x)$ with maximum n limit cycles. By perturbing the Hamiltonian centre given by $H = 0.5y^2 + x^{n+1}/(n+1)$ in the polynomial differential systems of odd degree n, we can obtain $(n+1)(n+3)/8 - 1$ limit cycles [10]. Several other similar investigations have been done using the perturbed polynomial differential systems of second, third, or even more degree. For example, see [11–13] and references therein.

Based on the above studies, some of the authors of this article investigated the number of limit cycles of perturbed quintic Hamiltonian systems with different degree polynomials [14, 15]. In these former articles a weakened Hilbert's 16th problem in the following form is considered:

$$\begin{aligned} \frac{dx}{dt} &= H_y + \varepsilon P(x,y), \\ \frac{dy}{dt} &= -H_x + \varepsilon Q(x,y). \end{aligned} \tag{1.2}$$

In system (1.2) $H(x,y)$ is a real polynomial of degree n, and $P(x,y)$ and $Q(x,y)$ are two real polynomial of degree m. Moreover, system (1.2) contains at least a family of closed orbits for any level curve $H(x,y) = h$ with $h \in R^2$ and $0 < \varepsilon \ll 1$. A full investigation of this planar system for the number of limit cycles and their stabilities can be found in [15]. In this article we study the existence of limit cycles and their stabilities for such system in the form of Fractional Differential Equations (FDEs). Recently great considerations have been made to the systems of FDE. The most essential property of these systems is their *nonlocal property* which does not exist in the integer-order differential operators. We mean by this property that the next state of a system depends not only upon its current state but also upon all of its historical states. This is a more realistic and is one reason why fractional calculus has become more and more popular. On the other hand, the integer-order differential operator is indifferent to its history. Furthermore, there have been several recent mathematical discoveries that have helped to unlock the power of the fractional derivative [16]. One such discovery is that of fractal functions. Indeed, most of the functions that we are familiar with are smooth. This means that locally they can be approximated by a straight line segment. For example, the function $f(x) = x^2$ is well approximated by $2x - 1$ at the point $x = 1$. The derivative of the function at a particular point provides the slope of the straight line approximation or tangent to the curve. Fractal functions are not smooth. They have details on all scales and they cannot be approximated locally by straight line segments. An example is the Weierstrass function which can be written as the infinite sum of cosine functions, $f(x) = \sum_{n=0}^{\infty}(1/2)^n cos(3^n x)$. For this function at the point $x = 1$, the tangent changes orientation under increasing magnification. Functions such as the Weierstrass function cannot be differentiated (a whole number of times). But it turns out that these fractal functions can be differentiated a fractional number of times, and the fractional calculus is important for studying these differentiability properties. Fractals are characterized by scaling laws and the fractional derivative at a point can reveal this law. In recent research, scientists at the Mount Sinai School of Medicine have shown that the surfaces of breast cells are fractals and they have found clear differences in the scaling laws for benign cells and malignant cells. The different scaling laws have enabled accurate diagnosis of breast cancers. Another important new discovery that has brought fractional calculus into prominence is that many physical processes are modeled by fractional differential equations. Obviously, the importance of a mathematical model is that it can be used to make predictions and to give insight into the physical process that underlies the behavior. One area where mathematical models have been employed extensively is that of diffusion and transport processes. For example, the dispersion of pollutants in the ocean and the motion of electronic charges in conductors are diffusion processes. Here, a probabilistic description leads to a (whole number) differential equation which can be solved to predict average properties of the system. Similar types of equations are used by financial analysts to model stock prices. It has recently been discovered that processes governed by diffusion which is enhanced or hindered in some fashion are better modeled by FDEs than by integer-order differential equations. These FDEs are finding numerous applications in areas ranging from financial mathematics to ocean-atmosphere dynamics to mathematical biology [16].

These and the other applications of FDEs provide a good motivation for study such Hilbert's 16th problem of system (1.2) in the form of FDE. So, in the next section we will consider system (1.2) in the form of FDEs and to be more specific we will take H_x, H_y as polynomials of degree 1 and $P(x,y)$, $Q(x,y)$ as polynomials of degrees 3 and 5, respectively. Due to the existence of Riemann-Liouville integral operator in the definition of FDE in the Caputo sense [17], direct analytical solution for FDE is too rare, and so using the numerical methods is inevitable. In order to use a reliable numerical method we should first discretized the given FDE. However, discretization schemes that produce difference equations whose dynamics resemble that of their continuous counterparts are a major challenge in numerical analysis. To this end we will apply the Mickens nonstandard discretization scheme [18] to the Grunwald-Letnikov discretization process for our system of FDE. As we will see in Section 3 this discretization scheme leads to the fast convergence with more accurate results in solving the original system (1.2) with integer-order derivative one. Therefore, we are expecting the same accurate results for system (1.2) in the form of FDE with different noninteger-order derivative. Then in Section 4 we will discuss the stability of limit cycle which exists in our system and illustrate the numerical results. We will summarize the results with some final comments in Section 5.

2. Specific Case of the Weakened Hilbert's 16th Problem

We consider the specific case of system (1.2) as

$$\begin{aligned} \frac{dx}{dt} &= y + \varepsilon\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)|y|, \\ \frac{dy}{dt} &= -x - x^5, \end{aligned} \tag{2.1}$$

where a_0, a_1, a_2, and a_3 are real constants. As discussed in [15] there is a closed relation between the number of the limit cycles in (2.1) and the number of zeros of its related Abelian integral [19]. The related Abelian integral of (2.1) stated as

$$A(h) = \oint_{\Gamma_h} P(x,y)dy = \iint_{H<h} \frac{\partial P}{\partial x} dy\, dx = \iint_{D_h} \sum_{j=1}^{3} a_j x^{j-1} |y| dy\, dx, \tag{2.2}$$

where D_h is the area surrounded by the first integral curves of (2.1), that is,

$$\Gamma_h : H(x,y) = \frac{x^2}{2} + \frac{x^6}{6} + \frac{y^2}{2} = h, \quad h > 0 \tag{2.3}$$

as $\varepsilon \to 0$ and $P(x,y) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3)|y|$. Now to evaluate the Abelian integral in (2.2), first we note that the limits of the double integrals can be found by solving the first integral (2.3) for y, that is, $y_{1,2} = \pm\sqrt{2h - x^2 - x^3/3}$ and then for x by solving $3x^2 + x^6 - 6h = 0$ when $y = 0$, which yields $x_1 = -\sqrt{\sqrt[3]{(3h + \sqrt{9h^2+1})^2} - 1/\sqrt[3]{3h + \sqrt{9h^2+1}}}$ with $x_2 = -x_1$. Hence, with the symmetry of D_h which exists with respect to $y = 0$ and noting that $\int_{x_1}^{x_2} y_2^2 \sum_{j=1}^{3} (2j) a_{2j} x^{2j-1}\, dx = 0$, integral (2.2) yields $A(h) = 2\sum_{j=1}^{3} a_{2j-1}(2j-1) \int_0^{x_2} x^{2j-2}(2h - x^2 - x^6/3)dx$. After evaluating and simplifying this equation as a polynomial of h, we get

$$A(\mu) = 4\mu^{3/2} \sum_{j=1}^{3} a_{2j-1} \left(\frac{\mu^{j+1}}{2j+5} + \frac{\mu^{j-1}}{2j+1} \right). \tag{2.4}$$

Note that here we replace $h = (1/6)\mu(\mu^2 + 3)$ where $\mu = x_2^2$. Finally, with this brief discussion the existence and stability of limit cycle for perturbed system (2.1) can be finalized in the following theorem.

Theorem 2.1. *The perturbed system* (2.1) *has no limit cycle for $a_1 a_3 > 0$ and one limit cycle for $a_1 a_3 < 0$. In the former case the unique limit cycle is stable for $a_3 < 0$ and unstable for $a_3 > 0$.*

For the proof of this theorem, as discussed above, we need to find the zero of the Abelian integral (2.4) which leads to a polynomial of degree 3 with respect to μ. Then it is straight forward to see that this polynomial has no positive root for $a_1 a_3 > 0$ and at least one positive real root for $a_1 a_3 < 0$. That is, in the first case system (2.1) has no limit cycle and in the former case there is one limit cycle. For the detail proof of this theorem refer to [15].

3. System (2.1) in the Form of Fractional Differential Equations and Its Discretization

In general, $D^\alpha y(t) = f(t, y(t))$, $T \geq t \geq 0$, $y(t_0) = y_0$, and $\alpha > 0$ is a single initial value FDE, where D^α denotes the fractional derivative in the Caputo sense [17] and defined by

$D^{\alpha}y(t) = J^{n-\alpha}D^{n}y(t)$. Here $-1 < \alpha \leq n$, $n \in N$ and J^{n} is the nth -order Riemann-Liouville integral operator defined as

$$J^{n}y(t) = \frac{1}{\Gamma(n)}\int_{0}^{t}(t-\tau)^{n-1}y(\tau)d\tau, \tag{3.1}$$

with $t > 0$. A limited number of methods have been utilized to solve this initial value problem. In order to apply Mickens' nonstandard discretization scheme [18] in our numerical scheme we choose the Grunwald-Letnikov method to approximate the one-dimensional fractional derivative as follows [20]:

$$D^{\alpha}y(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[t/h]}(-1)^{j}\binom{\alpha}{j}y(t-jh), \tag{3.2}$$

where $[t]$ denotes the integer part of t and h is the step size. In this case the above initial value problem is discretized as

$$\sum_{j=0}^{[t_n/h]} c_j^{\alpha} y(t_{n-j}) = f(t_n, y(t_n)), \quad n = 1, 2, 3, \ldots, \tag{3.3}$$

where $t_n = nh$ and c_j^{α} are the Grunwald-Letnikov coefficients defined as $c_j^{\alpha} = h^{-\alpha}(-1)^{j}\binom{\alpha}{j}$, $j = 0, 1, 2, \ldots$, or recursively $c_0^{\alpha} = h^{-\alpha}$ and $c_j^{\alpha} = (1-(1+\alpha)/j)c_{j-1}^{\alpha}$, $j = 1, 2, 3, \ldots$.

Now using this definition for FDE with Grunwald-Letnikov discretization method, system (2.1) in the form of FDE is discretized as follows:

$$\begin{aligned}\sum_{j=0}^{[t_n/h]} C_j^{\alpha_1} x(t_{n-j}) &= y(t_{n-1}) + \varepsilon\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)\left|y(t_{n-1})\right|,\\ \sum_{j=0}^{[t_n/h]} C_j^{\alpha_2} y(t_{n-j}) &= -x(t_{n-1}) - x^5(t_{n-1}).\end{aligned} \tag{3.4}$$

We assert that nonstandard discretization method is a numerical attempt which can be used in discretization process of FDE to get the better results and preserves their crucial property, that is, *nonlocal property*. In order to do this, we apply the Mickens nonstandard discretization scheme [18] to the Grunwald-Letnikov discretization process for FDE system (3.4). Indeed, the derivative term, $y'(t)$, in the Mickens schemes is replaced by $(y(t+h)-y(t))/\varphi(h)$, where $\varphi(h)$ is a continuous function of step size h. In addition the nonlinear terms such as $y(t)x(t)$ are either replaced by $y(t)x(t+h)$, $y(t+h)x(t)$ or left untouched depending upon the context of the differential equation. There is no appropriate general method for choosing the function $\varphi(h)$, but some special techniques may be found in [18, 21].

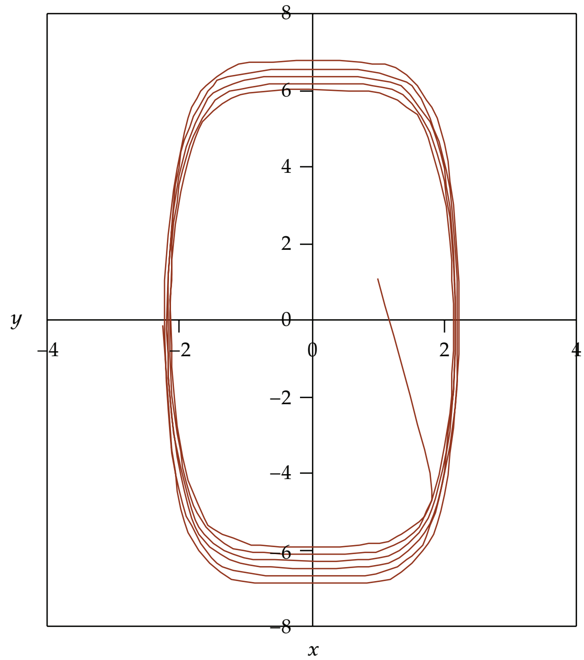

Figure 1: Stable limit cycle of system (3.5) for $\alpha = 1$, $a_1 = 1$, $a_3 = -10$, and $\varepsilon = 0.01$ with starting point $(1, 1)$.

Now we first write system (3.4) as follows:

$$x_{n+1} = \alpha x_n + h^\alpha y_n + h^\alpha \varepsilon\left(a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3\right)\left|y_n\right| - h^\alpha \sum_{j=1}^{[t_n/h_1]} C_j^\alpha x_{n-j},$$
$$y_{n+1} = \alpha y_n - h^\alpha\left(x_n + x_n^5\right) - h^\alpha \sum_{j=1}^{[t_n/h_2]} C_j^\alpha y_{n-j}. \tag{3.5}$$

Here, we replaced $x(t_n)$ and $y(t_n)$ by x_n and y_n, respectively. Later on, following Mickens' method in the next section, for finding the better results we replace the nonlinear terms in system (3.5) by appropriate combination of the variables in different levels of times.

4. Stability of the Limit Cycles in System (3.5) and Numerical Results

First we note that the linearized system (3.5), around a stationary point (x_n^*, y_n^*) or simply (x_n, y_n), will be $\left[\begin{smallmatrix} x_{n+1} \\ y_{n+1} \end{smallmatrix}\right] = \mathbf{L}\left[\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right]$ where matrix $\mathbf{L}$ is evaluated as

$$\mathbf{L} = \begin{bmatrix} \alpha + \varepsilon h^\alpha y_n\left(a_1 + 2a_2 x_n + 3a_3 x_n^2\right) & -h^\alpha\left(1 + 5x_n^4\right) \\ h^\alpha\left[1 + \varepsilon\left(a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3\right)\right] & \alpha \end{bmatrix}. \tag{4.1}$$

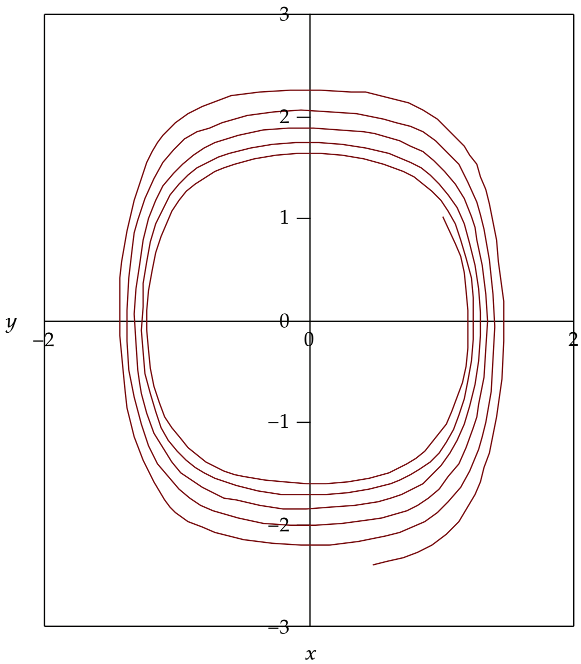

Figure 2: Unstable limit cycle of system (3.5) for $a_1 = 1, a_3 = 2$, and $\varepsilon = 0.01$ with starting point (1, 1).

Without losing our generality, we suppose here y_n to be positive. Now, from theory of dynamical systems, the limit cycle exists in system (3.5) whenever the characteristic equation of matrix,$|\mathbf{L} - \lambda \mathbf{I}| = 0$, has two solutions with module one.

In addition, for the stability of this limit cycle we can use the stability analysis which is thoroughly investigated by Matignon in [22]. To utilize this theorem for our problem, first we consider the linearization of system (2.1) in the form of FDE with the derivative order α in both equations around a given stationary point (x, y). This linearized system can be written as $D^{\alpha}\mathbf{X}(t) = \mathbf{M}\mathbf{X}(t)$ where matrix $\mathbf{M}$ is similar to matrix $\mathbf{L}$ in (4.1) with $\alpha = 0$ and $(x_n, y_n) = (x, y)$. Now the Matignon stability theorem for our problem can be stated as the following theorem.

Theorem 4.1. *The linearized system of fractional differential equations $D^{\alpha}\mathbf{X}(t) = \mathbf{M}\mathbf{X}(t)$ is asymptotically stable if and only if $|\arg(spec(\mathbf{M})| > \alpha\pi/2$.*

Note that the stability exists if and only if either asymptotically stability exists or those eigenvalues which satisfy $|\arg(\mathrm{spec}(M)| = \alpha\pi/2$ have geometric multiplicity one.

Now we will implement our numerical method described above for the existence of limit cycles in system (2.1) in the form of FDE for different values of fractional order α. In order to be consistence with the results in [15], in system (3.5) we let the constants a_0, a_2 be one and choose a_1, a_3 according to the following discussion with $\varepsilon = 0.01$. By these assumptions characteristic equation of matrix $\mathbf{L}$ in (4.1) at the point (1, 1) will be

$$|L - \lambda I| = \begin{vmatrix} \alpha + h^{\alpha}[0.01(a_1 + 2 + 3a_3)] - \lambda & -6h^{\alpha} \\ h^{\alpha}[1 + 0.01(a_1 + 2 + a_3)] & \alpha - \lambda \end{vmatrix} = 0, \tag{4.2}$$

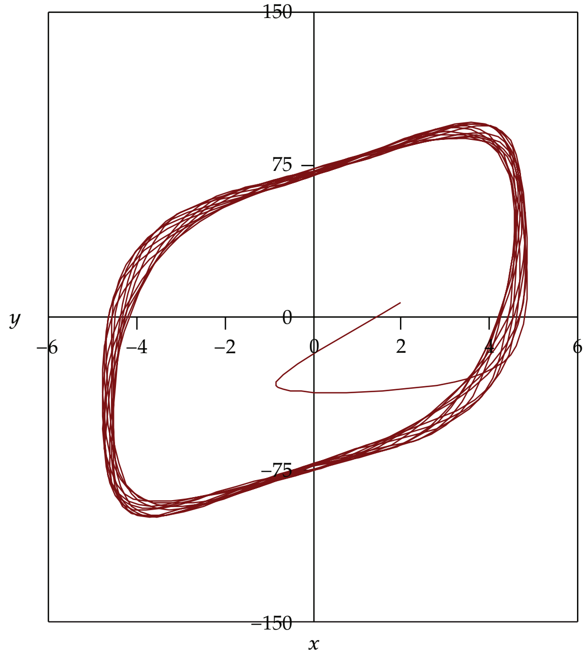

Figure 3: Stable limit cycle of system (3.5) for $\alpha = 0.97$, $a_1 = 1$, $a_3 = -10$, and $\varepsilon = 0.001$ with starting point $(1, 1)$.

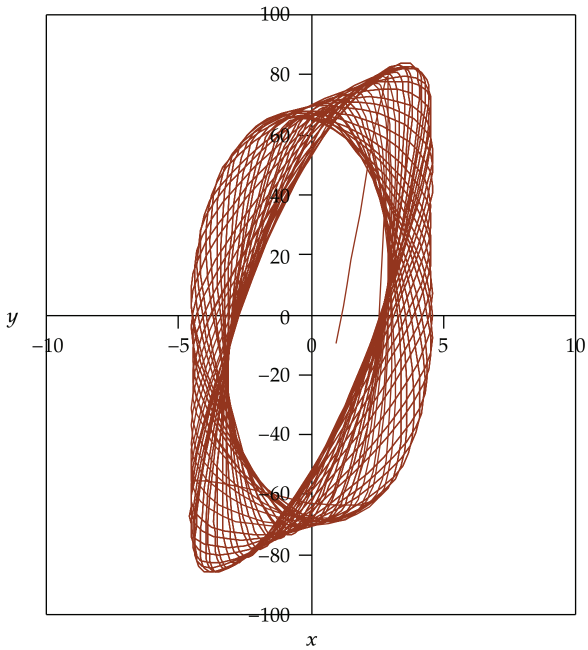

Figure 4: Stable limit cycle of system (3.5) for $\alpha = 0.95$, $a_1 = 1$, $a_3 = -10$, and $\varepsilon = 0.001$ with starting point $(1, 1)$.

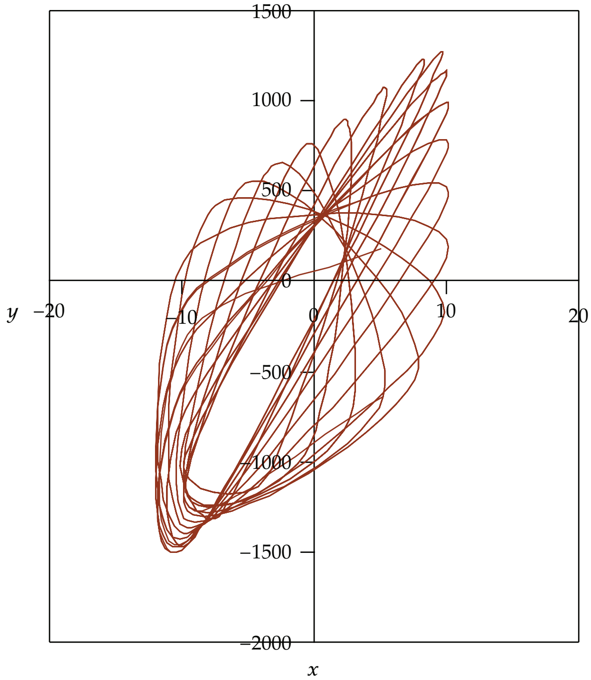

Figure 5: Stable but sensitive limit cycle of system (3.5) for $\alpha = 0.945$, $a_1 = 1$, $a_3 = -10$, and $\varepsilon = 0.001$ with starting point $(1, 1)$.

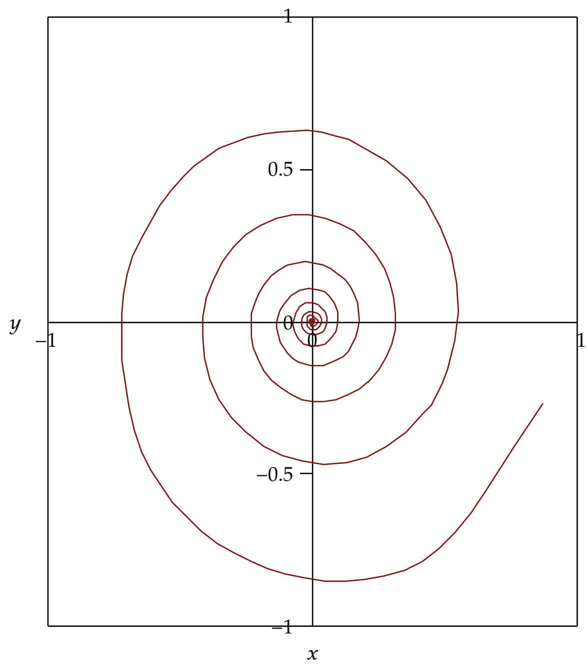

Figure 6: Numerical results of system (3.5) which is converging to zero for $\alpha = 0.94$, $a_1 = 1$, $a_3 = -10$, and $\varepsilon = 0.001$ with starting point $(1, 1)$.

which yields

$$\lambda^2 - \lambda\{2\alpha + h^\alpha[0.02 + 0.01(a_1 + 3a_3)]\} + \alpha h^\alpha[0.02 + 0.01(a_1 + 3a_3)] + 6h^{2\alpha}[1.02 + 0.01(a_1 + a_3)] + \alpha^2 = 0. \tag{4.3}$$

Equation (4.3) has two solutions with modular one if

$$\begin{gathered} |2\alpha + h^\alpha[0.02 + 0.01(a_1 + 3a_3)]| \le 2, \\ \alpha h^\alpha[0.02 + 0.01(a_1 + 3a_3)] + 6h^{2\alpha}[1.02 + 0.01(a_1 + a_3)] + \alpha^2 = 1. \end{gathered} \tag{4.4}$$

First, we note that for $\alpha = 1$ the inequality and equality occur in (4.4) if $a_1a_3 < 0$. That is, for some choice of a_1 and a_3 with opposite signs there is a limit cycle for system (3.5). This result agrees with the consequence of Theorem 2.1. Suppose that we choose $a_1 = 1$ and $a_3 = -10$; then the numerical solutions results of system (3.5) are illustrated in Figure 1. As we stated before, in order to get the better results in solving this system, following Mickens' method, we replace nonlinear terms x_n^2, x_n^3, and x_n^5 with x_nx_{n-1}, $x_n^2x_{n-1}$, and $x_n^3x_{n-1}^2$, respectively. Note that, given a_1, a_3, and α from conditions (4.4) we can evaluate the best choice for h (or $\varphi(h)$). Here, we get $h = 0.00487$.

For the stability of this limit cycle, since $\alpha = 1$, by the well-known theories of dynamical systems we should find the eigenvalues of the linearized system (2.1) at the given point (1,1). In this case, with the choice of the above parameters, these eigenvalues are $\lambda_{1,2} = -0.135 \pm i\,2.366057$. Obviously, since the signs of real parts of λ_1 and λ_2 are negative, the limit cycle in Figure 1 is stable. With similar discussion, if we choose a_3 to be a positive constant, say 2, then the related eigenvalues will be $\lambda_{1,2} = 0.045 \pm i\,2.50957666$ with the positive real parts. In this case, as we can see in Figure 2, the limit cycle is unstable. These results agree with consequence of Theorem 2.1.

Now for the fractional order $\alpha < 1$, say $\alpha = 0.97$, with the same values as above for a_i, $i = 0, 1, 2, 3$ and $\varepsilon = 0.001$, conditions (4.4) are satisfied. So by these values of the parameters, there is a limit cycle for the system (3.5). As illustrated in Figures 2, 3, 4, and 5 these limit cycles exist for different values of $\alpha \in (0.94, 1]$. For the stability of these limit cycles we may apply Theorem 4.1. For example, for $\alpha = 0.95$ corresponding eigenvalues of matrix $\mathbf{M}$ at the point $(1, 1)$ can be found as $|\mathbf{M} - \boldsymbol{\lambda}\mathbf{I}| = 0$ *or* $\lambda^2 + 0.027\lambda + 5.958 = 0$ which yields $\lambda_{1,2} = -0.0135 \pm i\,2.44086414$ with argument $\theta = 1.5652655^R$. Obviously, this value of θ is beiger than $\alpha(\pi/2)$ for $\alpha \in (0.94, 1)$, which proves the stability of the limit cycles whenever exist.

5. Final Comments

In this article we discussed the existence and stability of the limit cycle for special case of perturbed Hilbert's 16th problem. We found these limit cycles for different values of fractional order $\alpha \in (0.94, 1]$ using discretized system (3.5), provided by Grunwald-Letnikov numerical method for solving FDEs and applying Mickens' nonstandard method for more accurate results. The difficulties that we are facing here are in solving system (3.5) for values of $\alpha < 0.94$. Though this is the case for different nonlinear systems of FDEs, existing numerical methods are not capable for solving such these systems for small fractional derivative order α. In other words, for the small *efficient dimension*, which is the sum of fractional derivatives of the equations in the systems such as system (2.1), the numerical results are not accurate enough. Here, for the values of $\alpha < 0.94$ with the same values for other parameters as above, the results are found by solving system (3.5) all converging to zero (see Figure 6).

Another difficulty exists in choosing a_1 and a_3 for conditions in (4.4) to be satisfied. That is, whenever $\alpha < 1$ by choosing small positive values for a_1 and a_3 conditions (4.4) are satisfied, but the numerical limit cycle cannot be found in system (3.5) even in unstable form. Nevertheless, as we saw the limit cycles exist for the values a_1 and a_3 with different signs. In particular these limit cycles are stable, easy to find for values $a_3 < -4$, and agreed with the stability condition in Theorem 4.1.

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